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A COMPARISON BETWEEN THIN SETS AND GENERALIZED AZARIN SETS

BY

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1. **Introduction.** Let $\mathbf{R}^p(p\geq 2)$ denote p-dimensional Euclidean space, D the half space defined by $\{P=(x_1, x_2, \ldots, x_p) \in \mathbf{R}^p: x_p>0\}$ and ∂D the frontier of D in $\mathbf{R}^p$. The Martin boundary (see [2]) of D can be identified with $\partial D \cup \{\infty\}$. We recall that the function $h(P)=x_p$ is a minimal harmonic function on D with pole at ∞. In 1949 (see [6]), Mme. Lelong defined thin sets at the boundary of D, including the point at ∞, of the type that we shall in future refer to as *minimally* thin sets. If u is a subharmonic function on D such that $\limsup u\leq 0$ everywhere on ∂D and $\sup u(P)/x_p=\beta<\infty$, she proved ([6], Théorème 1a) that there exists a set $E\subset D$, minimally thin at ∞, such that $\lim u(P)/x_p=\beta$, $P\to\infty$, $P \in D\backslash E$. Conversely (see [2] and [6], Reciproque 1a), her results imply that if $E\subset D$ is minimally thin at ∞, then there exists a subharmonic function u on D satisfying the above conditions such that

$$\limsup_{P\to\infty,\, P\in E} u(P)/x_p < \limsup_{P\to\infty,\, P\in D} u(P)/x_p = \beta.$$

If $P=(x_1, x_2, \ldots, x_p)$ and θ is defined such that $\cos\theta=x_p/|P|$, $0\leq\theta<\pi/2$, then Azarin [1] has obtained results which imply that if u is subharmonic on D and subject to the above restrictions, then there exists a set $F\subset D$ such that $\lim u(P)/|P|=\beta\cos\theta$, $P\to\infty$, $P \in D-F$, uniformly in θ where the exceptional set F satisfies the following thinness condition:

(1.1) There exists a sequence of balls $\{B_n\}$ which covers F such that $\sum_n (r_n/R_n)^{p-1} < \infty$, where r_n is the radius of B_n and R_n is the distance between its centre and the origin.

We note here that the results of Azarin have since been generalized by Essén and Lewis (see [4]).

If E is contained in a Stolz domain $K=\{P \in D: 0\leq\theta\leq\theta_0<\pi/2\}$, then $\lim((u(P)/x_p)-\beta)=0$, $P\to\infty$, $P \in K\backslash E$, if and only if $\lim((u(P)/|P|)-\beta\cos\theta)=0$, $P\to\infty$, $P \in K\backslash E$, uniformly in θ. If $E\subset K$ is a minimally thin set at ∞ with respect to D, it follows that E must also satisfy Azarin's condition (1.1). We shall show, however, that the converse does not hold (see in particular the corollaries of

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Theorem 5). Our main purpose here is to work out some of the relations between minimally thin sets at ∞ which are restricted to a Stolz domain, and those which satisfy a generalized form of Azarin's condition. We point out here that Azarin's work, which extends and unifies earlier generalizations of the Ahlfors-Heins version of the Phragmén-Lindelöf principle, is applicable to exceptional sets in the half space itself. We only claim to improve on the results of Azarin when the exceptional set in question is restricted to a Stolz domain.

DEFINITION 1. Let $h:[0, \infty)\to[0, \infty)$ be a continuous non-decreasing function such that $h(0)=0$. We define $E\subset D$ to be a generalized Azarin set with respect to h if and only if there exists a sequence of balls $\{B_n\}$ such that their union covers E and such that $\sum h(r_n/R_n)<\infty$, where r_n is the radius of B_n and R_n is the distance between its centre and the origin such that $R_n\to\infty$ as $n\to\infty$.

DEFINITION 2. If $\alpha>0$ and $h(r)=r^\alpha$, $r>0$, we shall say that E is a generalized Azarin set of order α if it is a generalized Azarin set with respect to h. In $\mathbf{R}^p$ $(p\geq 2)$, a generalized Azarin set of order $p-1$ shall simply be called an Azarin set (see (1.1)).

REMARK 1. Any generalized Azarin set of order α is also a generalized Azarin set of order α', whenever $\alpha'>\alpha$.

NOTATION:

(i) B_n means an open ball in $\mathbf{R}^p$ of radius r_n and centre P_n, where $|P_n|=R_n$

(ii) K is the Stolz domain $\{P\in D: 0\leq \theta\leq\theta_0<\pi/2\}$.

(iii) Let $s>1$ be fixed and I_n be the intersphere $\{P\in\mathbf{R}^p: s^n\leq|P|<s^{n+1}\}$.

(iv) $$\gamma(r)=\begin{cases}(\log(1/r))^{-1}, & 0<r<s^{-1},\\ (\log s)^{-1} & r\geq s^{-1}.\\ 0, & r=0.\end{cases}$$

(v) If $E\subset D$ is a given set, $c(E)$ denotes the ordinary (Newtonian if $p=3$) capacity in $\mathbf{R}^p$ of E and we define $E_n=E\cap I_n$ so that $c_n=c(E_n)$. In $\mathbf{R}^2$, $\lambda(E)$ denotes the logarithmic capacity of E and $\lambda_n=\lambda(E_n)$.

Let us also state known criteria for minimal thinness at ∞ for a set E contained in a Stolz domain: In $\mathbf{R}^p$, $p\geq 3$, a necessary and sufficient condition is that

$$\sum_{n=1}^{\infty} c_n/s^{n(p-2)}<\infty. \tag{1.2}$$

(see Mme. Lelong [6], p. 131). In $\mathbf{R}^2$, a necessary and sufficient condition is that

$$\sum_{n=1}^{\infty}\gamma(\lambda_n/s^n)<\infty. \tag{1.3}$$

(see Jackson [5], Theorem 1′).

2. **Some results on generalized Azarin sets.** When is a generalized Azarin set minimally thin at ∞? We start with two preliminary results.

THEOREM 1. *If $E=\bigcup_{n=1}^{\infty} B_n$ is a subset of $\mathbf{R}^p (p\geq 3)$ such that each ball $B_n \subset I_n \cap K$, then E is minimally thin at ∞ if and only if it is a generalized Azarin set of order $p-2$.*

Proof. Since E is contained in the Stolz domain K, we can use (1.2). $E \cap I_n = B_n$ is a ball of radius r_n, and it is clear that $c_n = r_n^{p-2}$. We also know that $s^n \leq R_n \leq s^{n+1}$. Hence $c_n/s^{n(p-2)} \geq (r_n/R_n)^{p-2} \geq c_n/s^{(n+1)(p-2)}$. Since the series $\sum c_n/s^{n(p-2)}$ and $\sum c_n/s^{(n+1)(p-2)}$ are co-convergent, the theorem is proved.

REMARK 2. When $p\geq 3$, it is clear that Azarin's condition (1.1) does not characterize minimally thin sets at ∞ which are restricted to a Stolz domain.

THEOREM 2. *If $E=\bigcup_n B_n$ is defined as in Theorem 1 subject to the modification that $E\subset \mathbf{R}^2$, then E is minimally thin at ∞ if and only if it is a generalized Azarin set with respect to γ. This means that*

$$\sum \gamma(r_n/R_n) < \infty. \tag{2.1}$$

Proof. E is minimally thin at ∞ if and only if (1.3) is true. Since $E \cap I_n = B_n$ is a disc of radius r_n, $\lambda_n = r_n$. Thus Theorem 2 will be proved if we can show that (2.1) is equivalent to

$$\sum \gamma(r_n/s^n) < \infty. \tag{2.2}$$

Since $s^n \leq R_n < s^{n+1}$ it is clear that $r_n/s^n \to 0$, $n\to\infty$, if either one of these two conditions hold. The function γ is nondecreasing and hence

$$\gamma(r_n/s^{n+1}) \leq \gamma(r_n/R_n) \leq \gamma(r_n/s^n). \tag{2.3}$$

Since

$$\gamma(xy) = \gamma(x)\gamma(y)(\gamma(x)+\gamma(y))^{-1}, \qquad x, y \in (0, s^{-1}],$$

we also know that if n is large enough,

$$\gamma(r_n/s^{n+1}) = \gamma(r_n/s^n)\gamma(1/s)(\gamma(r_n/s^n)+\gamma(1/s))^{-1}. \tag{2.4}$$

But $\gamma(1/s)>0$, and it is clear from (2.3) and (2.4) that conditions (2.1) and (2.2) are equivalent.

REMARK 3. Let E be constructed as in Theorem 2 so that $r_n/R_n = s^{-n}$ for all n. Then E is a generalized Azarin set of order α for all $\alpha>0$, but E is not minimally thin at ∞. We can conclude that in two dimensions a minimally thin set at ∞ which is restricted to a Stolz domain cannot be characterized as a generalized Azarin set of any order.

We shall now consider a sufficient condition for minimal thinness.

THEOREM 3. *If $E\subset K\subset \mathbf{R}^p$, $(p\geq 3)$ such that E is a generalized Azarin set of order $p-2$, then E is minimally thin at ∞.*

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Proof. By assumption, there exists a sequence of balls $\{B_n\}$ which covers E such that $\sum (r_n/R_n)^{p-2}<\infty$. We now re-label the sequence $\{B_n\}$ to form a double sequence $\{B_{nj}\}$ such that the centre of each ball B_{nj} lies in the intersphere I_n. Hence $s^n \le R_{nj} < s^{n+1}$, and it follows that $\sum_{n,j} (r_{nj}/R_{nj})^{p-2}<\infty$ if and only if $\sum_{n,j} (r_{nj}/s^n)^{p-2}<\infty$. Let c_{nj} denote the ordinary capacity in $\mathbf{R}^p$ $(p\ge 3)$ of $E_{nj}= E\cap B_{nj}$. Then $c_{nj}\le c(B_{nj})=r_{nj}^{p-2}$ so that

$$\sum_{n,j} c_{nj}s^{-n(p-2)} < \infty. \tag{2.5}$$

Since the capacity function is countably subadditive (see [2], chapters IV, VIII), $c(\bigcup_j E_{nj})\le \sum_j c_{nj}$. Hence with the possible exception of at most finitely many n, we have

$$c_n(E) = c(E_n) \le \sum_j (c_{n-1,j}+c_{nj}+c_{n+1,j}).$$

It then follows from (2.5) that $\sum c_n(E)s^{-n(p-2)}<\infty$. By (1.2), E is minimally thin at ∞, and Theorem 3 is proved.

THEOREM 4. *If $E\subset K\subset \mathbf{R}^2$ is a generalized Azarin set with respect to γ, then E is minimally thin at ∞.*

Proof. Arguing as in the previous proof, we have a sequence of balls $\{B_{nj}\}$ such that $s^n\le R_{nj}<s^{n+1}$ and such that $\sum_{n,j}\gamma(r_{nj}/s^n)<\infty$. If $E_{nj}=E\cap B_{nj}$, then $\lambda(E_{nj})\le r_{nj}$. Furthermore, $\lambda(E_{nj}/s^n)=s^{-n}\lambda(E_{nj})$ (see [7], p. 56) and therefore $\sum_{n,j}\gamma\circ\lambda(E_{nj}/s^n)<\infty$. The ordinary capacity function $\gamma\circ\lambda$ is countably subadditive (even though λ fails to have this property). Therefore

$$\sum_{n=2}^{\infty}\gamma(\lambda_n(E)/s^n) = \sum_{n=2}^{\infty}\gamma\circ\lambda(E_n/s^n) \le 3\sum_{n,j}\gamma\circ\lambda(E_{nj}/s^n) < \infty.$$

By applying (1.3) we conclude that E is minimally thin at ∞.

REMARK 4. The results of this section remain valid for minimally thin sets at the origin provided that we modify our sequence of balls $\{B_n\}$ in such a way that the distance between the origin and each B_n is greater than zero.

REMARK 5. If $p=3$, the axis which is normal to ∂D constitutes an example of a minimally thin set at ∞ which is not an Azarin set of order 1. The implication proved in Theorem 3 is therefore strict in general.

3. **A Lemma.** In the sequel, we need certain results from Carleson ([3], §§II–IV). Let h: $[0, \infty)\to[0, \infty)$ be a continuous, non-decreasing function such that $h(0)=0$. If E is a bounded set, consider a countable number of balls $\{B_n\}$ with radii $\{r_n\}$ such that $E\subset \bigcup_n B_n$, and define $M_h(E)=\inf\sum h(r_n)$ for all such coverings. For every compact set F, there is a non-negative set function μ depending on F such that

$$\mu(B) \le h(r) \tag{3.1}$$

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for every ball B of radius r, and such that the following inequality holds:

$$M_h(F) \leq \text{const.}\ \mu(F). \tag{3.2}$$

The constant depends only on the dimension p (see [3], Theorem 1 p. 7).

We want to study the relation between the outer measure M_h and outer capacity C_L with respect to a kernel L which is defined in the following way.

Let us define

$$\phi_p(r) = \begin{cases} \log(1/r), & p = 2, r > 0. \\ r^{2-p}, & p > 2, r > 0. \end{cases}$$

If H is a non-negative, continuous increasing function on R, we consider kernels of the form $L=H \circ \phi_p$ which are restricted so that

$$\int_0 L(r)r^{p-1}\, dr < \infty.$$

REMARK 6. Carleson ([3], p. 14) also assumes that H is convex, which induces a strong form of the maximum principle ([3], p. 15) for L-potentials. Since the potentials of our kernel L satisfy a weak form of the maximum principle as discovered by Ugaheri ([8], p. 38, Fundamental Lemma), the convexity property of H is not required in the proofs of those results in Carleson's book which we shall need.

LEMMA 1. *Let $E \subset \mathbf{R}^p$ $(p \geq 2)$ be a bounded set. If*

$$A = \int_0^\infty L(r)\, dh(r) < \infty. \tag{3.3}$$

there exists a constant depending only on A and p such that

$$M_h(E) \leq \text{const.}\ C_L(E) \tag{3.4}$$

where C_L is the outer capacity function with respect to L.

Proof. We first claim that (3.4) holds for compact sets. If the compact set F is given, let μ be a non-negative set function associated with F such that (3.1) and (3.2) are true. Let

$$u_\mu(x) = \int L(|x-y|)\, d\mu(y) \quad \text{be the } L\text{-potential of } \mu.$$

Arguing as in the second part of the proof of Theorem 1 in Carleson ([3], p. 28), we see that if x_0 is an arbitrary point and if $\Phi(r)=\mu(\{x : |x-x_0|<r\})$,

$$\mu_\mu(x_0) = \int_0^\infty L(r)\, d\Phi(r) \leq \int_0^\infty L(r)\, dh(r) = A.$$

Hence $\mu(F) \leq AC_L(F)$. It follows from (3.2) that

$$M_h(F) \leq \text{const.}\ C_L(F),$$

and we have completed the first step in the proof of Lemma 1.

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Secondly, we prove that (3.4) is true for open sets. We use the auxiliary function m_h' defined in Carleson ([3], pp. 6 and 11). It has the following properties: there exist constants C_1 and C_2 depending only on the dimension such that

$$C_1 M_h(E) \leq m_h'(E) \leq C_2 M_h(E).$$

If 0 is open,

$$m_h'(0) = \sup_{F \subset 0} m_h'(F),\ F \text{ compact}.$$

(see [3], §2, (1.3) and (3.6)). Hence, since the lemma is true for compact sets,

$$M_h(0) \leq \text{const. } m_h'(0) \leq \text{const. } \sup_{F \subset 0} m_h'(F) \leq \text{const. } \sup_{F \subset 0} M_h(F)$$
$$\leq \text{const. } \sup_{F \subset 0} C_L(F) = \text{const. } C_L(0).$$

Thus (3.4) holds for open sets.

The result for general bounded sets is now immediate since

$$M_h(E) = \inf_{0 \supset E} M_h(0), \qquad 0 \text{ open},$$

(see [3], p. 9) and

$$C_L(E) = \inf_{0 \supset E} C_L(0), \quad 0 \text{ open}.$$

4. **On sets which are minimally thin at ∞.** Let $E \subset K$ be minimally thin at ∞. How can we characterize E as a generalized Azarin set?

THEOREM 5. *If $E \subset K \subset \mathbf{R}^p$ ($p \geq 2$) is minimally thin at ∞, then E is a generalized Azarin set with respect to any function h such that $\int_0^\infty \phi_p^+(r)\, dh(r) < \infty$.*

Proof. Let c be the outer ordinary capacity function in $\mathbf{R}^p$ ($p \geq 2$). If $p=2$, $c = \gamma \circ \lambda$. Since $c(rE) = r^{p-2} c(E)$ if $p \geq 3$, and $\lambda(rE) = r\lambda(E)$, $p=2$, E is minimally thin at ∞ in $\mathbf{R}^p$ ($p \geq 2$) if and only if

$$\sum_n c(E_n/s^n) < \infty,$$

(see (1.2) when $p \geq 3$ and (1.3) when $p=2$). By Lemma 1, $M_h(E_n/s^n) \leq \text{Const.}\, c(E_n/s^n)$. Therefore, the minimal thinness of $E \subset K$ at ∞ implies that $\sum M_h(E_n/s^n) < \infty$. For each n, there exists a covering of E_n/s^n by a sequence of balls $\{B_{ni}'\}$, each of radius r_{ni}', such that $\sum_i h(r_{ni}') \leq M_h(E_n/s^n) + 2^{-n}$. We can now construct a sequence of $\{B_{ni}\}$ of balls, each of radius $r_{ni} = s^n r_{ni}'$ such that $\{B_{ni}\}$ covers E_n and $r_{ni}/R_{ni} \leq r_{ni}'$ for each i. Since h is non-decreasing, $h(r_{ni}/R_{ni}) \leq h(r_{ni}')$, and therefore

$$\sum_i h(r_{ni}/R_{ni}) \leq M_h(E_n/s^n) + 2^{-n}.$$

Summing over n, we get a double sequence of balls $\{B_{ni}\}$ which covers E such that the corresponding generalized Azarin sum is finite. The proof of Theorem 5 is complete.

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COROLLARY 1. *Let $\alpha>p-2$ be given. If $E\subset K$ is minimally thin at ∞, then E is a generalized Azarin set of order α.*

Proof. Choosing

$$h(r)=\begin{cases} r^{\alpha}, & 0\leq r\leq 1,\\ 1, & r>1,\end{cases}$$

it is easily checked that if $\alpha>p-2$, $\int_0^\infty \phi_p^+(r)\,dh(r)<\infty$, and the result is immediate from Theorem 5.

We introduce

$$\gamma_\alpha(r)=\begin{cases}\{\log(1/r)\}^{-\alpha}, & 0<r<e^{-1},\\ 1 \quad , & r\geq e^{-1}.\end{cases}$$

COROLLARY 2. *Let $\alpha>1$ be given. If $E\subset K\subset \mathbf{R}^2$ is minimally thin at ∞, then E is a generalized Azarin set with respect to γ_α.*

Proof. Choosing $h=\gamma_\alpha$, it is easily checked that if $\alpha>1$, $\int_0^\infty \phi_2^+(r)\,d\gamma_\alpha(r)<\infty$, and the result follows directly from Theorem 5.

REMARK 7. It is clear from these corollaries that a subset E of a Stolz domain which is minimally thin at ∞, is much smaller than most of those sets which can be covered by the coverings considered by Azarin ([1]).

5. **On radial projections.** Let $E\subset D$ be minimally thin at ∞. According to a result of Mme. Lelong (see [6], p. 132), the radial projection onto the unit sphere of those rays through the origin whose intersection with E have ∞ as a limit point has ordinary capacity zero. What can be said about the radial projection onto the unit sphere of those rays through the origin which hit infinitely many of the balls in a generalized Azarin covering of E with respect to a given function h?

We need one more concept. Let $h=[0,\infty)\to[0,\infty)$ be a continuous, non-decreasing function such that $h(0)=0$. If $\rho>o$ is given, let

$$\Lambda_h^{(\rho)}(E)=\inf\sum h(r_n)$$

where the set E is covered by a set of balls of radii $\{r_n\}$ where $r_n\leq\rho$ for all n. The limit $\Lambda_h(E)=\lim_{\rho\to 0}\Lambda_h^{(\rho)}(E)$ is the classical Hausdorff measure of E (see Carleson [3], p. 6).

THEOREM 6. (i) *Let $E\subset D\subset\mathbf{R}^p$ $(p\geq 2)$ be the union of a sequence of balls $\{B_n\}$ which has ∞ as a limit point and such that $\sum h(r_n/R_n)<+\infty$. If E^* is the projection onto the unit sphere of those rays whose intersections with E have ∞ as a limit point then $\Lambda_h(E^*)=0$.*

(ii) *Let $g\colon[0,+\infty)\to[0,+\infty)$ fulfil the same conditions as h and satisfy the inequality $h\leq g$. If there exists a subset $G\subset D\cap\{P:|P|=1\}$ such that $\Lambda_g(G)>0$ and $\Lambda_h(G)=0$, then there exists a union E of a sequence of balls $\{B_n\}$ as described in* (i) *such that $G\subset E^*$ and such that*

$$\sum h(r_n/R_n)<+\infty. \tag{5.1}$$

$$\sum g(r_n/R_n)=+\infty. \tag{5.2}$$

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Proof. The radial projections of the balls $\{B_n\}$ onto the unit sphere have radii $\{r_n/R_n\}$. If $\rho>0$ is given, E^* can be covered by balls whose radii $\{r_n/R_n\}_q^\infty$ are all $\leq\rho$, and such that

$$\Lambda_h^{(\rho)}(E^*) \leq \sum_q^\infty h(r_n/R_n).$$

Since the right hand member can be made arbitrarily small by choosing q arbitrarily large, it follows that $\Lambda_h(E^*)=0$, and hence (i) is proved.

If B is a ball of radius r and centre P, let T_nB be a ball of radius s^nr and of centre s^nP. Since $\Lambda_h(G)=0$, if n is given, we can cover G by a v-sequence of balls $\{B'_{nv}\}$ whose radii $\{r'_{nv}\}$ are all less than 2^{-n} and such that

$$\sum_v h(r'_{nv}) < 2^{-n}.$$

We define $B_{nv}=T_nB'_{nv}$ and $E=\bigcup_{n,v} B_{nv}$. It is clear from the construction that the sum (5.1) associated with this set converges. Since $G\subset E^*$ and $\Lambda_g(G)>0$, it follows from (i) that the sum (5.2) associated with the set E diverges. We have proved (ii) and hence our theorem.

As a corollary, we can characterize the size of projections of the type discussed here in terms of capacity. If $\alpha>0$ is given, let C_α be the capacity with respect to the kernel $|P|^{-\alpha}$.

COROLLARY 1. *If E is a generalized Azarin set of order $\alpha<p-1$ then $C_\alpha(E^*)=0$.*

Proof. It follows from the first part of Theorem 6 that $\Lambda_h(E^*)=0$ where $h(r)=r^\alpha$, $\alpha<p-1$. By making use of the first part of Theorem 1 in Carleson ([3], p. 28) it follows immediately that $C_\alpha(E^*)=0$.

REMARK 8. If $p-1\leq\alpha<p$ then Corollary 1 is still valid but it is no longer significant because $C_\alpha(S)=0$ in this case where S is the unit sphere.

COROLLARY 2. *For each $\alpha\leq p-1$, there exists a generalized Azarin set E of order α, such that $C_\beta(E^*)>0$ for all $\beta<\alpha$.*

Proof. Let $h(r)=r^\alpha$ and choose $g\geq h$ such that

$$r^{-\beta}g(r)\to 0, \quad r\to 0, \quad \text{for all } \beta, \quad 0<\beta<\alpha. \tag{5.3}$$

$$r^{-\alpha}g(r)\to\infty, \quad r\to 0, \tag{5.4}$$

$$\text{If} \quad L(r)=(g(r))^{-1}, \quad \bar{L}(r)\leq \text{Const.}\, L(r), \tag{5.5}$$

for all r sufficiently small.

For the definition of $\bar{L}$, see ([3], Theorem 1, p. 28). One possibility is to take $g(r)=r^\alpha(1+\psi(r)^2)$, where $\psi(r)=\max(\log 1/r, 1)$.

From (5.4) and (5.5), we see that the conditions for Theorem 4 in Carleson ([3], p. 34) are satisfied. It follows that there exists a subset G of the unit sphere such that $C_L(G)>0$ and $\Lambda_h(G)=0$. It is clear from Carleson ([3], Theorem 1,

p. 28), that $\Lambda_g(G)=\infty$. It is also clear from (5.3) and (5.5) that $C_\beta(G)>0$, all $\beta<\alpha$. We can now apply the second part of Theorem 6 to construct the set E with the desired properties.

COROLLARY 3. *For each h such that $\int_0^\delta (h(r)/r)\,dr<\infty$, there exists a generalized Azarin set E with respect to h such that $C_\psi(E^*)>0$, where $\psi(r)=\max(\log 1/r, 1)$.*

Proof. Apply Theorem 5 of Carleson ([3], p. 35) and then repeat the argument of Corollary 2.

REMARK 9. It is clear from Corollary 3 of Theorem 6 along with Mme. Lelong's radial limit theorem for minimally thin sets at ∞ in $\mathbf{R}^2$ (see [6], p. 132) that the implication proved in Theorem 5 is strict in general when $p=2$. One cannot hope therefore to characterize minimally thin sets at ∞ as an Azarin set with respect to some function h such that $\int_0 \phi_p^+(r)\,dh(r)<\infty$.

6. **On classically thin sets in $\mathbf{R}^2$.** A set $E\subset\mathbf{R}^2$ is thin at 0 in the *classical* sense, or simply thin at 0, if and only if there exists a positive superharmonic function u in some neighbourhood of 0 such that $u(0)<+\infty$, whereas $\lim_{\substack{x\to 0\\ x\to E}} u(x)=+\infty$. If I_n, $c_n=c(E_n)$, $\lambda_n=\lambda(E_n)$ are defined as in section 1 but subject to the modification that $0<s<1$, then E is thin at 0 if and only if $\sum_n nc_n<+\infty$ (see [2], p. 81) or $\sum_n \gamma(\lambda_n{}^{1/n})<+\infty$, since $c_n=\gamma(\lambda_n)$ and $\gamma\{(x)^{1/n}\}=n\gamma(x)$. Some elementary calculations show that E is thin at 0 if and only if

$$\sum_n n\gamma\left\{\frac{\lambda_n}{s^n}\right\}<+\infty. \tag{6.1}$$

We recall (see [5], Theorem 1′) that E is minimally thin at 0 with respect to a half plane if and only if $\sum_n \gamma\{\lambda_n/s^n\}<+\infty$, provided that E is restricted to a Stolz domain. It is now clear that thin sets at 0 which are restricted to a Stolz domain are necessarily minimally thin at 0 with respect to a half plane and that this implication is strict in general. Since ordinary thinness is invariant with respect to the inversion mapping, E is thin at ∞ in $\mathbf{R}^2$ if and only if

$$\sum_n n\gamma\circ\lambda\left\{\frac{E_n}{s^n}\right\}<+\infty, \quad \text{or} \quad \sum_n n\gamma\left\{\frac{\lambda_n}{s^n}\right\}<\infty, \tag{6.2}$$

where it is again required that $s>1$ as in earlier sections. We shall now prove some covering theorems for thin sets at ∞ in $\mathbf{R}^2$ which are analogous to our earlier theorems for minimally thin sets at ∞ with respect to the half plane.

THEOREM 7. (*Analogue of Theorem* 2). *Let $E=\bigcup_{n=1}^\infty B_n$ be defined as in Theorem* 2. *Then E is thin at ∞ if and only if*

$$\sum_n n\gamma\left\{\frac{r_n}{R_n}\right\}<+\infty. \tag{6.3}$$

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Proof. Since $s^n \leq R_n < s^{n+1}$ we can repeat the reasoning in the proof of Theorem 2 to show that the series $\sum_n n\gamma\{\lambda_n/s^n\}$ and $\sum_n n\gamma\{\lambda_n/s^{n+1}\}$ are co-convergent which in turn implies that E is thin at ∞ if and only if $\sum_n n\gamma\{r_n/R_n\} < +\infty$.

THEOREM 8. (*Analogue of Theorem* 4). *Assume that* $E \subset \mathbf{R}^2$ *can be covered by a sequence of balls* $\{B_n\}$ *where each* $B_n = \{z : |z-z_n| \leq r_n,\ |z_n| = R_n\}$ *and* $R_n \to \infty$ *as* $n \to \infty$. *If*

$$\sum_n (\log R_n)\gamma\left\{\frac{r_n}{R_n}\right\} < +\infty, \tag{6.4}$$

then E is thin at ∞ *in* $\mathbf{R}^2$.

Proof. We define the double sequence $\{B_{nj}\}$ as in the proof of Theorem 4. Then the argument in the proof of Theorem 4 may be repeated to obtain $\sum_{n=2}^{\infty} n(\log s)\gamma\{\lambda_n/s^n\} \leq 3 \sum_{n,j} (\log R_{nj})\gamma \circ \lambda\{E_{nj}/s^n\} < +\infty$. The theorem follows.

THEOREM 9. (*Analogue of Theorem* 5). *If E is thin at* ∞ *in* $\mathbf{R}^2$ *and if h is defined so that* $\int_0 \phi_p^+(r)\, dh(r) < +\infty$, *then there exists a sequence of disks* $\{B_n\}$ *which covers E such that* $\sum_n (\log R_n)h\{r_n/R_n\} < +\infty$.

Proof. Now E is thin at ∞ in $\mathbf{R}^2$ if and only if $\sum_n nc\{E_n/s^n\} < +\infty$, and if we apply Lemma 1 of section 3 it follows that $\sum_n nM_h\{E_n/s^n\} < +\infty$. We now proceed as in the proof of Theorem 5 except that we now define the covering sequence $\{B'_{ni}\}$ of E_n/s_n such that $1 \leq R'_{ni} \leq s$. The covering sequence $\{B_{ni}\}$ for each E_n will necessarily satisfy the restrictions that $s^n < R_{ni} < s^{n+1}$. It follows that $\sum_i h\{r_{ni}/R_{ni}\} < M_h\{E_n/s^n\} + 2^{-n}$ and therefore

$$\sum_i (\log R_{ni})h\,\{r_{ni}/R_{ni}\} < (n+1)\log s[M_h\,\{E_n/s^n\} + 2^{-n}].$$

If we now sum over n it is clear that the double sequence $\{B_{ni}\}$ satisfies the requirements of the theorem.

REMARK 10. We can conclude from Theorem 7 that a generalized Azarin set with respect to γ is not necessarily thin at ∞ in $\mathbf{R}^2$. We pointed out in Remark 5 that the implication proved in Theorem 3 is strict in general. We shall now demonstrate by example that the implication proved in Theorem 4 is also strict and that ordinary thin sets at ∞ in $\mathbf{R}^2$ are non-comparable in general with generalized Azarin sets with respect to γ. In defining γ we let $s=e$ and note that $1/\gamma(r) = \psi(r) = \max\{\log(1/r), 1\}$. We shall now construct a one-dimensional Cantor type of set $F \subset [0, 1]$ such that $C_\psi(F) = 0$ whereas $\Lambda_\gamma(F) > 0$ and note that the basic general idea is mentioned in Carleson ([3], p. 35).

We now proceed as in ([3], p. 31). Let $F = \bigcap_{n=1}^{\infty} F_n$ be the usual type of Cantor set with the special requirement that each F_n consists of 2^n closed intervals each of length $\varepsilon_n = 2^{-2^n}$. We write $F_n = \bigcup_{j=1}^{2^n} (J_{nj})$ where each J_{nj} is a component interval of length ε_n.

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LEMMA 2. *The Cantor set F that we have constructed above possesses the property that* $C_\psi(F)=0$ *whereas* $\Lambda_\gamma(F)>0$.

Proof. Since $\sum_n 2^{-n}\psi(\varepsilon_n)=\sum_n (2^{-n})2^n(\log 2)=\infty$ it follows from Carleson ([3], p. 31, Theorem 3) that $C_\psi(F)=0$. Let J_n be one of the component intervals of F_n. Then if $m>n$, $\sum \gamma(\varepsilon_m)=\gamma(\varepsilon_n)$, where we sum over those intervals of F_m which are contained in J_n. (In the case $m=n+1$, it is obvious that $2\gamma(\varepsilon_{n+1})=\gamma(\varepsilon_n)$. The general case follows by an induction argument). We claim that $m_\gamma(F)>0$, where m_γ is defined as in Carleson ([3], p. 6–7). In order to see this we observe that if F is covered by a finite collection of intervals of length $\{r_v\}_1^q$ such that $\min\{r_v\}\geq 2^{-2^n}$ and such that $r_v=2^{-p_v}$, $p_v=1, 2, 3, \ldots$, then $\sum_v \gamma(r_v)\geq 2^n\gamma(\varepsilon_n)=(\log 2)^{-1}$. Because of the equivalence of m_γ and M_γ (see [3], p. 7), we can therefore conclude that $M_\gamma(F)>0$, and hence $\Lambda_\gamma(F)>0$ which proves our lemma.

REMARK 11. If we constructed a Cantor set F such that each $\varepsilon_n=2^{-2^n/n}$, then a slight modification of the proof of Lemma 2 shows that $C_\psi(F)=0$ but $\Lambda_\gamma(F)=\infty$.

THEOREM 10. *We can construct a set* $E\subset\mathbf{R}^2$ *which is thin at* ∞ *but is not a generalized Azarin set with respect to* γ.

Proof. Let E' be the image of the Cantor set F as defined in Lemma 2 under a standard one-one mapping which carries the unit interval [0, 1) onto the unit circle. We can still say that $C_\psi(E')=0$ and $\Lambda_\gamma(E')>0$. We now let $E_n=T_n(E')$ where we recall that T_n is defined in the proof of Theorem 6. Then $E=\bigcup_{n=1}^\infty E_n$ has the property that $E^*=E'=E_n^*$ for all n. Hence $c(E_n^*)=0$, and therefore $\sum_n nc(E_n^*)<+\infty$, which implies that E is thin at ∞. We also have $\Lambda_\gamma(E^*)>0$, and hence can conclude from Theorem 6 that E is not a generalized Azarin set with respect to γ.

REMARK 12. We can conclude from Theorems 7 and 10 that ordinary thin sets at ∞ in $\mathbf{R}^2$ are non comparable with generalized Azarin sets with respect to γ. This observation trivially implies that the implication of Theorem 4 is also strict.

REMARK 13. There is no difference between a classically thin set E at 0 in $\mathbf{R}^p$ ($p\geq 3$) and a minimally thin set at 0 with respect to a half space, if E is restricted to a Stolz domain [see [2], p. 150). The situation is somewhat different at ∞ because minimally thin sets are preserved by inversion maps whereas classically thin sets are not (see [2], p. 148). At ∞ in $\mathbf{R}^p$ ($p\geq 3$), classically thin sets are necessarily minimally thin but the converse does not hold in general. A necessary and sufficient condition that E be thin at ∞ in $\mathbf{R}^p$ ($p\geq 3$) is that $\sum_n c(E_n)<\infty$. For such sets an analogue of Theorem 1 would be that $\sum_n (r_n)^{p-2}<\infty$. Theorems 3 and 9 have similar analogues.

Some of these results have been announced in the Compt. Rend. Acad. Sci. Paris, t. 277 (30 juillet 1973), Série A, 241–242.

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