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Π'-CLOSURE OF FINITE Π-SOLVABLE GROUPS

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Abstract

The purpose of this paper is to present a proof of the following theorem: *Suppose* Π *is a set of odd primes, G is a finite* Π*-solvable group, and A is a nilpotent* Π*-subgroup of maximal order of G. Then G has a normal* Π*-complement, if and only if* $N_G(ZJ(A))$ *has a normal* Π*-complement.* ($J(A)$ *is the Thompson subgroup of A.*)

Glauberman-Thompson's theorem 8.3.1 (see Gorenstein (1968)) in the study of finite groups, states that if P is a Sylow p-subgroup of the finite group G,p odd, and if $N_G(ZJ(P))$ has a normal p-complement, then so also does G. In this note we will extend this result to allow P to be any nilpotent Π-subgroup of maximal order if G is Π-solvable. More precisely,

THEOREM. *Suppose* Π *is a set of odd primes, G is a finite* Π*-solvable group, and A is a nilpotent* Π*-subgroup of maximal order of G. Then G has a normal* Π*-complement, if and only if* $N_G(ZJ(A))$ *has a normal* Π*-complement.*

Our notation is standard and is taken mainly from Gorenstein (1968). In particular, $J(G)$ is the *Thompson subgroup* of G.

PROOF. If G has a normal Π-complement it is well known that $N_G(ZJ(A))$ has a normal Π-complement. Assume now that G is of minimal order where $N_G(ZJ(A))$ has a normal Π-complement but G has none. Let K be a maximal Π-subgroup of G which contains A. Clearly, K is an S_Π-subgroup of G. Bialostocki's theorem, implies that $AF(K)$ is nilpotent. Hence $F(K) \subseteq A$. It is well known that $C_K(F(K)) \subseteq F(K)$, since K is solvable by the method of Feit and Thompson (1963). Therefore $ZJ(A) \triangleleft K$ by Mann's theorem (1971) Theorem 1(c) and Theorem 4. Theorem 1 of Mann implies that the set of all nilpotent subgroups of maximal order of K is a conjugate class. Let A^* be an Abelian subgroup of maximal order of K. Proposition 1 of [1], implies that $A^*F(K)$ is nilpotent. According to Mann (1971), Theorem 1, implies that there exists $x \in K$ such that $A^* \subseteq A^x$. Hence $J(K) = \langle J(A^x)/x \in K\rangle$. So $ZJ(K) =$

$ZJ(A)$ as $ZJ(A) \lhd K$. By induction $0_{\Pi'}(G) = 1$. It is easy to verify that $1 \subset 0_{\Pi}(G) \subset K$. Now set $\bar{G} = G/0_{\Pi}(G)$ and let $\bar{K}$ be the image of K in $\bar{G}$. Since G is Π-solvable, $0_{\Pi'}(\bar{G}) \neq 1$. By Gorenstein (1968), Theorem 6.2.2, $\bar{K}$ normalizes an S_q-subgroup $\bar{Q} \neq 1$ of $0_{\Pi'}(\bar{G}$ for some prime q and so normalizes $Z(\bar{Q})$. Let G_1 be the inverse image of $\bar{K}Z(\bar{Q})$ in G, so that $G_1 = KQ_1$, where Q_1 is an Abelian q-group isomorphic to $Z(\bar{Q})$. If $G_1 \subset G$, then by induction G_1 has a normal Π-complement which in this case must be Q_1 itself. But then $[0_{\Pi}(G), Q_1] \subseteq 0_{\Pi}(G) \cap Q_1 = 1$ and so Q_1 centralizes $0_{\Pi}(G)$. However, since $0_{\Pi'}(G) = 1$,[4]. Theorem 6.3.2. yields $C_G(0_{\Pi}(G) \subseteq 0_{\Pi}(G)$. This contradiction shows that $G = G_1 = KQ_1$. Thus S_2-subgroups of G are Abelian and according to Arad and Glauberman (to appear), Theorem 2(c), $ZJ(K) \lhd G$. Therefore $G = N_G(ZJ(A))$ has a normal Π-complement by our hypothesis, and we have completed the proof.

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