SOME REVERSES OF THE JENSEN INEQUALITY
WITH APPLICATIONS

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Abstract

Two new reverses of the celebrated Jensen’s inequality for convex functions in the general setting of
the Lebesgue integral, with applications to means, Hőlder’s inequality and \( f \)-divergence measures in
information theory, are given.

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divergence measures, \( f \)-divergence measures.

1. Introduction

Let \((\Omega, \mathcal{A}, \mu)\) be a measurable space consisting of a set \(\Omega\), a \(\sigma\)-algebra \(\mathcal{A}\) of parts
of \(\Omega\) and a countably additive and positive measure \(\mu\) on \(\mathcal{A}\) with values in \(\mathbb{R} \cup \{\infty\}\).
For a \(\mu\)-measurable function \(w : \Omega \to \mathbb{R}\), with \(w(x) \geq 0\) for \(\mu\)-a.e. (almost every) \(x \in \Omega\),
consider the Lebesgue space

\[
L_w(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_{\Omega} w(x) |f(x)| \, d\mu(x) < \infty \right\}.
\]

For simplicity of notation, we write everywhere in the following \(\int_{\Omega} w \, d\mu\) instead of \(\int_{\Omega} w(x) \, d\mu(x)\).

If \(f, g : \Omega \to \mathbb{R}\) are \(\mu\)-measurable functions, \(\int_{\Omega} w \, d\mu = 1\) and \(f, g, fg \in L_w(\Omega, \mu)\),
then we may consider the \(\check{C}eby\v{s}e\v{v}\ functional\)

\[
T_w(f, g) := \int_{\Omega} wfg \, d\mu - \int_{\Omega} wf \, d\mu \int_{\Omega} wg \, d\mu.
\]

The following result is known in the literature as the Grüss inequality:

\[
|T_w(f, g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta),
\]

\(\gamma\), \(\Gamma\) and \(\delta\), \(\Delta\) are the means of functions \(w \circ f\) and \(w \circ g\), respectively.

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provided
\[ -\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \]
for \( \mu \)-a.e. \( x \in \Omega \).

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller constant.

If we assume that \( -\infty < \gamma \leq f(x) \leq \Gamma < \infty \) for \( \mu \)-a.e. \( x \in \Omega \), then, by the Grüss inequality for \( g = f \) and by Schwarz’s integral inequality,
\[
\int_{\Omega} w \left| f - \int_{\Omega} w f \, d\mu \right| \, d\mu \leq \left( \int_{\Omega} w f^2 \, d\mu - \left( \int_{\Omega} w f \, d\mu \right)^2 \right)^{1/2} \leq \frac{1}{2} (\Gamma - \gamma). \quad (1.1)
\]

To provide a reverse of the celebrated Jensen’s integral inequality for convex functions, in 2002, the author [12] obtained the following result.

**Theorem 1.1.** Let \( \Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R} \) be a differentiable convex function on \((m, M)\) and \( f : \Omega \rightarrow [m, M] \) such that \( \Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu) \), where \( w \geq 0 \) \( \mu \)-a.e. on \( \Omega \) with \( \int_{\Omega} w \, d\mu = 1 \). Then we have the inequality
\[
0 \leq \int_{\Omega} w (\Phi \circ f) \, d\mu - \Phi \left( \int_{\Omega} w f \, d\mu \right) \\
\leq \int_{\Omega} w (\Phi' \circ f) f \, d\mu - \int_{\Omega} w (\Phi' \circ f) \, d\mu \int_{\Omega} w f \, d\mu \quad (1.2)
\leq \frac{1}{2} (\Phi'(M) - \Phi'(m)) \int_{\Omega} w \left| f - \int_{\Omega} w f \, d\mu \right| \, d\mu.
\]

For a generalisation of the first inequality in (1.2) without the differentiability assumption and the derivative \( \Phi' \) replaced with a selection \( \varphi \) from the subdifferential \( \partial \Phi \), see Niculescu [27].

If \( \mu(\Omega) < \infty \) and \( \Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L(\Omega, \mu) \), then we have the inequality
\[
0 \leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi \circ f) \, d\mu - \Phi \left( \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu \right) \\
\leq \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi' \circ f) f \, d\mu - \frac{1}{\mu(\Omega)} \int_{\Omega} (\Phi' \circ f) \, d\mu \cdot \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu \\
\leq \frac{1}{2} (\Phi'(M) - \Phi'(m)) \frac{1}{\mu(\Omega)} \int_{\Omega} \left| f - \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu \right| \, d\mu.
\]

The following discrete inequality is of interest as well.

**Corollary 1.2.** Let \( \Phi : [m, M] \rightarrow \mathbb{R} \) be a differentiable convex function on \((m, M)\). If \( x_i \in [m, M] \) and \( w_i \geq 0 \) \( (i = 1, \ldots, n) \) with \( W_n := \sum_{i=1}^{n} w_i = 1 \), then we have the
counterpart of Jensen’s weighted discrete inequality:

\[
0 \leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i\right) \\
\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\
\leq \frac{1}{2}(\Phi'(M) - \Phi'(m)) \sum_{i=1}^{n} w_i \left|x_i - \sum_{j=1}^{n} w_j x_j\right|
\]

(1.3)

**Remark 1.3.** The inequality between the first and the second terms in (1.3) was proved in 1994 by Dragomir and Ionescu [15].

Using the results (1.2) and (1.1), we can state the following string of reverse inequalities:

\[
0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu - \Phi\left(\int_{\Omega} w f \, d\mu\right) \\
\leq \int_{\Omega} w(\Phi' \circ f) f \, d\mu - \int_{\Omega} w(\Phi' \circ f) \, d\mu \int_{\Omega} w f \, d\mu \\
\leq \frac{1}{2}(\Phi'(M) - \Phi'(m)) \int_{\Omega} w \left|f - \int_{\Omega} w f \, d\mu\right| \, d\mu \\
\leq \frac{1}{2}(\Phi'(M) - \Phi'(m)) \left(\int_{\Omega} w f^2 \, d\mu - \left(\int_{\Omega} w f \, d\mu\right)^2\right)^{1/2} \\
\leq \frac{1}{4}(\Phi'(M) - \Phi'(m))(M - m),
\]

provided that \(\Phi : [m, M] \subset \mathbb{R} \rightarrow \mathbb{R}\) is a differentiable convex function on \((m, M)\) and \(f : \Omega \rightarrow [m, M]\) such that \(\Phi \circ f, f, \Phi' \circ f, (\Phi' \circ f) \cdot f \in L_w(\Omega, \mu)\), where \(w \geq 0\) \(\mu\)-a.e. on \(\Omega\) with \(\int_{\Omega} w \, d\mu = 1\).

**Remark 1.4.** The inequality between the first, second and last terms from (1.4) was proved in the general case of positive linear functionals in 2001 by the author [11].

Motivated by the above results, we establish in the current paper two new reverses of Jensen’s integral inequality for a convex function. Some natural applications for inequalities between means, reverses of Hölder’s inequality and for the \(f\)-divergence measure that play an important role in information theory are given as well.

### 2. Reverse inequalities

The following reverse of Jensen’s inequality holds.

**Theorem 2.1.** Let \(\Phi : I \rightarrow \mathbb{R}\) be a continuous convex function on the interval of real numbers \(I\) and let \(m, M \in \mathbb{R}, \; m < M\), with \([m, M] \subset \hat{I}\) (where \(\hat{I}\) is the interior of \(I\)).
If \( f : \Omega \rightarrow \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds
\[-\infty < m \leq f(x) \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega\]
and is such that \( f, \Phi \circ f \in L_w(\Omega, \mu) \), where \( w \geq 0 \) \( \mu \text{-a.e.} \) on \( \Omega \) with \( \int_{\Omega} w \, d\mu = 1 \), then

\[
0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu - \Phi(\tilde{f}_{\Omega,w}) \\
\leq \frac{(M - \tilde{f}_{\Omega,w})(\tilde{f}_{\Omega,w} - m) - (\Phi'(M) - \Phi'(m))}{M - m} \\
\leq \frac{1}{4} (M - m)(\Phi'(M) - \Phi'(m)),
\]
where \( \tilde{f}_{\Omega,w} := \int_{\Omega} w(x)f(x) \, d\mu(x) \in [m, M] \) and \( \Psi_{\Phi}(\cdot; m, M) : (m, M) \rightarrow \mathbb{R} \) is defined by
\[
\Psi_{\Phi}(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.
\]

We also have the inequality

\[
0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu - \Phi(\tilde{f}_{\Omega,w}) \leq \frac{1}{4} (M - m)\Psi_{\Phi}(\tilde{f}_{\Omega,w}; m, M) \\
\leq \frac{1}{4} (M - m)(\Phi'(M) - \Phi'(m)),
\]
provided that \( \tilde{f}_{\Omega,w} \in (m, M) \).

**Proof.** By the convexity of \( \Phi \),
\[
\int_{\Omega} w(x)\Phi(f(x)) \, d\mu(x) - \Phi(\tilde{f}_{\Omega,w}) \\
= \int_{\Omega} w(x)\Phi\left(\frac{m(M - f(x)) + M(f(x) - m)}{M - m}\right) \, d\mu(x) \\
\leq \int_{\Omega} \frac{(M - \tilde{f}_{\Omega,w})\Phi(m) + (f(x) - m)\Phi(M) - \Phi\left(\frac{m(M - \tilde{f}_{\Omega,w}) + M(\tilde{f}_{\Omega,w} - m)}{M - m}\right)}{M - m} \\
\leq \int_{\Omega} \frac{(M - \tilde{f}_{\Omega,w})\Phi(m) + (f(x) - m)\Phi(M)}{M - m} - \Phi\left(\frac{m(M - \tilde{f}_{\Omega,w}) + M(\tilde{f}_{\Omega,w} - m)}{M - m}\right) := B.
\]
By denoting

\[ \Delta_\Phi(t; m, M) := \frac{(t - m)\Phi(M) + (M - t)\Phi(m)}{M - m} - \Phi(t), \quad t \in [m, M], \]

we have

\[ \Delta_\Phi(t; m, M) = \frac{(t - m)\Phi(M) + (M - t)\Phi(m) - (M - m)\Phi(t)}{M - m} \]

\[ = \frac{(t - m)\Phi(M) + (M - t)\Phi(m) - (M - t + t - m)\Phi(t)}{M - m} \]

\[ = \frac{(t - m)(\Phi(M) - \Phi(t)) - (M - t)(\Phi(t) - \Phi(m))}{M - m} \]

\[ = \frac{(M - t)(t - m)}{M - m} \Psi_\Phi(t; m, M) \]

for any \( t \in (m, M) \).

Therefore we have the equality

\[ B = \frac{(M - \tilde{f}_{\Omega,w})(\tilde{f}_{\Omega,w} - m)}{M - m} \Psi_\Phi(\tilde{f}_{\Omega,w}; m, M), \quad (2.4) \]

provided that \( \tilde{f}_{\Omega,w} \in (m, M) \).

For \( \tilde{f}_{\Omega,w} = m \) or \( \tilde{f}_{\Omega,w} = M \) the inequality (2.1) is obvious. If \( \tilde{f}_{\Omega,w} \in (m, M) \), then

\[ \Psi_\Phi(\tilde{f}_{\Omega,w}; m, M) \leq \sup_{t \in (m, M)} \Psi_\Phi(t; m, M) \]

\[ = \sup_{t \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M - t} \right) \cdot \left( \frac{\Phi(t) - \Phi(m)}{t - m} \right) \]

\[ \leq \sup_{t \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M - t} \right) + \sup_{t \in (m, M)} \left( \frac{\Phi(t) - \Phi(m)}{t - m} \right) \]

\[ = \sup_{t \in (m, M)} \left( \frac{\Phi(M) - \Phi(t)}{M - t} \right) - \inf_{t \in (m, M)} \left( \frac{\Phi(t) - \Phi(m)}{t - m} \right) \]

\[ = \Phi'_-(M) - \Phi'_+(m), \]

which by (2.3) and (2.4) produces the desired result (2.1).

Since, obviously,

\[ \frac{(M - \tilde{f}_{\Omega,w})(\tilde{f}_{\Omega,w} - m)}{M - m} \leq \frac{1}{4}(M - m), \]

then by (2.3) and (2.4) we deduce the first inequality (2.2). The second part is clear. \( \Box \)

**Corollary 2.2.** Let \( \Phi : I \rightarrow \mathbb{R} \) be a continuous convex function on the interval of real numbers \( I \) and \( m, M \in \mathbb{R} \), \( m < M \), with \( [m, M] \subset I \). If \( x_i \in [m, M] \) and \( p_i \geq 0 \)
for \( i \in \{1, \ldots, n\} \) with \( \sum_{i=1}^{n} p_i = 1 \), then we have the inequalities

\[
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \\
\leq \frac{(M - \bar{x}_p)(\bar{x}_p - m)}{M - m} \sup_{t \in (m,M)} \Psi_\Phi(t; m, M) \\
\leq \frac{1}{4} (M - m) (\Phi'_-(M) - \Phi'_+(m)),
\]

and

\[
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \leq \frac{1}{4} (M - m) \Psi_\Phi(\bar{x}_p; m, M) \\
\leq \frac{1}{4} (M - m) (\Phi'_-(M) - \Phi'_+(m)),
\]

where \( \bar{x}_p := \sum_{i=1}^{n} p_i x_i \in (m, M) \).

**Remark 2.3.** Define the weighted arithmetic mean of the positive \( n \)-tuple \( x = (x_1, \ldots, x_n) \) with the nonnegative weights \( w = (w_1, \ldots, w_n) \) by

\[
A_n(w, x) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i,
\]

where \( W_n := \sum_{i=1}^{n} w_i > 0 \), and the weighted geometric mean of the same \( n \)-tuple by

\[
G_n(w, x) := \left( \prod_{i=1}^{n} x_i^{w_i} \right)^{1/W_n}.
\]

It is well known that the following arithmetic mean–geometric mean inequality holds true:

\[
A_n(w, x) \geq G_n(w, x).
\]

Applying the inequality between the first and third terms in (2.5) for the convex function \( \Phi(t) = -\log t \), \( t > 0 \),

\[
1 \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \exp \left( \frac{1}{Mm} (M - A_n(w, x))(A_n(w, x) - m) \right) \\
\leq \exp \left( \frac{1}{4} \frac{(M - m)^2}{mM} \right),
\]

provided that \( 0 < m \leq x_i \leq M < \infty \) for \( i \in \{1, \ldots, n\} \).
Also, if we apply the inequality (2.6) for the same function $\Phi$ we obtain
\[
1 \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \left( \frac{M}{A_n(w, x)} \right)^{M - A_n(w, x) - m} \left( \frac{m}{A_n(w, x)} \right)^{A_n(w, x) - m - (M - m)/4} \leq \exp \left( \frac{1}{4} (M - m)^2 \right).
\]

The following result also holds.

**Theorem 2.4.** With the assumptions of Theorem 2.1, we have the inequalities
\[
0 \leq \int \Omega w(\Phi \circ f) \, d\mu(x) - \Phi(\bar{f}_\Omega, w)
\]
\[
\leq 2 \max \{ M - \bar{f}_\Omega, \bar{f}_\Omega, m \} \left( \frac{\Phi(m) + \Phi(M)}{2} - \Phi \left( \frac{m + M}{2} \right) \right) \tag{2.7}
\]
\[
\leq \frac{1}{2} \max \{ M - \bar{f}_\Omega, \bar{f}_\Omega, m \} (\Phi(M) - \Phi(m)).
\]

**Proof.** We first recall the following result obtained by the author in [14] that provides a refinement and a reverse for the weighted Jensen’s discrete inequality:
\[
\sum_{i=1}^{n} \min \{ p_i \mid \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \}
\]
\[
\leq \frac{1}{P_n} \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi \left( \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) \tag{2.8}
\]
\[
\leq n \max_{i \in \{1, \ldots, n\}} \{ p_i \mid \frac{1}{n} \sum_{i=1}^{n} \Phi(x_i) - \Phi \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \},
\]
where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset $C$ of the linear space $X$, $\{x_i\}_{i \in \{1, \ldots, n\}} \subset C$ are vectors and $\{p_i\}_{i \in \{1, \ldots, n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.8) that
\[
2 \min \{ t, 1 - t \} \left( \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right)
\]
\[
\leq r\Phi(x) + (1 - r)\Phi(y) - \Phi(rx + (1 - r)y)
\]
\[
\leq 2 \max \{ t, 1 - t \} \left( \frac{\Phi(x) + \Phi(y)}{2} - \Phi \left( \frac{x + y}{2} \right) \right) \tag{2.9}
\]
for any $x, y \in C$ and $t \in [0, 1]$. 

If we use the second inequality in (2.9) for the convex function $\Phi : I \to \mathbb{R}$ and $m, M \in \mathbb{R}$, with $[m, M] \subset I$, we have for $t = (M - \bar{f}_{\Omega,w})/(M - m)$ that

$$
\frac{(M - \bar{f}_{\Omega,w})\Phi(m) + (\bar{f}_{\Omega,w} - m)\Phi(M)}{M - m} - \Phi\left(\frac{m(M - \bar{f}_{\Omega,w}) + M(\bar{f}_{\Omega,w} - m)}{M - m}\right)
\leq 2 \max\left\{\frac{M - \bar{f}_{\Omega,w} + \bar{f}_{\Omega,w} - m}{M - m}\left(\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right)\right)\right\}.
$$

(2.10)

Using (2.3) and (2.10) we deduce the first inequality in (2.7).

Since

$$
\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right) = \frac{1}{4}\left(\Phi(M) - \Phi\left(\frac{m + M}{2}\right) - \Phi\left(\frac{m + M}{2} - \frac{m}{2}\right)\right)
$$

and, by the gradient inequality,

$$
\frac{\Phi(M) - \Phi\left(\frac{m + M}{2}\right)}{M - \frac{m + M}{2}} \leq \Phi'(M)
$$

and

$$
\frac{\Phi\left(\frac{m + M}{2}\right) - \Phi(m)}{M - \frac{m + M}{2}} \geq \Phi'(m),
$$

then

$$
\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right) \leq \frac{1}{4}(\Phi'(M) - \Phi'(m)).
$$

(2.11)

Making use of (2.10) and (2.11), we deduce the last part of (2.7).

□

**Corollary 2.5.** With the assumptions in Corollary 2.2, we have the inequalities

$$
0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p)
\leq 2 \max\left\{\frac{M - \bar{x}_p}{M - m}, \frac{\bar{x}_p - m}{M - m}\right\}\left(\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right)\right)
\leq \frac{1}{2}\max\{M - \bar{x}_p, \bar{x}_p - m\}(\Phi'(M) - \Phi'(m)).
$$

**Remark 2.6.** Since, obviously,

$$
\frac{M - \bar{f}_{\Omega,w}}{M - m}, \frac{\bar{f}_{\Omega,w} - m}{M - m} \leq 1,
$$

we obtain from the first inequality in (2.7) the simpler but coarser inequality

$$
0 \leq \int_{\Omega} w(\Phi \circ f) \, d\mu(x) - \Phi(\bar{f}_{\Omega,w}) \leq 2\left(\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right)\right).
$$
The discrete version of this result, namely

$$0 \leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \leq 2 \left( \frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m + M}{2}\right) \right),$$

was obtained in 2008 by Simic [34].

**Remark 2.7.** With the assumptions in Remark 2.3 we have the following reverse of the arithmetic mean–geometric mean inequality

$$1 \leq \frac{A_n(w, x)}{G_n(w, x)} \leq \left( \frac{A(m, M)}{G(m, M)} \right)^{2 \max((M-A_n(w, x))/(M-m), (A_n(w, x)-m)/(M-m))}, \quad (2.12)$$

where $A(m, M)$ is the arithmetic mean and $G(m, M)$ is the geometric mean of the positive numbers $m$ and $M$.

### 3. Applications for the Hölder inequality

It is well known that if $f \in L^p(\Omega, \mu)$, $p > 1$, where the Lebesgue space $L^p(\Omega, \mu)$ is defined by

$$L^p(\Omega, \mu) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid f \text{ is } \mu\text{-measurable and } \int_{\Omega} |f(x)|^p \, d\mu(x) < \infty \right\},$$

and $g \in L^q(\Omega, \mu)$ with $1/p + 1/q = 1$ then $fg \in L^1(\Omega, \mu) = L^1(\Omega, \mu)$ and the Hölder inequality holds true:

$$\int_{\Omega} |fg| \, d\mu \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q}.$$

Assume that $p > 1$. If $h : \Omega \rightarrow \mathbb{R}$ is $\mu$-measurable, satisfies the bounds

$$0 < m \leq |h(x)| \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega$$

and is such that $h, |h|^p \in L_w(\Omega, \mu)$, for a $\mu$-measurable function $w : \Omega \rightarrow \mathbb{R}$, with $w(x) \geq 0$ for $\mu$-a.e. $x \in \Omega$ and $\int_{\Omega} w \, d\mu > 0$, then, from (2.1),

$$0 \leq \frac{\int_{\Omega} |h|^p \, w \, d\mu}{\int_{\Omega} w \, d\mu} - \left( \frac{\int_{\Omega} |h| \, w \, d\mu}{\int_{\Omega} w \, d\mu} \right)^p \leq \frac{M - |h|_{\Omega,w}}{M - m} \frac{|h|_{\Omega,w} - m}{B_p(m, M)} \quad \text{(3.1)}$$

$$\leq p \frac{M^{p-1} - m^{p-1}}{M - m} \frac{1}{(M - |h|_{\Omega,w})(|h|_{\Omega,w} - m)} \leq p \frac{M^{p-1} - m^{p-1}}{M - m} \frac{1}{4}.$$


where $|h|_{w} := \int_{\Omega} |h| w \, d\mu/\int_{\Omega} w \, d\mu \in [m, M]$, $\Psi_{p}(\cdot, m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_{p}(t; m, M) = \frac{M^{p} - t^{p}}{M - t} - \frac{t^{p} - m^{p}}{t - m},$$

and

$$B_{p}(m, M) := \sup_{t \in (m, M)} \Psi_{p}(t; m, M).$$

From (2.2) we also have the inequality

$$0 \leq \frac{\int_{\Omega} |h|^{p} w \, d\mu}{\int_{\Omega} w \, d\mu} - \left(\frac{\int_{\Omega} |h| w \, d\mu}{\int_{\Omega} w \, d\mu}\right)^{p} \leq \frac{1}{4} (M - m) \Psi_{p}(|h|_{w}; m, M) \leq \frac{1}{4} p(M - m)(M^{p-1} - m^{p-1}). \tag{3.2}$$

**Proposition 3.1.** If $f \in L_{p}(\Omega, \mu)$, $g \in L_{q}(\Omega, \mu)$ with $p > 1$, $1/p + 1/q = 1$, and there exist constants $\gamma, \Gamma > 0$ such that

$$\gamma \leq \frac{|f|}{|g|^{q-1}} \leq \Gamma \mu\text{-a.e on } \Omega,$$

then

$$0 \leq \frac{\int_{\Omega} |f|^{p} d\mu}{\int_{\Omega} |g|^{q} d\mu} - \left(\frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^{q} d\mu}\right)^{p} \leq \frac{B_{p}(\gamma, \Gamma)}{\Gamma - \gamma} \left(\frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^{q} d\mu} - \gamma\right) \leq \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \left(\frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^{q} d\mu} - \gamma\right) \leq \frac{1}{4} p(\Gamma - \gamma)(\Gamma^{p-1} - \gamma^{p-1}), \tag{3.3}$$

and

$$0 \leq \frac{\int_{\Omega} |f|^{p} d\mu}{\int_{\Omega} |g|^{q} d\mu} - \left(\frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^{q} d\mu}\right)^{p} \leq \frac{1}{4} (\Gamma - \gamma) \Psi_{p} \left(\frac{\int_{\Omega} |f| g \, d\mu}{\int_{\Omega} |g|^{q} d\mu} ; \gamma, \Gamma\right) \leq \frac{1}{4} p(\Gamma - \gamma)(\Gamma^{p-1} - \gamma^{p-1}), \tag{3.4}$$

where $B_{p}(\cdot, \cdot)$ and $\Psi_{p}(\cdot, \cdot, \cdot)$ are defined above.

**Proof.** The inequalities (3.3) and (3.4) follow from (3.1) and (3.2) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^{q}.$$ 

The details are omitted. \qed
Remark 3.2. We observe that for \( p = q = 2 \) we have \( \Psi_2(t; \gamma, \Gamma) = \Gamma - \gamma = B_2(\gamma, \Gamma) \) and then from the first inequality in (3.3) we get the following reverse of the Cauchy–Bunyakovsky–Schwarz inequality:

\[
\int_{\Omega} |g|^2 \, d\mu \int_{\Omega} |f|^2 \, d\mu - \left( \int_{\Omega} |fg| \, d\mu \right)^2 \\
\leq \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^2 \, d\mu} \right) \left( \int_{\Omega} |g|^2 \, d\mu - \gamma \right) \left( \int_{\Omega} |g|^2 \, d\mu \right),
\]

provided that \( f, g \in L_2(\Omega, \mu) \), and there exist constants \( \gamma, \Gamma > 0 \) such that

\[
\gamma \leq \frac{|f|}{|g|} \leq \Gamma \text{ a.e. on } \Omega.
\]

Corollary 3.3. With the assumptions of Proposition 3.1 we have the following additive reverses of the Hölder inequality:

\[
0 \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q} - \int_{\Omega} |fg| \, d\mu \\
\leq \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \frac{\int_{\Omega} |fg| \, d\mu}{\frac{\int_{\Omega} |g|^q \, d\mu}{\gamma}} \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right) \left( \frac{\int_{\Omega} |fg| \, d\mu}{\frac{\int_{\Omega} |g|^q \, d\mu}{\gamma}} - \gamma \right)^{1/p} \\
\leq p^{1/p} \left( \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \right) \left( \frac{\int_{\Omega} |fg| \, d\mu}{\frac{\int_{\Omega} |g|^q \, d\mu}{\gamma}} \right)^{1/p} \left( \frac{\int_{\Omega} |fg| \, d\mu}{\frac{\int_{\Omega} |g|^q \, d\mu}{\gamma}} - \gamma \right)^{1/p} \\
\times \int_{\Omega} |g|^q \, d\mu
\]

and

\[
0 \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q} - \int_{\Omega} |fg| \, d\mu \\
\leq \frac{1}{4^{1/p}} (\Gamma - \gamma)^{1/p} \Psi_p^{1/p} \left( \frac{\int_{\Omega} |fg| \, d\mu}{\frac{\int_{\Omega} |g|^q \, d\mu}{\gamma}} ; m, M \right) \int_{\Omega} |g|^q \, d\mu
\]

\[
\leq \frac{1}{4^{1/p}} p^{1/p} (\Gamma - \gamma)^{1/p} (\Gamma^{p-1} - \gamma^{p-1})^{1/p} \int_{\Omega} |g|^q \, d\mu,
\]

where \( p > 1 \) and \( 1/p + 1/q = 1 \).
PROOF. By multiplying in (3.3) with \((\int_{\Omega} |g|^q \, d\mu)^p\),
\[
\int_{\Omega} |f|^p \, d\mu \left( \int_{\Omega} |g|^q \, d\mu \right)^{p-1} - \left( \int_{\Omega} |fg| \, d\mu \right)^p 
\leq \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \int_{\Omega} |fg| \, d\mu \right)^{p-1} - \gamma \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\leq p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \int_{\Omega} |fg| \, d\mu \right)^{p-1} - \gamma \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
which is equivalent to
\[
\int_{\Omega} |f|^p \, d\mu \left( \int_{\Omega} |g|^q \, d\mu \right)^{p-1} 
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \int_{\Omega} |fg| \, d\mu \right)^{p-1} - \gamma \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\times \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + p \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right) \left( \int_{\Omega} |fg| \, d\mu \right)^{p-1} - \gamma \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
\[
\times \left( \int_{\Omega} |g|^q \, d\mu \right)^p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}
\]
\[
\leq \left( \int_{\Omega} |fg| \, d\mu \right)^p + \frac{1}{4} p(\Gamma - \gamma)(\Gamma^{p-1} - \gamma^{p-1}) \left( \int_{\Omega} |g|^q \, d\mu \right)^p
\]
Raising to the power \(1/p\) with \(p > 1\) and employing the elementary inequality that for \(p > 1\) and \(\alpha, \beta > 0\),
\[(\alpha + \beta)^{1/p} \leq \alpha^{1/p} + \beta^{1/p},\]
we have from the first part of (3.7) that
\[
\left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1-1/p} 
\leq \int_{\Omega} |fg| \, d\mu + \left( \frac{B_p(\gamma, \Gamma)}{\Gamma - \gamma} \right)^{1/p} \left( \Gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^{1/p} \left( \int_{\Omega} |fg| \, d\mu \right)^{1-1/p} - \gamma \left( \int_{\Omega} |g|^q \, d\mu \right)^{1-1/p}
\]
\[
\times \int_{\Omega} |g|^q \, d\mu,
\]
and since \(1 - 1/p = 1/q\) we get from (3.8) the first inequality in (3.5). The rest is obvious.

The inequality (3.6) can be proved in a similar manner; the details are omitted. \(\Box\)
If \( h : \Omega \to \mathbb{R} \) is \( \mu \)-measurable, satisfies the bounds
\[
0 < m \leq |h(x)| \leq M < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega
\]
and is such that \( h, |h|^p \in L_w(\Omega, \mu) \), for a \( \mu \)-measurable function \( w : \Omega \to \mathbb{R} \), with \( w(x) \geq 0 \) for \( \mu \)-a.e. \( x \in \Omega \) and \( \int_{\Omega} w \, d\mu > 0 \), then from (2.7) we also have the inequality
\[
0 \leq \frac{\int_{\Omega} |h|^p w \, d\mu}{\int_{\Omega} w \, d\mu} - \left( \frac{\int_{\Omega} |h| w \, d\mu}{\int_{\Omega} w \, d\mu} \right)^p
\]
\[
\leq 2 \left( \frac{m^p + M^p}{2} - \left( \frac{m + M}{2} \right)^p \right) \max \left\{ \frac{M - |h|_{\Omega,w}}{M - m}, \frac{|h|_{\Omega,w} - m}{M - m} \right\}
\]
(3.9)
\[
\leq \frac{1}{2} p (M^{p-1} - m^{p-1}) \max \{ M - |h|_{\Omega,w}, |h|_{\Omega,w} - m \},
\]
where, as above, \( |h|_{\Omega,w} := \int_{\Omega} |h| w \, d\mu / \int_{\Omega} w \, d\mu \in [m, M] \).

From (3.9) we can state the following result.

**Proposition 3.4.** With the assumptions of Proposition 3.1 we have
\[
0 \leq \frac{\int_{\Omega} |f|^p \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \left( \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^p
\]
\[
\leq 2 \cdot \frac{\gamma p + \Gamma p}{\Gamma - \gamma} \max \left\{ \gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu}, \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right\}
\]
\[
\leq \frac{1}{2} p (\Gamma^{p-1} - \gamma^{p-1}) \max \left\{ \gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu}, \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right\}.
\]

Finally, the following additive reverse of the Hölder inequality can also be stated.

**Corollary 3.5.** With the assumptions of Proposition 3.1,
\[
0 \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q} - \int_{\Omega} |fg| \, d\mu
\]
\[
\leq 2^{1/p} \cdot \left( \frac{\gamma p + \Gamma p}{\Gamma - \gamma} \right)^{1/p}
\]
\[
\times \max \left\{ \left( \gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^{1/p}, \left( \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right)^{1/p} \right\} \int_{\Omega} |g|^q \, d\mu
\]
\[
\leq \frac{1}{2^{1/p}} \max \left\{ \left( \gamma - \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} \right)^{1/p}, \left( \frac{\int_{\Omega} |fg| \, d\mu}{\int_{\Omega} |g|^q \, d\mu} - \gamma \right)^{1/p} \right\}
\]
\[
\times (\Gamma^{p-1} - \gamma^{p-1})^{1/p} \int_{\Omega} |g|^q \, d\mu.
\]
As a simpler but coarser inequality we have the following result:

\[
0 \leq \left( \int_{\Omega} |f|^p \, d\mu \right)^{1/p} \left( \int_{\Omega} |g|^q \, d\mu \right)^{1/q} - \int_{\Omega} |fg| \, d\mu \leq 2^{1/p} \cdot \left( \frac{\gamma^p + \Gamma^p}{2} - \left( \frac{\gamma + \Gamma}{2} \right)^p \right)^{1/p} \int_{\Omega} |g|^q \, d\mu,
\]

where \( f \) and \( g \) are as above.

### 4. Applications for \( f \)-divergence

One of the important issues in many applications of probability theory is finding an appropriate measure of distance (or difference or discrimination) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [19], Kullback and Leibler [24], Rényi [30], Havrda and Charvat [17], Kapur [22], Sharma and Mittal [32], Burbea and Rao [4], Rao [29], Lin [25], Csiszár [7], Ali and Silvey [1], Vajda [39], Shioya and Da-Te [33] and others (see, for example, [26], and the references therein).

These measures have been applied in a variety of fields such as: anthropology [29], genetics [26], finance, economics and political science [31, 36, 37], biology [28], the analysis of contingency tables [16], approximation of probability distributions [6, 23], signal processing [20, 21] and pattern recognition [2, 5]. A number of these measures of distance are specific cases of Csiszár \( f \)-divergence and so further exploration of this concept will have a flow-on effect to other measures of distance and to areas in which they are applied.

Assume that a set \( \Omega \) and the \( \sigma \)-finite measure \( \mu \) are given. Consider the set of all probability densities on \( \mu \) to be \( \mathcal{P} := \{ p \mid p : \Omega \to \mathbb{R}, p(x) \geq 0, \int_{\Omega} p(x) \, d\mu(x) = 1 \} \). The Kullback–Leibler divergence [24] is well known among the information divergences. It is defined as

\[
D_{KL}(p, q) := \int_{\Omega} p(x) \log \left( \frac{p(x)}{q(x)} \right) \, d\mu(x), \quad p, q \in \mathcal{P},
\]

where \( \log \) is to base \( e \).

In information theory and statistics, various divergences are applied in addition to the Kullback–Leibler divergence. These are, for example, the variation distance \( D_v \), Hellinger distance \( D_H \) [18], \( \chi^2 \)-divergence \( D_{\chi^2} \), \( \alpha \)-divergence \( D_\alpha \), Bhattacharyya distance \( D_B \) [3], harmonic distance \( D_{Ha} \), Jeffreys distance \( D_J \) [19], triangular discrimination \( D_\Delta \) [38]. They are defined as follows:

\[
D_v(p, q) := \int_{\Omega} |p(x) - q(x)| \, d\mu(x), \quad p, q \in \mathcal{P};
\]

\[
D_H(p, q) := \int_{\Omega} |\sqrt{p(x)} - \sqrt{q(x)}| \, d\mu(x), \quad p, q \in \mathcal{P};
\]
For other divergence measures, see Kapur [22] or the book online by Taneja [35].

Csiszár $f$-divergence is defined as follows [8]:

$$I_f(p, q) := \int_{\Omega} p(x)f\left(\frac{q(x)}{p(x)}\right) d\mu(x), \quad p, q \in \mathcal{P},$$

where $f$ is convex on $(0, \infty)$. It is assumed that $f$ is strictly convex and satisfies the condition that $f(1) = 0$. By appropriately defining this convex function, various divergences are derived. Most of the above distances (4.1)–(4.9) are particular instances of Csiszár $f$-divergence. There are also many others which are not in this class (see, for example, [35]). For the basic properties of Csiszár $f$-divergence, see [8, 9] and [39].

The following result holds.

**Proposition 4.1.** Suppose that $f : (0, \infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exist constants $0 < r < 1 < R < \infty$ such that

$$r \leq \frac{q(x)}{p(x)} \leq R \quad \text{for } \mu\text{-a.e. } x \in \Omega.$$

Then we have the inequalities

$$I_f(p, q) \leq \frac{(R - 1)(1 - r)}{R - r} \sup_{t \in (r, R)} \Psi_f(t; r, R)$$

$$\leq (R - 1)(1 - r)\frac{f'_+(-R) - f'_-(R)}{R - r}$$

$$\leq \frac{1}{4} (R - r)(f'_-(R) - f'_+(r)),$$

where $\Psi_f(\cdot; r, R) : (r, R) \to \mathbb{R}$ is defined by

$$\Psi_f(t; r, R) = \frac{f(R) - f(t)}{R - t} - \frac{f(t) - f(r)}{t - r}.$$
We also have the inequality

\[ I_f(p, q) \leq \frac{1}{4}(R - r) \frac{f(R)(1 - r) + f(r)(R - 1)}{(R - 1)(1 - r)} \]

\[ \leq \frac{1}{4}(R - r)(f_r'(R) - f_r'(r)). \] (4.11)

The proof follows by Theorem 2.1 by choosing \( w(x) = p(x), f(x) = q(x)/p(x), m = r \) and \( M = R \) and performing the required calculations. The details are omitted.

Using the same approach and Theorem 2.4 we can also state the following result.

**Proposition 4.2.** With the assumptions of Proposition 4.1,

\[ I_f(p, q) \leq 2 \max \left\{ \frac{R - 1}{R - r}, \frac{1 - r}{R - r} \right\} \left( \frac{f(r) + f(R)}{2} - f\left(\frac{r + R}{2}\right) \right) \]

\[ \leq \frac{1}{2} \max\{R - 1, 1 - r\}(f_r'(R) - f_r'(r)). \] (4.12)

The above results can be used to obtain various inequalities for divergence measures in information theory that are particular instances of \( f \)-divergence.

Consider the Kullback–Leibler divergence

\[ D_{KL}(p, q) := \int_{\Omega} p(x) \log \left( \frac{p(x)}{q(x)} \right) d\mu(x), \quad p, q \in \mathcal{P}, \]

which is an \( f \)-divergence for the convex function \( f : (0, \infty) \to \mathbb{R}, f(t) = -\log t \).

If \( p, q \in \mathcal{P} \) such that there exist constants \( 0 < r < 1 < R < \infty \) with

\[ r \leq \frac{q(x)}{p(x)} \leq R \quad \text{for } \mu\text{-a.e. } x \in \Omega, \]

then we get from (4.10) that

\[ D_{KL}(p, q) \leq \frac{(R - 1)(1 - r)}{rR}, \]

from (4.11) that

\[ D_{KL}(p, q) \leq \frac{1}{4}(R - r) \log(R^{-1/(R-1)}r^{-1/(1-r)}) \]

and from (4.12) that

\[ D_{KL}(p, q) \leq 2 \max \left\{ \frac{R - 1}{R - r}, \frac{1 - r}{R - r} \right\} \log \left( \frac{A(r, R)}{G(r, R)} \right) \]

\[ \leq \frac{1}{2} \max\{R - 1, 1 - r\}(R - r) \frac{1}{rR}, \]

where \( A(r, R) \) is the arithmetic mean and \( G(r, R) \) is the geometric mean of the positive numbers \( r \) and \( R \).
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References

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