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LUSTERNIK–SCHNIRELMANN CATEGORY BASED ON THE DISCRETE CONLEY INDEX THEORY

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Abstract. We study Lusternik–Schnirelmann type categories for isolated invariant sets by the use of the discrete Conley index.

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1. Introduction. We are primarily concerned with a discrete dynamical system from an open subset of a locally compact, separable metrizable space X to X (called a *local map*).

The *category* of a space is introduced by Lusternik and Schnirelmann [**16**]. The invariant gives an information about the existence of critical points; that is, if there is a real-valued smooth function on a compact closed manifold with n critical points, then the manifold is covered by n contractible open subsets [**3**]. The Conley index theory is a generalization of Morse theory, which is a tool to analyse the topology of manifold from the view point of critical points. This theory (or theory of isolated invariant sets) is introduced and developed by Conley, Easton, Salamon and others for continuous dynamical systems (flows) [**8**, **24**], and is later extended to the discrete case [**12**, **19**, **23**, **28**]. In this paper, we study Lusternik–Schnirelmann type categories for isolated invariant sets based on the discrete homotopy Conley index, and give a modification of well-known results concerning Lusternik–Shnirelmann category type estimates for the number of rest points of flows [**20**, **21**, **25**]. By modifying a result of Szymczak, Wójcik and Zgliczyński [**30**], we also study the relations between the Conley indeies/categories in the invariant subspace and in the entire space.

2. The discrete Conley index. We begin with a brief review of the basic definitions and properties of the discrete Conley index theory in the style of Franks and Richeson [**12**].

DEFINITION 2.1. Let $f : O \subseteq X \to X$ be a local map, where O is open in a locally compact, separable metrizable space X. For a given set $N \subseteq O$, we define the *maximal invariant set* $InvN$ to be the set of $x \in N$, such that there exists an orbit $\{x_n\} \subseteq N$ with $x_0 = x$ and $f(x_n) = x_{n+1}$ for $n \in \mathbb{Z}$. A compact invariant set S is *isolated* if there exists a compact neighbourhood N of S such that $S = InvN \subseteq \operatorname{Int} N$. The neighbourhood N is called an *isolating neighbourhood* of S. A compact set N is an *isolating block*, if $f(N) \cap N \cap f^{-1}(N) \subseteq \operatorname{Int} N$.

REMARK. We note that every isolating block N is an isolating neighbourhood of the set $S = InvN$, and every neighbourhood of an isolated invariant set S contains an isolating block N with $S = InvN$ (see [**10**, **11**]).

DEFINITION 2.2. Let S be an isolated invariant set and suppose (N, L) is a compact pair contained in the interior of the domain of f. The pair (N, L) is called a *filtration pair* for S provided N and L are each the closures of their interiors and

(1) $\mathrm{Cl}(N \setminus L)$ is an isolating neighbourhood of S,
(2) L is a neighbourhood of $N^- = \{x \in N \mid f(x) \notin \mathrm{Int}\, N\}$ in N, and
(3) $f(L) \cap \mathrm{Cl}(N \setminus L) = \emptyset$.

THEOREM 2.3 [**12**, Theorem 3.6]. *If N is an isolating block and L is any sufficiently small compact neighbourhood of N^- in N, the (N, L) is a filtration pair for $S = InvN$.*

THEOREM 2.4 [**12**, Theorem 3.7]. *Let $f : O \subseteq M \to M$ be a local map from an open subset O of a manifold M to M with an isolated invariant set S. Inside any neighbourhood of S, there exists a filtration pair (N, L) for S, which is homeomorphic to a finite polyhedral pair.*

Let $P = (N, L)$ be a filtration pair for f. $(N_L, *_L)$ denotes the pointed space that is obtained by collapsing L to a single point $*_L$. In case L is empty, N_L is defined to be the disjoint union of N and the single point space $\{*_L\}$. The definition of filtration pair permits us to define a base-point preserving map $f_P : N_L \to N_L$ that fixes the point $*_L$ and sends every point $z \in N \setminus L$ to $\pi(f(z))$, where $\pi : N \to N_L$ is the natural quotient map. This map f_P is called the *pointed space map associated to P*.

Suppose $\mathcal{K}$ is a category. Let X, Y be objects in $\mathcal{K}$ and $f : X \to X$, $g : Y \to Y$ be endmorphisms in $\mathcal{K}$. We say that (X, f) and (Y, g) are *shift equivalent* [**32**], if there exist a couple of morphisms $r : X \to Y$, $s : Y \to X$ in $\mathcal{K}$ and $m \in \mathbb{N}$ such that the following diagrams are commutative.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "r"'] & X \arrow[d, "r"] \\
Y \arrow[r, "g"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "s"'] & Y \arrow[d, "s"] \\
X \arrow[r, "f"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f^m"] \arrow[d, "r"'] & X \arrow[d, "r"] \\
Y \arrow[r, "g^m"'] \arrow[ur, "s"] & Y
\end{tikzcd}
$$

THEOREM 2.5 [**12**, Theorem 4.3]. *Suppose $P = (N, L)$ and $P' = (N', L')$ are filtration pairs for an isolated invariant set S. Then, (N_L, f_P) and $(N_{L'}, f_{P'})$ are shift equivalent, where f_P and $f_{P'}$ are the pointed space maps associated to P, P', respectively.*

REMARK. We note by [**12**, Definition 4.4, Lemma 4.5] that the maps $r : N_L \to N'_{L'}$, $s : N'_{L'} \to N_L$ giving a standard shift equivalence preserve a f-invariant subset of S.

DEFINITION 2.6. Let S be an isolated invariant set for a local map $f : O \subseteq X \to X$. Then, we define the *discrete homotopy Conley index $Ch(S, f)$* of S to be the shift equivalence class of $(N_L, [f_P])$, where $N_L \in Ob(\mathcal{H}Top_*)$ and $[f_P] \in Mor_{\mathcal{H}Top_*}(N_L, N_L)$. $\mathcal{H}Top_*$ means the pointed homotopy category of pointed topological spaces.

We here point out the following fact.

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THEOREM 2.7 [**12**, Proposition 8.2]. *Let $P = (N, L)$ be a filtration pair for an isolated invariant set S. Then, the isomorphism class of (N_L, f_P) in the sense of Szymczak* [**28**] *is precisely the Conley index of S.*

We next introduce some terminology, which are appeared in geometric topology. A locally compact separable metrizable space X is said to be an *absolute neighbourhood retract (ANR),* if X can be closed embedded in separable Hilbert space in such a way that there exists an open neighbourhood U of X that retracts to X [**2, 13**]. All locally contractible finite dimensional spaces are ANR's. A closed set A in a space X is said to be a *Z-set* provided that for every $\varepsilon > 0$ there exists a map $h : X \to X$ that is ε-close to the identity and whose image misses A [**1, 6**]. Some examples of Z-sets are a topological manifold X and the boundary $A = \partial X$; the subset $X \times \{0\}$ of $X \times [0, 1]$ for a space X. Recall that the Hilbert cube Q is the countable product of copies of the unit interval. A Hilbert cube manifold (Q-manifold) is a separable metrizable space in which each point has an open neighbourhood homeomorphic to an open subset of Q.

3. The Conley category and Morse decompositions.

DEFINITION 3.1. Let $f : X \to X$ be a map and $A \subseteq X$. We define the *weak category* of f reduced to A, $c^*_A(f)$, to be the smallest integer n such that $A = U_1 \cup \cdots \cup U_n$, where the U_i are open in A and each restriction $f^k|_{U_i} : U_i \to X$ is null-homotopic for some k.

REMARK. If X is a compact ANR, then $c^*_A(f)$ is bounded.

EXAMPLE 3.2 [**5, 15**]. Let f be a self-map on a one-dimensional ANR continuum X. Then, the shift space $\varprojlim\{X, f\}$ is tree-like if and only if $c^*_X(f) = 1$.

The following fact is an easy consequence of definitions.

PROPOSITION 3.3. *Let $f : (X, A) \to (X, A)$, $g : (Y, B) \to (Y, B)$ be maps of pairs. Suppose $((X, A), [f])$, $((Y, B), [g])$ are shift equivalent in the homotopy category of pairs. Then, $c^*_A(f) = c^*_B(g)$.*

DEFINITION 3.4. Let S be an isolated invariant set for a map f. We define the *Conley category $cc(S, f)$ of S* to be the weak category $c^*_{N_L}(f_P)$, where $P = (N, L)$ is a filtration pair of S and $f_P : N_L \to N_L$ is the pointed space map associated to P. For an f-invariant subset A of S, we define the *Conley category $cc_A(S, f)$ of S reduced to A* to be the weak category $c^*_A(f_P)$. We will write $cc(S)$ ($cc_A(S)$) for $cc(S, f)$ ($cc_A(S, f)$, respectively) when no confusion can arise.

REMARK. The Conley category of S (reduced to A) does not depend on the choice of a filtration pair of S and it is an invariant for the discrete homotopy Conley index, by Theorem 2.5, Remark following the theorem and Proposition 3.3.

Let $x \in X$. If $\sigma : \mathbb{Z} \to X$ is given by $\sigma(n) = x_n$ and $x_0 = x$ and $f(x_n) = x_{n+1}$ for $n \in \mathbb{Z}$, then we call σ *a solution through* x. Recall that we define the *ω-limit set* of x to be $\omega(x) = \cap_{n\in\mathbb{N}} \mathrm{Cl}(\cup_{k\geq n}\{f^k(x)\})$. For any solution $\sigma : \mathbb{Z} \to X$ through x, we define the *α-limit set* to be $\alpha_\sigma(x) = \cap_{n\in\mathbb{N}} \mathrm{Cl}(\cup_{k\geq n}\{\sigma(-k)\})$.

DEFINITION 3.5. The collection of pairwise disjoint isolated invariant sets $\{S_i \subseteq S \mid i = 1, \ldots, m\}$ is a *Morse decomposition* [**12**] of an isolated invariant set S if for every

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$x \in S$ and every solution $\sigma : \mathbb{Z} \to S$ through x, we have either $\sigma(\mathbb{Z}) \subseteq S_i$ for some i, or the ω-limit set $\omega(x) \subseteq S_i$ and the α-limit set $\alpha_\sigma(x) \subseteq S_j$ for some $i < j$.

We use a basic property of a Morse decomposition.

PROPOSITION 3.6. *Let S be an isolated invariant set with an isolating neighbourhood N and $\{S_1, \dots, S_m\}$ be a Morse decomposition of S. If $x \in N$ satisfies $f^n(x) \in N$ for $n \in \mathbb{N}$, then there exists ℓ $(1 \le \ell \le m)$ such that $\omega(x) \subseteq S_\ell \cup S_{\ell+1} \cup \dots \cup S_m$.*

Proof. We note that $\omega(x) \subseteq S$. Suppose on the contrary that the intersection $[S \setminus (S_1 \cup \dots \cup S_m)] \cap \omega(x)$ contains a point z. When choosing a solution $\sigma : \mathbb{Z} \to S$ through z with $\sigma(\mathbb{Z}) \subseteq \omega(x)$, the solution σ possesses $\omega(z) \subseteq S_p$ and $\alpha_\sigma(z) \subseteq S_q$ for some $1 \le p < q \le m$, and hence both sets $\omega(x) \cap S_p$, $\omega(x) \cap S_q$ are non-empty. Take open sets U and V satisfying $S_p \subseteq U$, $S_q \cup \dots \cup S_m \subseteq V$ and $\mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset$. Then, we can find a sequence of numbers $m_1 < n_1 < m_2 < n_2 < \dots$ such that $f^{m_i}(x) \in U$, $f^{n_i}(x) \in V$; if $m_i \le k < n_i$, then $f^k(x) \notin V$; if $n_i \le k < m_{i+1}$, then $f^k(x) \notin U$; the limit set of $\{f^{m_i}(x)\}$ $(\{f^{n_i}(x)\})$ intersects S_p (S_q, respectively).

Let $y = \lim_j f^{n_{i_j}}(x) \in S_q$. The unboundedness of $\{n_i - m_i\}$ yields the existence of a solution $\tilde{\sigma}$ through y with $\omega(y) \subseteq S_q$ and $\alpha_{\tilde{\sigma}}(y) \cap V = \emptyset$. This leads to a contradiction since $\{S_1, \dots, S_m\}$ is a Morse decomposition of S. □

We need a discrete version of a result concerning the Conley index theory for flows. We provide a sketch of a proof here because we have been unable to find a precise reference for the discrete case (continuous map); our strategy is essentially by Conley and Zehnder [**9**], Salamon [**24**] and Razvan [**22**].

LEMMA 3.7 (Salamon [**24**]). *Let S be an isolated invariant set with an isolating neighbourhood N and $\{S_1, \dots, S_m\}$ be a Morse decomposition of S. We define $K = \{x \in N \mid f^n(x) \in N$ for $n \in \mathbb{N}\}$ and $K_i = \{x \in K \mid \omega(x) \subseteq S_i \cup \dots \cup S_m\}$ for $i = 1, \dots, m$. Then, the collection $\{K_i \mid i = 1, \dots, m\}$ has the following properties:*

(1) *K_i is closed in N with $K_1 \supseteq \dots \supseteq K_m$, and*

(2) *if A is a closed (in K) subset of $K_i \setminus K_{i+1}$ (we let $K_{m+1} = \emptyset$) and U is a neighbourhood of S_i, then there exists $n_0 \in \mathbb{N}$ such that $f^n(A) \subseteq U \cap K$ for $n \ge n_0$.*

Proof. (1): It is clearly $K_1 = K$ by Proposition 3.6.

Let $i \ge 2$. We define,

$$S[i] = \bigcup_{k=i}^{m} S_k \cup \bigcup_{i \le p < q \le m} C(S_q, S_p; S),$$

where $C(S_q, S_p; S) = \{x \in S \setminus (S_q \cup S_p) \mid$ there exists a solution $\sigma : \mathbb{Z} \to S$ through x with $\alpha_\sigma(x) \subseteq S_q$ and $\omega(x) \subseteq S_p\}$. We note that $S[i]$ is closed in S by [**12**].

Let $\{x_n\} \subseteq K_i$ with $x_n \to x \in K$. Suppose on the contrary that $x \notin K_i$. Take a closure compact neighbourhood U of $S[i]$ in K such that $\mathrm{Cl}\, U \cap (S_1 \cup \dots \cup S_{i-1}) = \emptyset$, and then find $\ell_0 \in \mathbb{N}$ with $f^{\ell_0}(x) \in K \setminus \mathrm{Cl}\, U$ (use Proposition 3.6). By the continuity of f^{ℓ_0}, we have $n_0 \in \mathbb{N}$ such that $f^{\ell_0}(x_n) \in K \setminus \mathrm{Cl}\, U$ for $n \ge n_0$. Since $x_n \in K_i$, for each $n \ge n_0$, we can take $k_n \in \mathbb{N}$ such that $f^{k_n}(x_n) \in K \setminus U$ and $f^k(x_n) \in U$ for $k > k_n$. We may assume that $f^{k_n}(x_n) \to z \in K \setminus U$ and $k_1 \le k_2 \le \dots$.

We note that $\{k_n\}$ is unbounded, because of $\omega(z) \subseteq S_i \cup \dots \cup S_m$. This shows the existence of a solution $\sigma : \mathbb{Z} \to N$ through z. Since $z \in S \setminus (S_1 \cup \dots \cup S_m)$, z would be an element of $S[i]$ by the definition of a Morse decomposition. This contradicts $z \notin U \supseteq S[i]$ and therefore finishes the proof of (1).

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(2): We may assume that U is a compact neighbourhood of S_i such that $U \subseteq N$ and $U \cap K_{i+1} = \emptyset$ by (1) and $S_i \cap K_{i+1} = \emptyset$. Take an open neighbourhood V of S_i with $f(V) \subseteq U$. Choose $n_1 \in \mathbb{N}$ such that each $x \in (U \setminus V) \cap K_i$ has the property $f^{-n}(x) \cap \{f^{-(n-1)}(U) \cap \cdots \cap f^{-1}(U) \cap U\} = \emptyset$ for some $1 \leq n \leq n_1$. Find a neighbourhood W of S_i with $f^n(W) \subseteq V$ for any $0 \leq n \leq n_1$. Then, we see $f^n(W \cap K_i) \subseteq V$ for $n \in \mathbb{N}$. Let $n_2 \in \mathbb{N}$ be that each $x \in A$ satisfies $f^m(x) \in W$ for some $0 \leq m \leq n_2$. This clearly concludes $f^{n_2+n}(A) \subseteq V \subseteq U$ for $n \in \mathbb{N}$. □

We are now in a position to state our main theorem.

THEOREM 3.8. *Let $f : O \subseteq M \to M$ be a local map from an open subset O of a manifold M to M with an isolated invariant set S. Suppose $\{S_1, \ldots, S_m\}$ is a Morse decomposition of S. Then, $cc(S) \leq 1 + \sum_{i=1}^m cc_{S_i}(S)$.*

Proof. Let $P = (N, L)$ be a filtration pair of ANR's for S (Theorem 2.4). Since N_L is a compact ANR, the values $cc(S)$, $cc_{S_i}(S)$ are finite. We note that the following maps f, f_P are topological conjugate by π

$$\begin{array}{ccc} N \setminus \bigcup_{k=0}^{\infty} f^{-k}(L) & \xrightarrow{f} & N \setminus \bigcup_{k=0}^{\infty} f^{-k}(L) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ N_L \setminus \bigcup_{k=0}^{\infty} f_P^{-k}(*_L) & \xrightarrow{f_P} & N_L \setminus \bigcup_{k=0}^{\infty} f_P^{-k}(*_L), \end{array}$$

and $N \setminus \bigcup_{k=0}^{\infty} f^{-k}(L) = \{x \in \mathrm{Cl}(N \setminus L) \mid f^n(x) \in \mathrm{Cl}(N \setminus L) \text{ for } n \in \mathbb{N}\}$.

Write $K = N_L \setminus \bigcup_{k=0}^{\infty} f_P^{-k}(*_L)$ and let $K_i = \{x \in N_L \mid \omega_{f_P}(x) \subseteq S_i \cup \cdots \cup S_m\}$ for $i = 1, \ldots, m$. Since $\bigcup_{k=0}^{\infty} f_P^{-k}(*_L) = \bigcup_{k=0}^{\infty} \mathrm{Int} f_P^{-k}(*_L)$ by [**12**, Theorem 3.8], K is a compact set, and we see $K = K_1$ by Proposition 3.6. Under the identification above, we can use Lemma 3.7 for f_P.

For each i ($1 \leq i \leq m$), we shall successively construct an open neighbourhood V_i of S_i in N_L such that $c^*_{S_i}(f_P) = c^*_{V_i}(f_P)$ and $K_i \subseteq V_i \cup \cdots \cup V_m$.

Let $i = m$. When $c^*_{S_m}(f_P)$ has a value of k, then we can write $S_m = A_1 \cup \cdots \cup A_k$, where the A_ℓ ($1 \leq \ell \leq k$) are open in S_m and each restriction $f_P^n|_{A_\ell} : A_\ell \to N_L$ is null-homotopic for some n. By virtue of [**27**, Lemma 2.5], [**31**], we find an open neighbourhood B_ℓ of A_ℓ in N_L such that $B_\ell \subseteq N \setminus L$ and each restriction $f_P^n|_{B_\ell} : B_\ell \to N_L$ is null-homotopic. The open neighbourhood $U_m = B_1 \cup \cdots \cup B_k$ of S_m satisfies $U_m \subseteq N \setminus L$ and $c^*_{S_m}(f_P) \geq c^*_{U_m}(f_P)$.

Apply Lemma 3.7 to $A = K_m$ and $S_m \subseteq U_m$, then we obtain $n_m \in \mathbb{N}$ such that $f_P^{n_m}(K_m) \subseteq U_m \cap K$. Now, we let V_m be $f_P^{-n_m}(U_m)$. The set V_m is also an open neighbourhood of S_m in N_L such that $S_m \subseteq f_P^{n_m}(V_m) \subseteq U_m$ and $K_m \subseteq V_m$. Hence, we see $c^*_{S_m}(f_P) = c^*_{V_m}(f_P)$.

Suppose that we have constructed $V_m, \ldots, V_{i+1}$ satisfying our conditions. We consider the case i. In the same way as the case $i = m$, we obtain an open neighbourhood U_i of S_i in N_L, such that $U_i \subseteq N \setminus L$ and $c^*_{S_i}(f_P) \geq c^*_{U_i}(f_P)$. Apply Lemma 3.7 to $A = K_i \setminus \bigcup_{j=i+1}^m V_j$ and $S_i \subseteq U_i$ again, then we have $n_i \in \mathbb{N}$ such that $f_P^{n_i}(K_i \setminus \bigcup_{j=i+1}^m V_j) \subseteq U_i \cap K$. Then, it is easily seen that the set $V_i = f_P^{-n_i}(U_i)$ satisfies the condition in a similar way. We finish our construction.

Let $L \neq \emptyset$. $K_1 = K$, $N_L \setminus \bigcup_{j=1}^m V_j \subseteq \bigcup_{k=0}^{\infty} \mathrm{Int} f_P^{-k}(*_L)$ and the compactness show that $N_L \setminus \bigcup_{j=1}^m V_j \subseteq \mathrm{Int} f_P^{-k_0}(*_L)$ for some $k_0 \in \mathbb{N}$. By the homotopy extension

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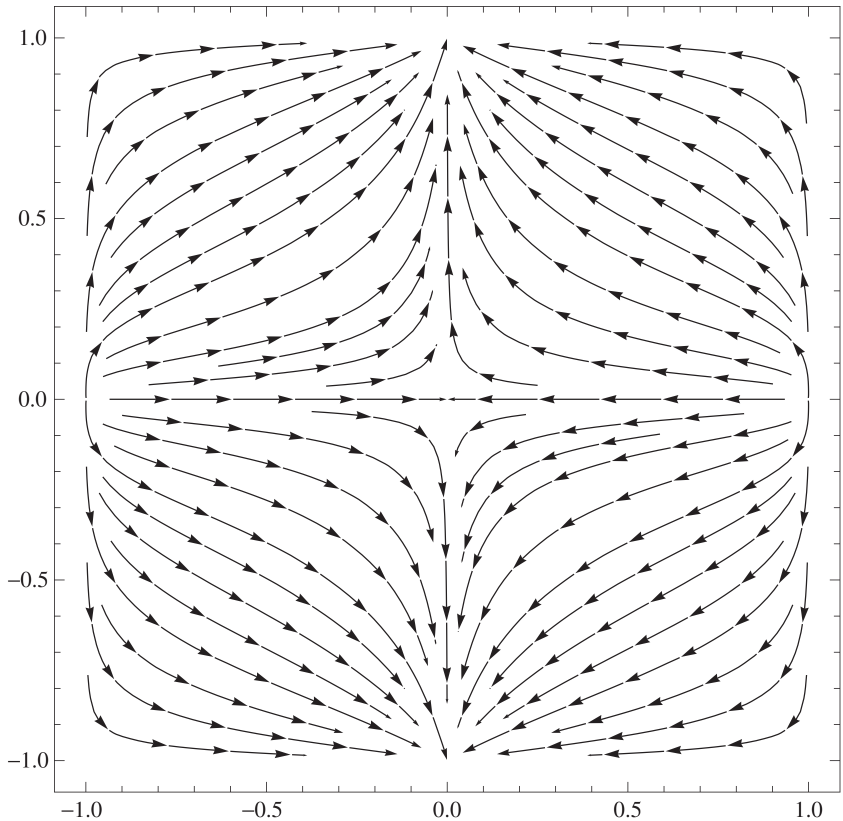

Figure 1. the stream of f.

theorem, we obtain an open neighbourhood of $N_L \setminus \bigcup_{j=1}^m V_j$ in N_L such that the restriction of $f_P^{k_0}$ to the open neighbourhood is also null-homotopic. Therefore, $c^*_{N_L}(f_P) \leq 1 + \sum_{i=1}^m c^*_{V_i}(f_P) = 1 + \sum_{i=1}^m c^*_{S_i}(f_P)$. In the case of $L = \emptyset$, then $N = K = \bigcup_{j=1}^m V_j$ and $N_L = N \cup \{*_L\}$. Hence, we have our inequality immediately and finally conclude that $cc(S) \leq 1 + \sum_{i=1}^m cc_{S_i}(S)$. $\square$

EXAMPLE 3.9. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a map defined by $f(\boldsymbol{x}) = 2\boldsymbol{x}$ for $\boldsymbol{x} \in \mathbb{R}^2$. Then, we see $cc(S) = 2 = 1 + cc_S(S)$, where $S = \{\mathbf{0}\}$.

EXAMPLE 3.10. Let f be a self-map on $[-1, 1] \times (-1, 1)$ defined by $f(x, y) = (x^3, \sin \frac{y\pi}{2})$ (see Figure 1). Let $\tilde{f}$ be a map on an open Möbius band (defined by $(-1, t) \sim (1, -t)$) induced by f. S is the centre circle of the open Möbius band and $S_1 = \{(0, 0)\}$, $S_2 = \{(1, 0)\}$ (Figure 1). We see $cc(S) = 3 = 1 + cc_{S_1}(S) + cc_{S_2}(S)$. The set N_L can be considered as the projective plane $\mathbb{R}P^2$, and note that $f_P \simeq id$ and the category of $\mathbb{R}P^2$ is 3 [**26**].

EXAMPLE 3.11. Suppose f is a natural map of the plane such that both rectangles get mapped as shown below (see Figure 2). This is a variation of the Smale horseshoe map (see [**22**]). Then, we see that $cc(S) = 2 < 3 = 1 + cc_{S_1}(S) + cc_{S_2}(S)$, where S_i and S are isolated invariant sets with $S_i = Inv N_i$ $(i = 1, 2)$ and $S = Inv(N_1 \cup N_2)$.

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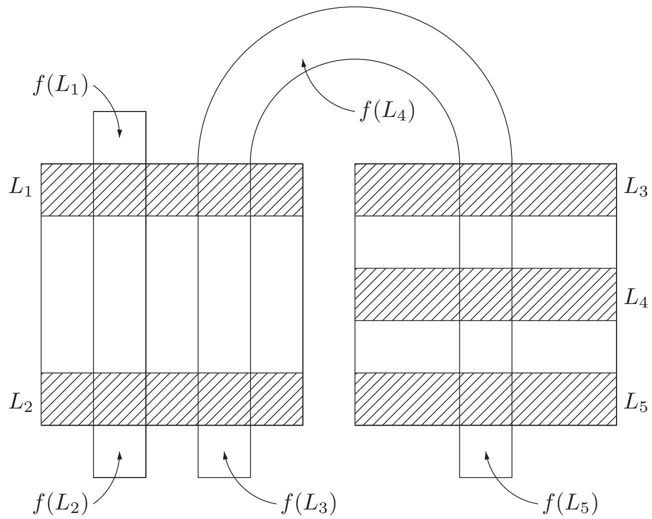

Figure 2. N_1: the left rectangle, N_2: the right rectangle.

4. The Conley indices and categories in the invariant subspace and in the entire space. The next lemma is extremely useful for proving main results in this section. The statement is similar in spirit to the concept of a representable index pair in a Euclidean space $\mathbb{R}^n$, i.e., an index pair composed of hypercubes [**29**, **30**].

LEMMA 4.1. *Let* $f : (U, A \times \{0\}) \to (A \times [0, 1), A \times \{0\})$ *be a map on an open set* U *of* $A \times [0, 1)$ *containing* $A \times \{0\}$*, and* $S \subseteq A \times \{0\}$ *be an isolated invariant set for* f*. Then, we have a filtration pair* $P = (N, L)$ *for* S *such that*

$$(N, L) \cap (A \times [0, \varepsilon]) = (J, K) \times [0, \varepsilon],$$

for some compact pair (J, K) *of* A *and* $0 < \varepsilon < 1$.

Proof. Let M be an isolating block for S in $A \times [0, 1)$, that is, the set M satisfies $f(M) \cap M \cap f^{-1}(M) \subseteq \operatorname{Int} M$.

Take a small compact neighbourhood $\widetilde{M}$ of M satisfying $f(\widetilde{M}) \cap \widetilde{M} \cap f^{-1}(\widetilde{M}) \subseteq \operatorname{Int} M$. Using the compactness, we can obtain a compact neighbourhood J of $pr(M \cap (A \times \{0\}))$ in A and a small number ε (> 0) such that $M \cap (A \times [0, \varepsilon]) \subseteq J \times [0, \varepsilon] \subseteq \widetilde{M}$, where pr means the projection map from $A \times [0, 1)$ to A. Note that for a sufficiently small $\gamma > 0$, the retraction of $M \cap (A \times [0, \gamma])$ to $A \times \{0\}$ is in $\widetilde{M} \cap (A \times \{0\})$. Then, the union $N = M \cup (J \times [0, \varepsilon])$ possesses the property $f(N) \cap N \cap f^{-1}(N) \subseteq \operatorname{Int} N$. Thus, N is an isolating block for S that satisfies $N \cap (A \times [0, \varepsilon]) = J \times [0, \varepsilon]$.

By modifying sufficiently small compact neighbourhood of $N^- = \{x \in N \mid f(x) \notin \operatorname{Int} N\}$ in a similar argument to the one of the set N and by retaking small ε if necessary, we find a compact neighbourhood L of N^- in N such that $L \cap (A \times [0, \varepsilon])$ is the product set of a compact set K in A and $[0, \varepsilon]$. (If $N^- = \emptyset$ or $N^- \cap (A \times \{0\}) = \emptyset$, then let $K = \emptyset$.) The construction together with Theorem 2.3 implies that (N, L) is a filtration pair for S having our desired property. □

As an application of Lemma 4.1, we give a variation of a series of results (in $\mathbb{R}^n$) by Szymczak, Wójcik and Zgliczyński [**30**]. Our arguments are essentially based on their

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proofs. We first consider the case of an isolated invariant set being of attracting type. For a map $f : X \to X$ and an isolated invariant set $S \subseteq X$, we define the unstable and stable sets of S by $W^u(S) = \{x \in X \mid \text{there exists } \sigma : \mathbb{Z}^- \to X \text{ such that } \sigma(0) = x, f(\sigma(n-1)) = \sigma(n) \text{ for } n \in \mathbb{Z}^- \text{ and } \emptyset \neq \alpha_\sigma(x) \subseteq S\}$ and $W^s(S) = \{x \in X \mid \emptyset \neq \omega(x) \subseteq S\}$, respectively.

LEMMA 4.2 [**30**, Theorem 1]. *Let $f : (X, A) \to (X, A)$ be a self-map defined on a pair (X, A) satisfying A is closed and collared in X, and $S \subseteq A$ be an isolated invariant set for f such that $W^u(S) \subseteq A$. Then, we hold $Ch(S, f) = Ch(S, f|_A)$, in particular $cc(S, f) = cc(S, f|_A)$.*

Proof. Let $\varphi : A \times [0, 1) \to X$ be an *open* embedding with $\varphi(a, 0) = a$ for $a \in A$. Choose an open set U in $A \times [0, 1)$ such that $A \times \{0\} \subseteq U$ and $f \circ \varphi(U) \subseteq \varphi(A \times [0, 1))$, and write $\tilde{f} = \varphi^{-1} \circ f \circ \varphi|_U : U \to A \times [0, 1)$ and $\widetilde{S} = \varphi^{-1}(S)$. We note that $\widetilde{S} \subseteq A \times \{0\}$ is isolated invariant for $\tilde{f}$ and $W^u_{\tilde{f}}(\widetilde{S}) \subseteq A \times \{0\}$.

$$\begin{tikzcd}
U \arrow[r, "\tilde{f}"] \arrow[d, hook, "\varphi|_U"'] & A \times [0, 1) \arrow[d, hook, "\varphi"] \\
X \arrow[r, "f"'] & X.
\end{tikzcd}$$

If we prove that $Ch(\widetilde{S}, \tilde{f}) = Ch(\widetilde{S}, \tilde{f}|_{A \times \{0\}})$, our assertion follows.

Let $P = (N, L)$ be a filtration pair for $\widetilde{S}$ such that

$$(N, L) \cap (A \times [0, \varepsilon]) = (J, K) \times [0, \varepsilon],$$

for some compact pair (J, K) of A and $0 < \varepsilon < 1$ (by Lemma 4.1). Using the product structure of the filtration pair, we can define the homotopy $H : ((N, L) \cap (A \times [0, \varepsilon])) \times [0, 1] \to (N, L) \cap (A \times [0, \varepsilon])$ from the identity map (on the 0-level) to the retraction map to $(N, L) \cap (A \times \{0\})$ (on the 1-level) relative to $(N, L) \cap (A \times \{0\})$ by $H((x, s), t) = (x, s(1 - t))$ $((x, s) \in N \cap (A \times [0, \varepsilon]), 0 \le t \le 1)$.

We consider the following subspaces of N_L:

$$(N_L)_\varepsilon = \{[(x, s)] \in N_L \mid (x, s) \in N, 0 \le s \le \varepsilon\} \cup \{*_L\},$$
$$(N_L)_0 = \{[(x, 0)] \in N_L \mid (x, 0) \in N\} \cup \{*_L\}.$$

Then the homotopy H induces a homotopy $\bar{H} : (N_L)_\varepsilon \times [0, 1] \to (N_L)_\varepsilon$ from the identity map (on the 0-level) to the retraction map to $(N_L)_0$ (on the 1-level) relative to $(N_L)_0$.

The assumption $W^u_{\tilde{f}}(\widetilde{S}) \subseteq A \times \{0\}$ shows the existence of a number $m \in \mathbb{N}$ satisfying $(\tilde{f}_P)^m(N_L) \subseteq (N_L)_\varepsilon$. Taking $r = \bar{H}_1 \circ (\tilde{f}_P)^m$ and the inclusion map $s : (N_L)_0 \to N_L$, we can easily check that the following diagrams are homotopically commutative.

$$\begin{tikzcd}
N_L \arrow[r, "\tilde{f}_P"] \arrow[d, "r"'] & N_L \arrow[d, "r"] \\
(N_L)_0 \arrow[r, "\tilde{f}_P|_{\dots}"'] & (N_L)_0
\end{tikzcd}
\qquad
\begin{tikzcd}
(N_L)_0 \arrow[r, "\tilde{f}_P|_{\dots}"] \arrow[d, "s"'] & (N_L)_0 \arrow[d, "s"] \\
N_L \arrow[r, "\tilde{f}_P"'] & N_L
\end{tikzcd}
\qquad
\begin{tikzcd}
N_L \arrow[r, "(\tilde{f}_P)^m"] \arrow[d, "r"'] & N_L \arrow[d, "r"] \\
(N_L)_0 \arrow[r, "(\tilde{f}_P|_{\dots})^m"'] \arrow[ur, "s"] & (N_L)_0.
\end{tikzcd}$$

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Since the pair $(N, L) \cap (A \times \{0\}) = (J \times \{0\}, K \times \{0\})$ is also a filtration pair for $\widetilde{S}$ in $A \times \{0\}$, and we may consider $(N_L)_0$ as $J \times \{0\}/K \times \{0\}$, the diagrams imply $Ch(\widetilde{S}, \tilde{f}) = Ch(\widetilde{S}, \tilde{f}|_{A\times\{0\}})$. This completes our proof. □

We give a statement in the form of a pair of ANR's.

THEOREM 4.3. *Let $f : (X, A) \to (X, A)$ be a self-map defined on a pair (X, A) of ANR's satisfying A is a Z-set in X, and $S \subseteq A$ be an isolated invariant set for f such that $W^u(S) \subseteq A$. Then, we hold $Ch(S, f) = Ch(S, f|_A)$, in particular $cc(S, f) = cc(S, f|_A)$.*

Proof. It follows from an Edwards' theorem [**6**, Theorem 44.1] that $(X \times Q, A \times Q)$ is a Q-manifolds pair, and $A \times Q$ is obviously a Z-set in $X \times Q$. Thus, $A \times Q$ is collared in $X \times Q$ by [**6**, Theorem 16.2]. We note that $S \times Q$ is also isolated invariant for $f \times id_Q$ and $W^u(S \times Q) \subseteq A \times Q$.

Using the general fact (Lemma 4.2) above, we conclude that

$$\begin{aligned} Ch(S, f) &= Ch(S \times Q, f \times id_Q) \\ &= Ch(S \times Q, f \times id_Q|_{A\times Q}) \\ &= Ch(S, f|_A), \end{aligned}$$

and this is precisely the assertion of the theorem. □

By the collaring theorem of Brown [**4**], we have the following.

COROLLARY 4.4. *Let $f : (M, \partial M) \to (M, \partial M)$ be a self-map defined on a manifold M and $S \subseteq \partial M$ be an isolated invariant set for f such that $W^u(S) \subseteq \partial M$. Then, we hold $Ch(S, f) = Ch(S, f|_{\partial M})$, in particular $cc(S, f) = cc(S, f|_{\partial M})$.*

We next consider the case of an isolated invariant set being of repelling type.

LEMMA 4.5 [**30**, Theorem 2]. *Let $f : (X, A) \to (X, A)$ be a self-map defined on a pair (X, A) satisfying A is closed and collared in X, and $S \subseteq A$ be an isolated invariant set for f such that $W^s(S) \subseteq A$. Then, $Ch(S, f)$ is trivial, i.e. $cc(S, f) = 1$.*

Proof. Given $\tilde{f}$ and $\widetilde{S}$ as in the proof of Lemma 4.2, we obtain a filtration pair $P = (N, L)$ for $\widetilde{S}$ such that $(N, L) \cap (A \times [0, \varepsilon]) = (J, K) \times [0, \varepsilon]$ for some compact pair (J, K) of A and $0 < \varepsilon < 1$, by Lemma 4.1 again. Using the product structure of the filtration pair, we can define the homotopy $G : (N, L) \times [0, 1] \to (N, L)$ from the identity map (on the 0-level) to the retraction map to $(N, L) \cap (A \times [\varepsilon, 1))$ (on the 1-level) relative to $(N, L) \cap (A \times [\varepsilon, 1))$ by $G((x, s), t) = (x, \max\{s, t\varepsilon\})$ $((x, s) \in N, 0 \leq t \leq 1)$.

Let $(N_L)^\varepsilon = \{[(x, s)] \in N_L \mid (x, s) \in N, \varepsilon \leq s < 1\} \cup \{*_L\}$. Then, the homotopy G induces a homotopy $\bar{G} : N_L \times [0, 1] \to N_L$ from the identity map (on the 0-level) to the retraction map to $(N_L)^\varepsilon$ (on the 1-level) relative to $(N_L)^\varepsilon$.

The assumption $W^s(S) \subseteq A \times \{0\}$ shows the existence of a number $m \in \mathbb{N}$ satisfying $(\tilde{f}_P)^m((N_L)^\varepsilon) = *_L$. Taking $r = (\tilde{f}_P)^m \circ H_1$ and the trivial map $s(N_L) = *_L$,

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we can easily check that the following diagrams are homotopically commutative.

$$\begin{tikzcd} N_L \arrow[r, "\tilde{f}_P"] \arrow[d, "r"'] & N_L \arrow[d, "r"] \\ N_L \arrow[r, "s"'] & N_L \end{tikzcd} \qquad \begin{tikzcd} N_L \arrow[r, "s"] \arrow[d, "s"'] & N_L \arrow[d, "s"] \\ N_L \arrow[r, "\tilde{f}_P"'] & N_L \end{tikzcd} \qquad \begin{tikzcd} N_L \arrow[r, "(\tilde{f}_P)^m"] \arrow[d, "r"'] & N_L \arrow[d, "r"] \\ N_L \arrow[r, "s^m"'] \arrow[ur, "s"] & N_L. \end{tikzcd}$$

Thus, $Ch(S, f) = Ch(\widetilde{S}, \tilde{f})$ is trivial. □

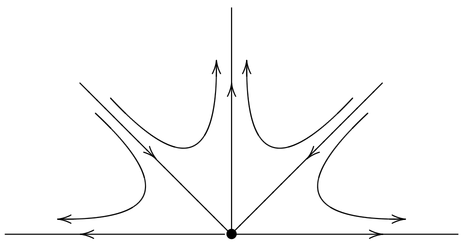
THEOREM 4.6. *Let $f : (X, A) \to (X, A)$ be a self-map defined on a pair of ANR's satisfying A is a Z-set in X, and $S \subseteq A$ be an isolated invariant set for f such that $W^s(S) \subseteq A$. Then $Ch(S, f)$ is trivial, i.e. $cc(S, f) = 1$.*

Proof. We can proceed our proof in a way similar to what we did in Theorem 4.3. □

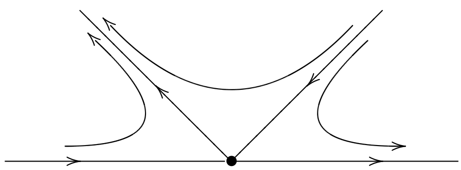
COROLLARY 4.7. *Let $f : (M, \partial M) \to (M, \partial M)$ be a self-map defined on a manifold M and $S \subseteq \partial M$ be an isolated invariant set for f such that $W^s(S) \subseteq \partial M$. Then, $Ch(S, f)$ is trivial, i.e., $cc(S, f) = 1$.*

EXAMPLE 4.8. f is a natural map of the half-plane, which has the one fixed point as shown below.

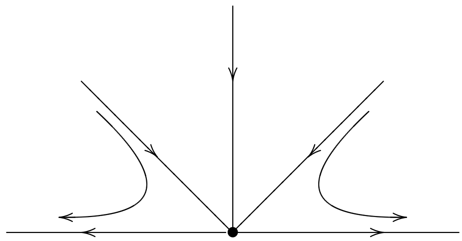
(1) $Ch(S, f) \neq Ch(S, f|_A)$, $cc(S, f) = cc(S, f|_A)$, where $S = \{\bullet\}$ and $A =$ the horizontal line



(2) $cc(S, f) \neq cc(S, f|_A)$, where $S = \{\bullet\}$ and $A =$ the horizontal line



(3) $Ch(S, f) = Ch(S, f|_A)$, $cc(S, f) = cc(S, f|_A)$, where $S = \{\bullet\}$ and $A =$ the horizontal line



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