THE ADJOINTS OF DIFFERENTIABLE MAPPINGS

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The notion of *symmetric* (non-linear) mappings has been introduced by Vainberg [3, p. 56]. However, symmetric mappings of this type have not played any important rôle in non-linear functional analysis. Naturally, as in the case of linear mappings, the symmetric mappings should be defined in such a way that they are easy to handle and belong to the most elementary class of non-linear mappings.

In this paper, we shall introduce the notion of *adjoint* mappings of non-linear mappings and define symmetric mappings as the mappings which coincide with their adjoints. It will be seen that a mapping is symmetric if and only if it is potential. (See Theorem 1.) This means that our definition gives a natural generalization of the notion of symmetry for linear mappings, because it is evident that a linear mapping is symmetric if and only if it is potential.

An extensive study on the potential mappings can be found in Vainberg's book [3]. We shall spend most of this paper for the study on the notion of adjoint mappings.

1. Preliminaries

Let $\boldsymbol{E}$ be a real Hilbert space. A mapping f of $\boldsymbol{E}$ into itself is said to be *(Fréchet-)differentiable at* $a \in \boldsymbol{E}$ if there exists a continuous linear mapping l of $\boldsymbol{E}$ into $\boldsymbol{E}$ such that

$$f(a+x)-f(a) = l(x)+r(a, x) \qquad \text{for every } x \in E$$

where

$$\lim_{||x||\to 0} ||r(a, x)||/||x|| = 0.$$

The linear mapping l is determined uniquely and depends on the element a. We call it *the derivative of* f *at* a and denote it by $f'(a)$.

If a mapping f is differentiable at every point of $\boldsymbol{E}$, f is said to be *differentiable*. In this case, $f'(x)$ is a mapping of $\boldsymbol{E}$ into the set $\mathscr{L}$ of all continuous linear mappings of $\boldsymbol{E}$ into $\boldsymbol{E}$. As is well known, the set $\mathscr{L}$ is a Banach algebra with the norm:

$$||l|| = \sup_{||x||=1} ||l(x)|| \qquad \text{for every } l \in \mathscr{L}.$$

If the mapping $f'(x)$ is continuous with respect to this norm topology, f is said to be *continuously differentiable*.

Throughout this paper, we denote by $\mathscr{D}$ the set of all continuously differentiable mappings f such that $f(0) = 0$. Real numbers are denoted by the Greek letters.

2. *-admissibility

A mapping $f \in \mathscr{D}$ is said to be **-admissible* if there exists $g \in \mathscr{D}$ such that

(1) $$g'(x) = (f'(x))^* \qquad \text{for every } x \in \boldsymbol{E},$$

where $(f'(x))^*$ denotes the adjoint of the linear mapping $f'(x)$. In other words, $f \in \mathscr{D}$ is *-admissible if there exists $g \in \mathscr{D}$ such that

$$(g'(x)(y), z) = (y, f'(x)(z))$$

for any x, y and z in $\boldsymbol{E}$. ($f'(x)(z)$ is the value of $f'(x)$ at z, and $(y, f'(x)(z))$ is the inner product of y and $f'(x)(z)$.)

(2) *If $f \in \mathscr{D}$ is *-admissible, the mapping g in* (1) *is determined uniquely.*

To prove this, we have only to prove generally that, if f, $g \in \mathscr{D}$ and

$$f'(x) = g'(x) \qquad \text{for every } x \in E,$$

we have $f(x) = g(x)$ for every $x \in \boldsymbol{E}$. Let us consider abstract functions $f(\xi x)$ and $g(\xi x)$ of real variable ξ. Since

$$\frac{d}{d\xi} f(\xi x) = f'(\xi x)(x) \text{ and } \frac{d}{d\xi} g(\xi x) = g'(\xi x)(x),$$

it follows from [Theorem 2.7, p. 34, [3]] that

$$f(x) = \int_0^1 f'(\xi x)(x) d\xi = \int_0^1 g'(\xi x)(x) d\xi = g(x).$$

In the sequel, we denote this uniquely determined mapping by f^* and call it *the adjoint of f*.

A linear mapping $l \in \mathscr{L}$ is always *-admissible because $\mathscr{L} \subset \mathscr{D}$ and

$$l'(x) = l \qquad \text{for every } x \in \boldsymbol{E},$$

and l^* defined in this way coincides with the usual adjoint.

The following properties of *-admissible mappings are easily proved:

(3) *If f is *-admissible, f^* is also *-admissible and*

$$(f^*)^* = f.$$

(4) *If f and g are *-admissible, $f+g$ is *-admissible and*

$$(\alpha f+\beta g)^* = \alpha f^*+\beta g^*.$$

(5) *If f is *-admissible,*

$$f^*(x) = \int_0^1 (f'(\xi x))^*(x)d\xi \qquad \text{for every } x \in \boldsymbol{E}.$$

The following property will be used frequently.

(6) *If f is *-admissible,*

$$(f(x), x) = (f^*(x), x) \qquad \text{for every } x \in \boldsymbol{E}.$$

In fact, by making use of properties of the abstract integral (§ 2, Chapter I, [3]), we have

$$\begin{aligned}(f(x), x) &= \left(\int_0^1 f'(\xi x)(x)d\xi, x\right)\\ &= \int_0^1 (f'(\xi x)(x), x)d\xi\\ &= \int_0^1 ((f'(\xi x))^*(x), x)d\xi\\ &= \left(\int_0^1 (f'(\xi x))^*(x)d\xi, x\right) = (f^*(x), x).\end{aligned}$$

3. Symmetry and skew-symmetry

A mapping $f \in \mathscr{D}$ is said to be *symmetric* if it is *-admissible and $f = f^*$. If f is *-admissible and $f^* = -f$, it is said to be *skew-symmetric*.

From (3) and (4) it follows immediately that, if f is *-admissible, $f+f^*$ is symmetric and $f-f^*$ is skew-symmetric.

The following theorem is a paraphrase of [Theorem 5.1, p. 56, [3]].

THEOREM 1. *A mapping f is symmetric if and only if it is potential; in other words, f is symmetric if and only if there exists a real-valued function $\phi(x)$ on $\boldsymbol{E}$ such that*

$$\phi(x+y)-\phi(x) = (f(x), y)+r(x, y)$$

for any x and y in $\boldsymbol{E}$ and

$$\lim_{||y||\to 0} |r(x, y)|/||y|| = 0.$$

Next, we give a characterization for skew-symmetric mappings.

THEOREM 2. *A mapping f is skew-symmetric if and only if it is linear and $(f(x), x) = 0$ for every $x \in \boldsymbol{E}$.*

PROOF. Let $f \in \mathscr{D}$ be skew-symmetric. Then, it follows from (6) that

$$(f(x), x) = (f^*(x), x) = -(f(x), x)$$

for every $x \in \boldsymbol{E}$. Therefore, $(f(x), x) = 0$ for every $x \in \boldsymbol{E}$. Thus, we have only to prove that f is linear. At first, we prove that, if f is skew-symmetric, $(f(x+\xi y), y)$ is constant with respect to ξ. In fact,

$$\begin{aligned}\frac{d}{d\xi}(f(x+\xi y), y) &= (f'(x+\xi y)(y), y)\\ &= ((f'(x+\xi y))^*(y), y)\\ &= ((f^*)'(x+\xi y)(y), y)\\ &= -(f'(x+\xi y)(y), y)\\ &= -\frac{d}{d\xi}(f(x+\xi y), y),\end{aligned}$$

hence it follows that

$$\frac{d}{d\xi}(f(x+\xi y), y) = 0 \qquad \text{for every } \xi.$$

Therefore, especially, we have

$$(f(x+y), y) = (f(x), y) \qquad \text{for any } x \text{ and } y \text{ in } \boldsymbol{E}.$$

Similarly,

$$(f(x+y), x) = (f(y), x) \qquad \text{for any } x \text{ and } y \text{ in } \boldsymbol{E}.$$

On the other hand, we have

$$0 = (f(x+y), x+y) = (f(x+y), x)+(f(x+y), y).$$

These three equalities imply that

$$(f(x), y) = -(f(y), x) \qquad \text{for any } x \text{ and } y \text{ in } \boldsymbol{E},$$

from which the linearity of f follows.

Conversely, if l is linear and $(l(x), x) = 0$ for any $x \in \boldsymbol{E}$, we have

$$(l(x), y)+(l(y), x) = (l(x+y), x+y)-(l(x), x)-(l(y), y) = 0,$$

from which it follows that $l^* = -l$.

4. Conditions for *-admissibility

The adjoints of non-linear mappings, unlike the adjoints of continuous linear mappings, cannot always be defined. Theorem 1 suggests the existence of close connection between *-admissibility and potentiality. The following theorem makes the connection clear.

THEOREM 3. *$f \in \mathscr{D}$ is *-admissible if and only if there exists a skew-symmetric mapping $l \in \mathscr{L}$ such that $f+l$ is potential.*

PROOF. If $f \in \mathscr{D}$ is *-admissible, we have that

$$f = \tfrac{1}{2}(f+f^*)+\tfrac{1}{2}(f-f^*),$$

$\frac{1}{2}(f+f^*)$ is symmetric (= potential by Theorem 1) and $\frac{1}{2}(f-f^*)$ is skew-symmetric. Therefore, we can take $\frac{1}{2}(f-f^*)$ as the mapping l in the theorem. Conversely, let us assume that $f \in \mathscr{D}$ and $f+l$ is potential for some skew-symmetric mapping $l \in \mathscr{L}$. Then, since $f+l$ and l are *-admissible, it follows from (4) that f is *-admissible.

It should be remembered that, if f is *-admissible, the skew-symmetric mapping in the above theorem can be determined uniquely. In fact, if there are two skew-symmetric mappings l_1 and l_2 such that $f+l_1$ and $f+l_2$ are symmetric, $l_1-l_2 = (f+l_1)-(f+l_2)$ is also symmetric. Therefore,

$$l_1-l_2 = (l_1-l_2)^* = l_1^*-l_2^* = l_2-l_1,$$

from which it follows that $l_1 = l_2$.

Thus, since non-zero skew-symmetric mappings are not symmetric, it is meant by

$$f = \tfrac{1}{2}(f+f^*)+\tfrac{1}{2}(f-f^*)$$

that the set of all *-admissible mappings is the direct sum of the set of all symmetric mappings and the set of all skew-symmetric mappings. In other words, if we put

$$f_S = \tfrac{1}{2}(f+f^*) \text{ and } l_f = \tfrac{1}{2}(f-f^*),$$
$$f = f_S+l_f$$

is the unique expression of a *-admissible mapping f as the sum of a symmetric mapping and a skew-symmetric mapping, and it is easy to see that

$$f^* = f-2l_f. \tag{7}$$

Now, we give another characterization for the *-admissibility.

THEOREM 4. *$f \in \mathscr{D}$ is *-admissible if and only if*

$$f'(x)-(f'(x))^*$$

is independent of $x \in \boldsymbol{E}$.

PROOF. If f is *-admissible, we have by (7) that

$$f-f^* = 2l_f.$$

Therefore, since l_f is a linear mapping,

$$f'(x)-(f'(x))^* = (f-f^*)'(x) = 2l_f'(x) = 2l_f,$$

which means that $f'(x)-(f'(x))^*$ is independent of $x \in \boldsymbol{E}$. Conversely, if $f'(x)-(f'(x))^*$ is independent of $x \in \boldsymbol{E}$, we can put

$$f'(x)-(f'(x))^* = l.$$

Then, for $g = f-l$, we have

$$g'(x) = (f-l)'(x) = f'(x)-l = (f'(x))^*.$$

Therefore, by the definition of the *-admissibility, f is *-admissible.

The following fact follows immediately from this theorem.

(8) *If $f \in \mathscr{D}$ is *-admissible and $f'(a)$ is symmetric for some $a \in \boldsymbol{E}$, then f itself is symmetric.*

In fact, if f satisfies this condition, we have $f = f_S$, because

$$l_f = \tfrac{1}{2}(f'(a)-(f'(a))^*) = 0.$$

In particular, a *-admissible mapping f is symmetric if $f'(a) = 0$ for some $a \in \boldsymbol{E}$.

5. *-admissibility for the twice differentiable mappings

For the twice differentiable mappings, we can have a simpler criterion for the *-admissibility. Let f be twice continuously differentiable; in other words, there exists a continuous linear mapping $f''(x)$ of $\boldsymbol{E}$ into $\mathscr{L}$ such that

$$f'(x+y)-f'(x) = f''(x)(y)+r(x, y)$$

for every x and y in $\boldsymbol{E}$, where

$$\lim_{||y||\to 0} ||r(x, y)||/||y|| = 0,$$

and $f''(x)$ is continuous with respect to $x \in \boldsymbol{E}$. (Therefore, $f''(x)(y) \in \mathscr{L}$ for any x and y in $\boldsymbol{E}$.)

THEOREM 5. *Let f be twice continuously differentiable. Then, f is *-admissible if and only if $f''(x)(x)$ is a symmetric mapping for every $x \in \boldsymbol{E}$.*

PROOF. Let f be *-admissible. Then, by Theorem 4,

$$\begin{aligned} f''(x)(x)-(f''(x)(x))^* &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[(f'(x+\varepsilon x)-f'(x))-(f'(x+\varepsilon x)-f'(x))^*\right] \\ &= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left[(f'(x+\varepsilon x)-(f'(x+\varepsilon x)^*)-(f'(x)-(f'(x))^*)\right] \\ &= 0. \end{aligned}$$

Conversely, if $f''(x)(x)$ is symmetric for every $x \in \boldsymbol{E}$, since

$$f'(x)-f'(0) = \int_0^1 f''(\xi x)(x)d\xi,$$

we have

$$f'(x)-(f'(x))^* = \int_0^1 f''(\xi x)(x)d\xi+f'(0) - \left(\int_0^1 f''(\xi x)(x)d\xi+f'(0)\right)^*$$
$$= \int_0^1 f''(\xi x)(x)d\xi+f'(0) - \int_0^1 (f''(\xi x)(x))^*d\xi-(f'(0))^*.$$

Therefore, by Theorem 4, f is *-admissible.

6. Products of *-admissible mappings

The product fg of two mappings $f \in \mathscr{D}$ and $g \in \mathscr{D}$ is defined by

$$(fg)(x) = f(g(x)) \qquad \text{for every } x \in \boldsymbol{E}.$$

It is well-known that $fg \in \mathscr{D}$ and

$$(fg)'(x) = f'(g(x))g'(x) \qquad \text{for every } x \in \boldsymbol{E}.$$

As is easily seen from this equality, the product of two *-admissible mappings is not always *-admissible. In this section, we hall take a deeper look into this fact. We begin with some lemmas.

(9) *Let* $l \in \mathscr{L}$ *and* $\boldsymbol{E}_x = \{\xi x | -\infty < \xi < \infty\}$ *be a one-dimensional closed subspace generated by a single element* $x \in \boldsymbol{E}$. *If* $l(x) \in \boldsymbol{E}_x$ *for every* $x \in \boldsymbol{E}$, *there exists a number* α *such that* $l = \alpha 1$, *where* 1 *is the identity mapping.*

Proof. By the assumption, there exists a real-valued function $\phi(x)$ such that

$$l(x) = \phi(x)x \qquad \text{for every } x \in \boldsymbol{E}.$$

We have only to prove that $\phi(x)$ is a constant function. Let x and y are arbitrary non-zero elements. If $(x, y) = 0$, since

$$\phi(x)x+\phi(y)y = l(x)+l(y) = l(x+y) = \phi(x+y)x+\phi(x+y)y,$$

we have

$$(\phi(x)x, x)+(\phi(y)y, x) = (\phi(x+y)x, x)+(\phi(x+y)y, x),$$

hence it follows that

$$\phi(x) = \phi(x+y). \tag{10}$$

If $y \in \boldsymbol{E}_x$, since $y = \alpha x$ for some α and

$$\phi(y)y = l(y) = l(\alpha x) = \alpha\phi(x)x = \phi(x)(\alpha x) = \phi(x)y,$$

we have

$$\phi(x) = \phi(y). \tag{11}$$

Now generally, since E_x is a closed linear subspace, there exist $y_1 \in E_x$ and $y_2 \in E_x^\perp = \{z \in E | (x, z) = 0\}$ such that

$$y = y_1 + y_2.$$

Then,

$$\begin{aligned} \phi(y) &= \phi(y_1+y_2) \\ &= \phi(y_1) && \text{(by (10))} \\ &= \phi(x) && \text{(by (11))}, \end{aligned}$$

which means that $\phi(x)$ is a constant function.

(12) *Let f be symmetric. If lf is symmetric for every $l \in \mathscr{L}$, then $f = 0$.*

Proof. From the assumption that $(lf)^* = lf$, we have

$$lf'(x) = f'(x)l^* \qquad \text{for every } x \in E.$$

Now, let a be a fixed element. Since

$$lf'(a) = f'(a)l^* \qquad \text{for every } l \in \mathscr{L}$$

$f'(a)$ cannot be in the form of $\alpha 1$. Therefore, by (9), there exists a non-zero element b such that

$$f'(a)(b) \notin E_b.$$

Since E_b is a closed linear subspace, there exists a non-zero element c such that

$$(f'(a)(b), c) = 0 \text{ and } (b, c) = 1.$$

Now, let x be an arbitrary element and consider the linear mapping $l(y) = (y, c)x$, which is obviously symmetric. Then,

$$\begin{aligned} f'(a)(x) &= f'(a)((b, c)x) = f'(a)l(b) \\ &= lf'(a)(b) = (f'(a)(b), c)x = 0. \end{aligned}$$

Since x is arbitrary, $f'(a) = 0$, which is true for every $a \in E$. Therefore, $f = 0$, because $f(0) = 0$.

Now, we can prove the following theorems.

Theorem 6. *Let $f \in \mathscr{D}$. If lf is *-admissible for every $l \in \mathscr{L}$, the mapping f is linear.*

Proof. Let an element a be fixed and let us consider the mapping

$$g = f - f'(a).$$

It is easy to see that $g'(a) = 0$ and lg is *-admissible for every $l \in \mathscr{L}$. Therefore, g is symmetric by (8), and it follows from Theorem 4 that

$$(lg)'(x)-((lg)'(x))^* = (lg)'(a)-((lg)'(a))^* = lg'(a)-(lg'(a))^* = 0,$$

which means that lg is symmetric for every $l \in \mathscr{L}$. Therefore, $g = 0$ by (12), or $f = f'(a)$; in other words, f is linear.

THEOREM 7. *Let $f \in \mathscr{D}$. If fl is *-admissible for every $l \in \mathscr{L}$, the mapping f is linear.*

PROOF. We consider the same g as in the proof of Theorem 6. It is clear that $g'(a) = 0$ and gl is *-admissible for every $l \in \mathscr{L}$. Therefore, for any $l \in \mathscr{L}$ such that

$$a \in l(\boldsymbol{E}),$$

we have, for $a = l(b)$,

$$\begin{aligned}(gl)'(x)-((gl)'(x))^* &= (gl)'(b)-((gl)'(b))^* \\ &= g'(l(b))\,l-l^*g'(l(b)) \\ &= g'(a)\,l-l^*g'(a) = 0.\end{aligned}$$

In other words, we have

$$g'(l(x))\,l = l^*g'(l(x)) \qquad \text{for every } l \in \mathscr{L} \text{ such that } a \in l(\boldsymbol{E}),$$

or

$$g'(y)l = l^*g'(y) \qquad \text{for every } l \in \mathscr{L} \text{ such that } a \in l(\boldsymbol{E}) \text{ and } y \in l(\boldsymbol{E}). \tag{13}$$

Now, let c be an arbitrary element and let us consider the following symmetric linear mapping

$$l(x) = x+(c, x)c.$$

Since

$$l(y-(c, y)(1+||c||^2)^{-1}c) = y \qquad \text{for every } y \in \boldsymbol{E},$$

we have $l(\boldsymbol{E}) = \boldsymbol{E}$. Therefore, for this mapping l, we have

$$g'(y)l(x) = lg'(y)(x) \qquad \text{for every } x \in \boldsymbol{E} \text{ and } y \in \boldsymbol{E},$$

which is equivalent to

$$g'(y)(x)+(c, x)g'(y)(c) = g'(y)(x)+(c, g'(y)(x))\,c,$$

or

$$(c, x)g'(y)(c) = (c, g'(y)(x))\,c.$$

Therefore, $g'(y)$ has the following property: if $(c, x) = 0$, then $(c, g'(y)(x)) = 0$. Now, let us assume that $g'(y) \neq 0$. Then, $g'(y)$ satisfies the condition of (9). Therefore, $g'(y)$ should be in the form of $\alpha 1$ for some α. However, this is impossible, because it should satisfy the equality (13).

REMARK. Let f be *-admissible and $l \in \mathscr{L}$ be symmetric. Let us consider the mapping

$$g = lfl.$$

This is *-admissible, because, since

$$g'(x) = lf'(l(x))\,l,$$

we have

$$(g'(x))^* = (lf'(l(x))l)^* = l(f'(l(x))^* l$$
$$= l(f^*)'(l(x))\,l = (lf^*l)'(x),$$

which means that

$$(lfl)^* = lf^*l.$$

Although the mappings of Hammerstein type are not always *-admissible, we can sometimes associate the mapping of the above type to the original mapping of Hammerstein type. In fact, if l is a positive definite, symmetric mapping, to the mapping lf, which is the general form of the mapping of Hammerstein type, we can associate the mapping $l_1 f l_1$ where l_1 is the square root of l. This method has been effectively used in § 10 of [**3**].

7. Ranges and null sets

For a mapping f of $\boldsymbol{E}$ into itself we denote its range and null set by $R(f)$ and $N(f)$, respectively; in other words, we put

$$R(f) = f(E) \text{ and } N(f) = \{x \in \boldsymbol{E} \mid f(x) = 0\}.$$

For a linear mapping l, it is well known that

$$(14) \qquad (\overline{R(l)})^\perp = N(l^*), \quad \overline{R(l)} = N(l^*)^\perp,$$
$$(\overline{R(l^*)})^\perp = N(l), \quad \overline{R(l^*)} = N(l)^\perp.$$

Naturally, in the case of non-linear mappings, we cannot have such precise relations like these.

(15) *For any* $f \in \mathscr{D}$, *we have the following relations*:

$$R(f)^\perp = \bigcap_{x \in \boldsymbol{E}} R(f'(x))^\perp = \bigcap_{x \in \boldsymbol{E}} N(f'(x)^*).$$

PROOF. Let $a \in R(f)^\perp$. Then, for any $x \in \boldsymbol{E}$ and $y \in \boldsymbol{E}$, we have

$$(a, f'(x)(y)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[(a, f(x+\varepsilon y) - f(x)) \right] = 0,$$

which means that $a \in R(f'(x))^\perp$ for every $x \in \boldsymbol{E}$. Conversely, if $a \in R(f'(x))^\perp$ for every $x \in \boldsymbol{E}$,

$$(a, f(x)) = \left(a, \int_0^1 f'(\xi x)(x) d\xi\right)$$
$$= \int_0^1 (a, f'(\xi x)(x))\, d\xi = 0,$$

hence it follows that $a \in R(f)^\perp$. The second equality follows immediately from (14).

For any subset M of $\boldsymbol{E}$, we denote the smallest linear subset containing M by $[M]$. Then, it is obvious that the following equalities follow from (15).

(16) *For any $f \in \mathscr{D}$ we have*

$$[\overline{R(f)}] = \bigcup_{x \in \boldsymbol{E}} \overline{R(f'(x))} = \bigcup_{x \in \boldsymbol{E}} N((f'(x))^*)^\perp.$$

On the other hand, as to the relation between $N(f)$ and $N(f'(x))$, we have only the following inequality.

(17) *For any $f \in \mathscr{D}$ we have*

$$\bigcap_{x \in \boldsymbol{E}} N(f'(x)) \subset N(f).$$

Proof. If $f'(x)(a) = 0$ for every $x \in \boldsymbol{E}$, since

$$f'(\xi a)(a) = 0 \qquad \text{for every } \xi,$$

we have

$$f(a) = \int_0^1 f'(\xi a)(a) d\xi = 0,$$

which means that $a \in N(f)$.

By the relations (15), (16) and (17), the following theorem can be easily proved.

Theorem 8. *Let f be *-admissible. Then,*

$$R(f)^\perp \subset N(f^*), \quad N(f^*)^\perp \subset \overline{[R(f)]},$$
$$R(f^*)^\perp \subset N(f), \quad N(f)^\perp \subset \overline{[R(f^*)]}.$$

Remark. Each of the relations of the above theorem cannot be replaced by the equality. For example, for the mapping

$$f(x) = (\xi_1^2, \xi_1) \text{ where } x = (\xi_1, \xi_2)$$

of a two-dimensional Euclidean space into itself, we have

$$R(f)^\perp = \{0\} \text{ and } N(f^*) = \{x = (\xi_1, \xi_2) \mid \xi_1^2 + \xi_2 = 0\}.$$

As is easily seen, Theorem 8 can be expressed in the following form.

(18) *Let f be a *-admissible mapping such that $R(f)$ (resp. $R(f^*)$) is closed. Then, either*

1° *for any* $y \in \boldsymbol{E}$ *there exist* $x_i \in \boldsymbol{E}$ $(i = 1, 2, \cdots, n)$ *and numbers* α_i $(i = 1, 2, \cdots, n)$ *such that*

$$y = \sum_{i=1}^{n} \alpha_i f(x_i) \text{ (respectively } y = \sum_{i=1}^{n} \alpha_i f^*(x_i))$$

or

2° *there exists an element* $a \neq 0$ *such that*

$$f^*(a) = 0 \text{ (respectively } f(a) = 0).$$

In fact, if $[R(f)] = \boldsymbol{E}$, we have 1° and, if $[R(f)] \neq \boldsymbol{E}$, since $R(f) \neq 0$, any non-zero element in $R(f)$ satisfies 2°.

8. The mapping degree

For the definition of the mapping degree we refer to [2], in which the following theorem has been proved:

Let $f \in \mathscr{D}$ *be completely continuous (i.e., continuous and transforms every bounded set into a compact set). For a real number* λ *which is not a proper value of* $f'(0)$, *we consider the vector field*

$$f_\lambda(x) = \lambda x - f(x) \qquad \text{for every } x \in \boldsymbol{E}.$$

Then, there exists a sphere $S = S_r = \{x \in \boldsymbol{E} | \, ||x|| \leqq r\}$ *such that the mapping degree* $d(f_\lambda, S, 0)$ *of* f_λ *at* 0 *relative to* S *is equal to* $(-1)^\beta$, *where* β *is the sum of the multiplicities of all the proper values* λ' *of* $f'(0)$ *such that* $\lambda\lambda' > 0$ *and* $|\lambda'| < |\lambda|$. (cf. Theorem 4.7, p. 136, [1]).

The purpose of this section is to obtain a relation between the mapping degrees of f and f^* by making use of the above theorem of Leray and Schauder. We begin with the following theorem.

THEOREM 9. *Let* f *be *-admissible. Then,* f *is completely continuous if and only if* f^* *is completely continuous.*

PROOF. Let f be completely continuous. Then, by [Theorem 7, p. 51, [3]], $f'(x)$ is a completely continuous linear mapping for each $x \in \boldsymbol{E}$. Therefore, $(f^*)'(x) = (f'(x))^*$ is also completely continuous for each $x \in \boldsymbol{E}$. On the other hand, by Theorem 4, we have

$$f - f^* = f'(x) - (f'(x))^* \qquad \text{for every } x \in \boldsymbol{E}.$$

Therefore, f^* is completely continuous. The converse can be proved similarly.

Now, let f be a *-admissible completely continuous mapping. Since f^* is also completely continuous, $(f^*)'(0) = (f'(0))^*$, $(f'(0))^*$ has the same proper value as those of $f'(0)$ and the multiplicities of the common proper values coincide, the following theorem follows from the above theorem of Leray and Schauder.

THEOREM 10. *Let f be a *-admissible completely continuous mapping. For a real number λ which is not a proper value of $f'(0)$, let us consider the vector field $f_\lambda(x) = \lambda x - f(x)$ for every $x \in \boldsymbol{E}$. Then,*

$$f^*_\lambda(x) = \lambda x - f^*(x) \qquad \text{for every } x \in \boldsymbol{E},$$

and there exists a sphere S such that

$$d(f_\lambda, S, 0) = d(f^*_\lambda, S, 0).$$

References

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