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Separable Reduction and Supporting Properties of Fréchet-Like Normals in Banach Spaces

Marián Fabian and Boris S. Mordukhovich

Abstract. We develop a method of separable reduction for Fréchet-like normals and ϵ-normals to arbitrary sets in general Banach spaces. This method allows us to reduce certain problems involving such normals in nonseparable spaces to the separable case. It is particularly helpful in Asplund spaces where every separable subspace admits a Fréchet smooth renorm. As an applicaton of the separable reduction method in Asplund spaces, we provide a new direct proof of a nonconvex extension of the celebrated Bishop-Phelps density theorem. Moreover, in this way we establish new characterizations of Asplund spaces in terms of ϵ-normals.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space with dual X^*, $\Omega \subset X$ a nonempty set, $x \in \Omega$, and $\epsilon \geq 0$. Using the notation $u \xrightarrow{\Omega} x$: $u \to x$ with $u \in \Omega$, we define the set of *ϵ-normals*

$$\hat{N}_\epsilon(x;\Omega) = \left\{ \xi \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle \xi, u - x\rangle}{\|u - x\|} \leq \epsilon \right\} \tag{1.1}$$

to Ω at x [KM]. When $\epsilon = 0$, the set $\hat{N}_0(x;\Omega)$ in (1.1) is a cone called the *Fréchet normal cone* to Ω at x. For $\epsilon > 0$, we also consider the set

$$\tilde{N}_\epsilon(x;\Omega) = \{\xi \in X^* : \exists\, \nu > 0 \text{ with } \langle \xi, u - x\rangle \leq \epsilon\|u - x\| \text{ if } u \in \Omega \text{ and } \|u - x\| < \nu\}$$

called the *local ϵ-support* of Ω at x [EL]. One can easily observe the relationships

$$\hat{N}_\epsilon(x;\Omega) = \bigcap_{\gamma>0} \tilde{N}_{\epsilon+\gamma}(x;\Omega) \quad \text{and} \quad \hat{N}_\epsilon(x;\Omega) \supset \hat{N}_0(x;\Omega) + \epsilon B_{X^*} \tag{1.2}$$

for any $\epsilon \geq 0$. Moreover, if Ω is convex, the inclusion in (1.2) becomes the equality since

$$\hat{N}_\epsilon(x;\Omega) = \{\xi \in X^* : \langle \xi, u - x\rangle \leq \epsilon\|u - x\| \text{ for all } u \in \Omega\}. \tag{1.3}$$

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It is well known that the sets $\hat{N}_\epsilon(x;\Omega)$ and $\tilde{N}_\epsilon(x;\Omega)$ are closely related to the corresponding subdifferential sets

$$\hat{\partial}_\epsilon f(x) = \left\{ \xi \in X^* : \liminf_{u \to x} \frac{f(u) - f(x) - \langle \xi, u - x \rangle}{\|u - x\|} \geq -\epsilon \right\},$$

$\tilde{\partial}_\epsilon f(x) = \{\xi \in X^* : \exists \nu > 0 \text{ with } f(u) - f(x) \geq \langle \xi, u - x\rangle - \epsilon\|u - x\| \text{ if } u \in X$ and $\|u - x\| < \nu\}$ defined for any extended-real-valued function $f\colon X \to (-\infty, \infty]$ with $f(x) < \infty$. Indeed,

$$\hat{N}_\epsilon(x;\Omega) = \hat{\partial}_\epsilon \delta(x;\Omega) \quad \text{and} \quad \tilde{N}_\epsilon(x;\Omega) = \tilde{\partial}_\epsilon \delta(x;\Omega) \tag{1.4}$$

where $\delta(\,\cdot\,;\Omega)$ is the indicator function of the set Ω:

$$\delta(u;\Omega) = 0 \quad \text{if } u \in \Omega \quad \text{and} \quad \delta(u;\Omega) = \infty \text{ otherwise.}$$

Note that, in contrast to the hat-constructions $\hat{N}_\epsilon$ and $\hat{\partial}_\epsilon$, the tilde-constructions $\tilde{N}_\epsilon$ and $\tilde{\partial}_\epsilon$ carry little information when $\epsilon = 0$, even in finite dimensional spaces. For example, $\tilde{\partial}_0 f(x) = \varnothing$ everywhere for quite a nice function $f(x) = -x^2$ on R, while $\hat{\partial}_0 f(x) = \{f'(x)\}$ for any function $f\colon X \to$ R Fréchet differentiable at $x \in X$ on a Banach space X. We refer the reader to [BS], [L], [MS2] and their bibliographies for various useful properties of the Fréchet subdifferential $\hat{\partial}_0 f(x)$. In what follows we consider the hat-constructions for all $\epsilon \geq 0$ and the tilde-constructions only for $\epsilon > 0$.

It is shown in [FZ] that the construction of $\tilde{\partial}_\epsilon f$ is separably determined in the following sense: *Given a separable subspace $Y_0 \subset X$ and any $f\colon X \to (-\infty, \infty]$ locally bounded from below, there exists a separable subspace $Y \subset X$ such that $Y_0 \subset Y$ and*

$$\tilde{\partial}_\epsilon f(x) \neq \varnothing \quad \text{whenever} \quad x \in Y \quad \text{and} \quad \tilde{\partial}_\epsilon (f_{|Y})(x) \neq \varnothing \tag{1.5}$$

where $f_{|Y}$ denotes the restriction of f to the subspace Y. If X is Asplund (that is, every convex continuous function on X is generically Fréchet differentiable), this fact made it possible [FZ] to establish that, for any $\epsilon > 0$, the set

$$\{x \in X : \tilde{\partial}_\epsilon f(x) \neq \varnothing\} \tag{1.6}$$

is dense in $\operatorname{dom} f = \{x \in X : f(x) < \infty\}$ for each lower semicontinuous function $f\colon X \to (-\infty, \infty]$. Indeed, the separable reduction (1.5) allows to reduce the density result to the case of spaces admitting a Fréchet smooth renorm. In the latter case the density of (1.6) was earlier proved in [EL] with the help of Ekeland's variational principle. Moreover, it follows from [EL] that the density of (1.6) for every such f and every $\epsilon > 0$ implies the Asplund property of X.

An analogue of the separable reduction result (1.5) for the case of $\hat{\partial}_0 f$ (actually for $\hat{\partial}_\epsilon f$ with any $\epsilon \geq 0$) is obtained in [F]. This made it possible to characterize Asplund spaces in terms of the density of the set

$$\{(x, f(x)) \in X \times \text{R} : x \in \operatorname{dom} f, \hat{\partial}_0 f(x) \neq \varnothing\}$$

in the graph of f, for every lower semicontinuous function $f\colon X \to (-\infty, \infty]$. The latter fact is proved in [F] by a separable reduction to the case of spaces admitting a Fréchet smooth renorm which was previously resolved in [BP] on the base of the Borwein-Preiss smooth variational principle.

It is well known that if X is not Asplund, the condition $\hat{\partial}_0 f(x) \neq \varnothing$ may be violated at each $x \in \operatorname{dom} f$ even for concave continuous functions. However, in the case of indicator functions $f = \delta(\,\cdot\,;\Omega)$ we have

$$\hat{\partial}_0 \delta(x;\Omega) = \hat{N}_0(x;\Omega) \neq \varnothing \quad \forall x \in \Omega$$

(the cone $\hat{N}_0$ always contains $\xi = 0$) for every set $\Omega \subset X$ in any Banach space. Therefore, in this case the above separable reduction result for $\hat{\partial}_0 f$ does not carry any information. It follows from (1.2) and (1.4) that

$$\epsilon B_{X^*} \subset \tilde{\partial}_\epsilon \delta(x;\Omega) = \tilde{N}_\epsilon(x;\Omega) \subset \hat{N}_\epsilon(x;\Omega)$$

i.e., for $\epsilon > 0$ the separable reduction (1.5) also provides, even to a greater extent, an empty statement when $f = \delta(\cdot\,;\Omega)$.

The primary goal of this paper is to obtain valuable separable reduction results for the constructions $\tilde{N}_\epsilon$ and $\hat{N}_\epsilon$ to cover, first of all, the most important case of $\hat{N}_0$. To achieve this, we are going to further elaborate the separable reduction results from [FZ] and [F] so that they can be nontrivially applied for the set indicator functions $f = \delta(\,\cdot\,;\Omega)$. A natural framework for such an improvement of (1.5) is to provide a separable reduction in the form

$$\tilde{\partial}_\epsilon f(x) \setminus \mathrm{M}\mathrm{B}_{X^*} \neq \varnothing \quad \text{whenever} \quad x \in Y \quad \text{and} \quad \tilde{\partial}_\epsilon (f_{|Y})(x) \setminus \mathrm{M}\mathrm{B}_{Y^*} \neq \varnothing$$

where M is *any fixed number greater than* ϵ and Y is an appropriate separable subspace of the Banach space X. Having a separable reduction of $\tilde{\partial}_\epsilon f(x)$ in this form, we can apply it for the case of $f = \delta(\,\cdot\,;\Omega)$ and get

$$\tilde{N}_\epsilon(x;\Omega) \setminus \mathrm{M}\mathrm{B}_{X^*} \neq \varnothing \quad \text{whenever} \quad x \in Y \quad \text{and} \quad \tilde{N}_\epsilon(x;\Omega \cap Y) \setminus \mathrm{M}\mathrm{B}_{Y^*} \neq \varnothing \tag{1.7}$$

which is a valuable separable reduction for $\tilde{N}_\epsilon$. Indeed, (1.7) allows us to find elements of $\tilde{N}_\epsilon(x;\Omega)$ with the norm as large as $\sup\{\|\xi\| : \xi \in \tilde{N}_\epsilon(x;\Omega \cap Y)\}$. A similar separable reduction result for the case of $\hat{N}_0$ makes it possible to justify that the condition $\hat{N}_0(x;\Omega) \neq \{0\}$ is separably determined in any Banach space. Note that to get separable reduction results in the required new form for both tilde- and hat-constructions, we need to overcome essential technical difficulties in comparison with [FZ] and [F].

The separable reduction results obtained in this vein are notably efficient in Asplund spaces, where every separable subspace is Asplund and thus admits a Fréchet smooth renorm; see [D], [P]. Based on (1.7) and involving Ekeland's variational principle in the separable case, we can show that, in any Asplund space X, the set

$$\{x \in \operatorname{bd}\Omega : \tilde{N}_\epsilon(x;\Omega) \setminus \mathrm{M}\mathrm{B}_{X^*} \neq \varnothing\}$$

and its $\hat{N}_\epsilon$-counterpart are *dense* in the boundary $\operatorname{bd}\Omega$ of Ω for every closed set $\Omega \subset X$, where $\epsilon > 0$ is as small as we wish and $M > \epsilon$ is as large as we wish. Moreover, we are going to justify that the validity of these properties (even the *nonemptiness* of the corresponding sets) for every $\epsilon > 0$, every $M > \epsilon$, and every closed set $\Omega \subset X$ of a special nature is also *necessary* for X to be Asplund.

In the case of $\epsilon = 0$ we get characterizations of Asplund spaces through the existence of $x \in \operatorname{bd} \Omega$ satisfying

$$\hat{N}_0(x; \Omega) \neq \{0\} \tag{1.8}$$

for any closed sets $\Omega \subset X$, as well as through the density of such points in the boundary of Ω. If Ω is convex, (1.8) means that x is a *support point* of Ω, in the classical sense, due to representation (1.3) as $\epsilon = 0$. Thus the density of points $x \in \operatorname{bd} \Omega$ satisfying (1.8) is a natural extension of the celebrated Bishop-Phelps theorem [P, Theorem 3.18] to the case of nonconvex sets in Asplund spaces. This result was first proved in [MS1] with the help of Ekeland's variational principle and "fuzzy calculus" for Fréchet subdifferentials. (See also [MS2, Section 3] for another proof based on the same ideas.) The new proof given below is more direct and allows us to avoid "fuzzy calculus". This proof is based on the separable reduction of (1.8) to the case of spaces having a Fréchet smooth renorm and admitting the usage of the Borwein-Preiss smooth variational principle. The reverse result that the existence (and density) of points satisfying (1.8) for any closed set $\Omega \subset X$ implies the Asplund property of X was proved in [FM]. This also ensues from the corresponding fact for $\tilde{N}_\epsilon$ and $\hat{N}_\epsilon$, $\epsilon > 0$, established in the present paper.

The rest of the paper is organized as follows. In Section 2 we provide the basic separable reduction for the tilde-constructions $\tilde{\partial}_\epsilon$ and $\tilde{N}_\epsilon$ with $\epsilon > 0$. (As we mentioned above, the tilde-constructions for $\epsilon = 0$ do not make much sense.) The methods and results developed for the tilde-case are of some independent interest and also prepare the reader to the handling of more complicated hat-constructions. Section 3 is devoted to the separable reduction for the hat-constructions $\hat{\partial}_\epsilon$ and $\hat{N}_\epsilon$ with $\epsilon \geq 0$ which are considered as a limiting case of the tilde-constructions as $\epsilon' \downarrow \epsilon$. Section 4 contains applications of the separable reductions results and characterizations of Asplund spaces via supporting properties of Fréchet-like normals and ϵ-normals to nonconvex closed sets.

Everywhere we use standard notation except special symbols introduced where they are defined. Recall that $\overline{S}$ denotes the closure of a set S, and that $\operatorname{sp}(S)$ means the span of S, *i.e.*, the collection of all linear combinations of elements of S.

2 Separable Reduction for the Tilde-Constructions

The purpose of this section is to conduct an appropriate separable reduction for the construction $\tilde{\partial}_\epsilon$ which allows us to cover the case of $\tilde{N}_\epsilon$ as in (1.7) with $\epsilon > 0$. According to the discussion in Section 1, we need to provide a separable reduction of the assertion

$$\tilde{\partial}_\epsilon f(x) \setminus M B_{X^*} \neq \varnothing \tag{2.1}$$

for any given number $M > \epsilon$. Note that this assertion involves elements of the dual space X^*, while the method of separable reduction requires working only with elements of the initial Banach space X. So our first step is *to translate assertion (2.1) equivalently into the language of the space X*. To furnish this, we apply a certain *convexification* procedure, based on the definition of $\tilde{\partial}_\epsilon$, and then the classical separation theorem for convex sets; *cf.* [FZ].

Given a proper function $f\colon X \to (-\infty, \infty]$, a point $x \in \operatorname{dom} f$, and positive numbers δ and ϵ, we define a function $\varphi_{f,x,\delta,\epsilon}\colon X \to [-\infty, \infty]$ by

$$\varphi_{f,x,\delta,\epsilon}(h) = \inf\Big\{\sum_{i=1}^{m} \alpha_i \big[f(x+h_i) + \epsilon\|h_i\|\big] : m \in \mathrm{N}, h_i \in X, \tag{2.2}$$
$$\|h_i\| < \delta, \alpha_i \geq 0, i = 1, \dots, m, \sum_{i=1}^{m} \alpha_i = 1, \sum_{i=1}^{m} \alpha_i h_i = h\Big\}$$

if $\|h\| < \delta$ and $\varphi_{f,x,\delta,\epsilon}(h) = \infty$ otherwise. Note that $\varphi_{f,x,\delta,\epsilon}$ is convex if $\varphi_{f,x,\delta,\epsilon} > -\infty$, and that $\varphi_{f,x,\delta,\epsilon}(0) \leq f(x)$. We can easily check that if $\tilde{\partial}_\epsilon f(x) \neq \varnothing$, then $\varphi_{f,x,\delta,\epsilon}$ is proper and one has

$$\varphi_{f,x,\delta,\epsilon}(0) = f(x) \quad \text{and} \quad \varnothing \neq \partial\varphi_{f,x,\delta,\epsilon}(0) \subset \tilde{\partial}_\epsilon f(x)$$

for all $\delta > 0$ small enough, where $\partial\varphi$ denotes the subdifferential of convex analysis. On the other hand, if $\partial\varphi_{f,x,\delta,\epsilon}(0) \neq \varnothing$ for some $\delta > 0$ and $\varphi_{f,x,\delta,\epsilon}(0) = f(x)$, then $\partial\varphi_{f,x,\delta,\epsilon}(0) \subset \tilde{\partial}_\epsilon f(x)$ as well.

The following lemma provides an equivalent translation of (2.1) into the language of the initial space X.

Lemma 2.1 *Let $f\colon X \to (-\infty, \infty]$ be a function on a Banach space X, $x \in \operatorname{dom} f$, $\epsilon > 0$, and $M > \epsilon$. Then one has (2.1) if and only if there are numbers $\delta > 0$, $\gamma > 0$, $c \geq 0$, and a nonempty open set $U \subset X$ such that*

(i) $\varphi_{f,x,\delta,\epsilon}(h) \geq f(x) - c\|h\|$ *whenever $h \in X$, and*
(ii) $\varphi_{f,x,\delta,\epsilon}(th) \geq f(x) + (M+\gamma)t\|h\|$ *whenever $h \in U$ and $t \in (0,1]$.*

Proof First let us prove the *necessity*. Take $\xi \in \tilde{\partial}_\epsilon f(x) \setminus M B_{X^*}$ and find $\delta > 0$ from the definition of $\tilde{\partial}_\epsilon f(x)$. Then (i) is clearly satisfied with $c = \|\xi\|$. To establish (ii), we choose $\gamma > 0$ with $\|\xi\| > M + \gamma$ and find a nonempty open set $U \subset X$ so that $\langle \xi, h\rangle > (M+\gamma)\|h\|$ for every $h \in U$. Then (ii) is satisfied, and we get the necessity.

Let us prove the *sufficiency*. In what follows we replace $\varphi_{f,x,\delta,\epsilon}$ by φ for simplicity. Assuming (i) and (ii), we take c, γ, and U satisfying these conditions. Fix $0 \neq h \in U$ and find by (ii) a nonempty open convex set $0 \notin U_0 \subset U$, containing h, and a nonempty open convex set $0 \notin U_1 \subset \mathrm{R}$ such that

$$M < \tau/\|u\| < M + \gamma \quad \text{whenever} \quad (u, \tau) \in U_0 \times U_1.$$

Since φ is convex and $\varphi(0) \leq f(x)$, we get from (ii)

$$\varphi'_+(0)(u) \geq (M+\gamma)\|u\| \quad \text{whenever} \quad u \in U_0$$

for the right-hand directional derivative of φ at 0. Now let us consider the two nonempty convex sets

$$C_1 = \{(u,t) \in X \times \mathrm{R} : \varphi(u) \leq t\} \quad \text{and} \quad C_2 := \bigcup_{\lambda > 0} \lambda(U_0 \times U_1)$$

and observe that $C_1 \cap C_2 = \varnothing$. Indeed, if $\lambda(u,\tau) \in C_1 \cap C_2$ for some $\lambda > 0$, then

$$\lambda\tau \geq \varphi(\lambda u) \geq \varphi'_+(0)(\lambda u) = \lambda\varphi'_+(0)u \geq (M+\gamma)\lambda\|u\| > \lambda\tau$$

due to the choice of $\tau < (M+\gamma)u$, *i.e.*, we get a contradiction. Since C_2 is open, we apply the classical separation theorem and find $(0,0) \neq (\xi, s) \in (X \times \mathrm{R})^* = X^* \times \mathrm{R}$ such that

$$l := \inf\langle(\xi,s), C_1\rangle \geq \sup\langle(\xi,s), C_2\rangle =: r.$$

Note that $l \leq 0$ due to $(0,0) \in C_1$ and that $r \geq 0$ due to the structure of C_2. Thus $l = r = 0$, and we have

$$\begin{aligned} &\inf\{\langle\xi,u\rangle + st : (u,t) \in X \times \mathrm{R}, \varphi(u) \leq \varphi(0) + t\} \\ &\qquad = \sup\{\lambda\langle\xi,u\rangle + \lambda\tau s : (u,\tau) \in U_0 \times U_1, \lambda > 0\} = 0. \end{aligned} \tag{2.3}$$

Since $st = \langle\xi, 0\rangle + st \geq 0$ for all $t \geq 0$, we get $s \geq 0$.

To proceed, we first assume that $s > 0$. Then, putting $t = \varphi(u)$ in (2.3), we have $-\langle\xi,u\rangle \leq s\varphi(u)$ and

$$\left\langle -\frac{1}{s}\xi, u \right\rangle \leq \varphi(u) = \varphi(u) - \varphi(0)$$

if $u \in \operatorname{dom}\varphi$. This also obviously holds if $\varphi(u) = \infty$. Since $\varphi(0) = f(x)$ by (ii), we conclude that

$$-\frac{1}{s}\xi \in \partial\varphi(0) \subset \tilde{\partial}_\epsilon f(x).$$

On the other hand, from (2.3) for $\tau \in U_1$ and $u = h$ we have $\langle\xi,h\rangle + \tau s \leq 0$, and hence

$$\left\| -\frac{1}{s}\xi \right\| \geq \left\langle -\frac{1}{s}\xi, \frac{h}{\|h\|} \right\rangle \geq \frac{\tau}{\|h\|} > M$$

due to the choice of $\tau > M\|h\|$. Thus we obtain

$$\left\langle -\frac{1}{s}\xi, h \right\rangle > M\|h\| \quad \text{and} \quad -\frac{1}{s}\xi \in \tilde{\partial}_\epsilon f(x) \setminus M B_{X^*}$$

which justifies (2.1) in the case of $s > 0$. Note that we have not used (i) so far.

Next let us consider the remaining case of $s = 0$ in (2.3) and justify (2.1) using (i). If $s = 0$, we necessarily have $\xi \neq 0$ and get from (2.3) the following two conditions:

$$\langle\xi,u\rangle \geq 0 \quad \text{for all } u \in \operatorname{dom}\varphi \quad \text{and} \quad \langle\xi,u\rangle \leq 0 \quad \text{for all } u \in U_0. \tag{2.4}$$

Since $\xi \neq 0$ and U_0 is a neighborhood of h, the second condition in (2.4) yields $\langle\xi,h\rangle < 0$.

Further, let us form a closed convex subset of $X \times \mathrm{R}$ as follows:

$$C_3 = \{(u,t) \in X \times \mathrm{R} : t < -c\|u\|\}.$$

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Then (i) ensures that $C_1 \cap C_3 = \varnothing$. Employing again the separation theorem, we find $(0,0) \neq (\eta,\alpha) \in X^* \times \mathrm{R}$ such that

$$l := \inf\langle(\eta,\alpha), C_1\rangle \geq \sup\langle(\eta,\alpha), C_3\rangle =: r.$$

It is easy to check that $l = r = 0$, and thus

$$\begin{aligned}\inf\{\langle\eta,u\rangle + \alpha t : (u,t) \in X \times \mathrm{R}, \varphi(u) \leq \varphi(0) + t\} \\ = \sup\{\langle\eta,u\rangle + \alpha t : (u,t) \in X \times \mathrm{R}, t < -c\|u\|\} = 0.\end{aligned} \tag{2.5}$$

It follows from (2.5) that $\alpha \geq 0$. In fact $\alpha > 0$, since for $\alpha = 0$ condition (2.5) yields $\langle\eta,u\rangle \leq 0$ whenever $u \in X$, which contradicts $(\eta,\alpha) \neq (0,0)$. Thus (2.5) implies $-\frac{1}{\alpha}\eta \in \partial\varphi(0)$ similarly to the case of (2.3). Now put

$$\zeta := -\frac{1}{\alpha}\eta - K\xi \quad \text{with} \quad K > \max\left\{0, -\frac{M\|h\| + \langle\frac{1}{\alpha}\eta, h\rangle}{\langle\xi,h\rangle}\right\}. \tag{2.6}$$

Then using the definition of $\partial\varphi(0)$ and the first condition in (2.4), we get

$$\varphi(u) - \varphi(0) \geq \left\langle -\frac{1}{\alpha}\eta, u\right\rangle \geq \langle\zeta,u\rangle \quad \text{if } u \in \operatorname{dom}\varphi$$

and hence $\zeta \in \partial\varphi(0) \subset \tilde{\partial}_\epsilon f(x)$. Moreover, using (2.6) and $\langle\xi,h\rangle < 0$, we conclude that

$$\langle\zeta,h\rangle = \left\langle -\frac{1}{\alpha}\eta, h\right\rangle - K\langle\xi,h\rangle > M\|h\|. \tag{2.7}$$

Thus (2.7) yields $\|\zeta\| > M$, and we finally get $\zeta \in \tilde{\partial}_\epsilon f(x) \setminus M\mathrm{B}_{X^*}$. This justifies (2.1) for the case of $s = 0$ in (2.3) and completes the proof of the lemma. ■

Corollary 2.2 *Let Ω be a nonempty set in a Banach space X, $x \in \Omega$, $\epsilon > 0$, and $M > \epsilon$. Then*

$$\tilde{N}_\epsilon(x;\Omega) \setminus M\mathrm{B}_{X^*} \neq \varnothing \tag{2.8}$$

holds if and only if there exist $\delta > 0$, $\gamma > 0$, and a nonempty open set $U \subset X$ such that

$$\sum_{i=1}^m \alpha_i\epsilon\|h_i\| \geq (M+\gamma)\Big\|\sum_{i=1}^m \alpha_i h_i\Big\| \tag{2.9}$$

whenever $m \in \mathrm{N}$, $h_i \in X$, $\|h_i\| < \delta$, $x + h_i \in \Omega$, $\alpha_i \geq 0$, $i = 1,\dots,m$, $\sum_{i=1}^m \alpha_i = 1$, and $\sum_{i=1}^m \alpha_i h_i \in (0,1]U$.

Proof To establish this fact, we apply Lemma 2.1 where f is the indicator function of the set Ω. One can easily see that (2.1) reduces to (2.8) due to (1.4). It also follows from the definition of φ in (2.2) that condition (i) in the lemma automatically holds with $c = 0$, while condition (ii) coincides with (2.9) in this case. ∎

Now, based on Lemma 2.1 and Corollary 2.2, we are ready to perform the desired separable reduction for the constructions of $\tilde{\partial}_\epsilon$ and $\tilde{N}_\epsilon$ with $\epsilon > 0$. The next theorem contains the main result of this section.

Theorem 2.3 *Let Y_0 be a separable subspace of an arbitrary Banach space X, let $f\colon X \to (-\infty, \infty]$ be a proper function locally bounded from below, and let $\epsilon > 0$. Then there exists a separable subspace $Y \subset X$ such that $Y_0 \subset Y$ and one has*

$$\tilde{\partial}_\epsilon f(x) \setminus M\mathrm{B}_{X^*} \neq \varnothing \quad \textit{whenever} \quad x \in Y \quad \textit{and} \quad \tilde{\partial}_\epsilon (f_{|Y})(x) \setminus M\mathrm{B}_{Y^*} \neq \varnothing \tag{2.10}$$

for any $M > \epsilon$.

Proof Let $\mathcal{A}$ be the *countable* set of all sequences $(\alpha_i)_{i=1}^\infty$ with rational nonnegative entries satisfying $\alpha_i = 0$ for all large $i \in \mathrm{N}$ and $\sum_{i=1}^\infty \alpha_i = 1$. Given $x \in X$, we take $\rho(x) > 0$ such that f is bounded from below on the ball around x with radius $\rho(x)$. ∎

For $x \in X$, for $a = (\alpha_i) \in \mathcal{A}$, for rational numbers $r > 0$ and $\delta \in \big(0, \rho(x)\big)$, and for $k \in \mathrm{N}$ we find $u_i(x, a, r, \delta, k) \in X$, $i \in \mathrm{N}$, such that $\|u_i(x, a, r, \delta, k)\| < \delta$ for all $i \in \mathrm{N}$, that $\| \sum_{i=1}^\infty \alpha_i u_i(x, a, r, \delta, k)\| < r$, and that

$$\sum_{i=1}^\infty \alpha_i \big[f\big(x + u_i(x, a, r, \delta, k)\big) + \epsilon \|u_i(x, a, r, \delta, k)\|\big] - \frac{1}{k}$$
$$\leq \inf\Big\{ \sum_{i=1}^\infty \alpha_i \big[f(x + h_i) + \epsilon \|h_i\|\big] : h_i \in X, \|h_i\| < \delta, \|\sum_{i=1}^\infty \alpha_i h_i\| < r \Big\}.$$

Further, for $x \in X$, for $a = (\alpha_i) \in \mathcal{A}$, for rational numbers $r > 0$ and $\delta \in \big(0, \rho(x)\big)$, for $k \in \mathrm{N}$, and for $h \in X$, with $\|h\| < \delta$, we find $g_i(x, h, a, r, \delta, k) \in X$, $i \in \mathrm{N}$, such that $\|g_i(x, h, a, r, \delta, k)\| < \delta$ for all $i \in \mathrm{N}$, that $\| \sum_{i=1}^\infty \alpha_i g_i(x, h, a, r, \delta, k) - h\| < r$, and that

$$\sum_{i=1}^\infty \alpha_i \big[f\big(x + g_i(x, h, a, r, \delta, k)\big) + \epsilon \|g_i(x, h, a, r, \delta, k)\|\big] - \frac{1}{k}$$
$$\leq \inf\Big\{ \sum_{i=1}^\infty \alpha_i \big[f(x + h_i) + \epsilon \|h_i\|\big] : h_i \in X, \|h_i\| < \delta, \|\sum_{i=1}^\infty \alpha_i h_i - h\| < r \Big\}.$$

Let us construct a separable subspace $Y \subset X$ for which (2.10) holds. To proceed, we start with the given subspace $Y_0 \subset X$ and build by induction a sequence of separable subspaces

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$Y_1, Y_2, \ldots$ of X as follows. If a subspace $Y_n \subset X$ is already constructed for some $n \in \mathrm{N} \cup \{0\}$, we take any countable and dense subset C_n of Y_n and define Y_{n+1} by

$$\begin{aligned} Y_{n+1} := \overline{\mathrm{sp}}\Big(C_n \cup \{u_i(x,a,r,\delta,k) : x \in C_n, a \in \mathcal{A}, r > 0, \delta \in \big(0,\rho(x)\big) \text{ rational}, k, i \in \mathrm{N}\} \\ \cup \{g_i(x,h,a,r,\delta,k) : x, h \in C_n, a \in \mathcal{A}, \|h\| < \delta, r > 0, \\ \delta \in \big(0,\rho(x)\big) \text{ rational}, k, i \in \mathrm{N}\}\Big). \end{aligned}$$

Finally we put $Y = \overline{\bigcup\{Y_n : n \in \mathrm{N}\}}$ and $C = \bigcup\{C_n : n \in \mathrm{N}\}$. It immediately follows from the presented construction that $\overline{C} = Y$ and that Y is a separable subspace of X containing Y_0.

Now let us fix an arbitrary number $M > \epsilon$ and prove that (2.10) holds with the separable subspace $Y \subset X$ constructed above. Our goal is to show that for any $x \in Y$ with $\tilde{\partial}_\epsilon(f_{|Y})(x) \setminus M B_{Y^*} \neq \varnothing$ one has (2.1). To furnish this, we need to verify conditions (i) and (ii) in Lemma 2.1 providing a complete characterization of (2.1) in terms of the function $\varphi_{f,x,\delta,\epsilon}$ defined in (2.2). Take $\xi \in \tilde{\partial}_\epsilon(f_{|Y})(x) \setminus M B_{Y^*}$ and, using the definition of $\tilde{\partial}_\epsilon$, find a rational number $\delta > 0$ so that

$$f(x+y) + \epsilon\|y\| \geq f(x) + \langle \xi, y\rangle \quad \text{for all } y \in Y \quad \text{with} \quad \|y\| < 2\delta. \tag{2.11}$$

Since $x \in Y$, for every $n \in \mathrm{N}$ we can find $x_n \in C_n$ and a rational number γ_n satisfying $\|x - x_n\| \to 0$ as $n \to \infty$ and

$$\|x - x_n\| \leq \gamma_n \leq 2\|x - x_n\|, \quad n \in \mathrm{N}. \tag{2.12}$$

Further, fix any $h \in X$ with $\|h\| < \delta$ and consider any $a = (\alpha_i) \in \mathcal{A}$ and any $h_i \in X$, $\|h_i\| < \delta$, $i \in \mathrm{N}$, such that $\sum_{i=1}^\infty \alpha_i h_i = h$. Since $\alpha_i = 0$ for all large $i \in \mathrm{N}$, we may take $h_i = 0$ for these i. Then taking an arbitrary rational number $r \in (\|h\|, \delta)$, one has

$$\|h\| + \gamma_n < r, \quad \|h_i\| + \gamma_n < \delta, \quad \text{and} \quad \gamma_n + \delta < 2\delta \tag{2.13}$$

for all $i \in \mathrm{N}$ and for all sufficiently large $n \in \mathrm{N}$. Thus putting $h_i^n := h_i + x - x_n$ for these n, i and taking into account (2.11)–(2.13) and the construction of Y, we get

$$\begin{aligned} l(h) &:= \sum_{i=1}^\infty \alpha_i\big[f(x+h_i) + \epsilon\|h_i\|\big] = \sum_{i=1}^\infty \alpha_i\big[f(x_n+h_i^n) + \epsilon\|h_i\|\big] \\ &\geq \sum_{i=1}^\infty \alpha_i\big[f(x_n+h_i^n) + \epsilon\|h_i^n\|\big] - \epsilon\gamma_n \\ &\geq -\frac{1}{n} - \epsilon\gamma_n + \sum_{i=1}^\infty \alpha_i\Big[f\big(x_n + u_i(x_n,a,r,\delta,n)\big) + \epsilon\|u_i(x_n,a,r,\delta,n)\|\Big] \\ &\qquad \Big(\text{as } \|h_i^n\| \leq \|h_i\| + \gamma_n < \delta \text{ and } \Big\|\sum_{i=1}^\infty \alpha_i h_i^n\Big\| \leq \|h\| + \gamma_n < r\Big) \end{aligned}$$

$$
\begin{aligned}
&\geq -\frac{1}{n} - 2\epsilon\gamma_n + \sum_{i=1}^{\infty} \alpha_i \big[f\big(x + x_n - x + u_i(x_n, a, r, \delta, n)\big) \\
&\qquad\qquad\qquad + \epsilon \|x_n - x + u_i(x_n, a, r, \delta, n)\| \big] \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n + \Big\langle \xi, x_n - x + \sum_{i=1}^{\infty} \alpha_i u_i(x_n, a, r, \delta, n) \Big\rangle + f(x) \\
&\quad (\text{as } x_n - x + u_i(x_n, a, r, \delta, n) \in Y \text{ and } \|x_n - x + u_i(x_n, a, r, \delta, n)\| < \gamma_n + \delta < 2\delta) \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n - \|\xi\|\gamma_n - \|\xi\| \Big\| \sum_{i=1}^{\infty} \alpha_i u_i(x_n, a, r, \delta, n) \Big\| + f(x) \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n - \|\xi\|\gamma_n - \|\xi\| r + f(x).
\end{aligned}
$$

Now passing to the limit as $n \to \infty$, one has

$$l(h) \geq -\|\xi\| r + f(x).$$

Letting there $r \to \|h\|$ and putting $c := \|\xi\|$, we get $l(h) \geq -c\|h\| + f(x)$. Finally comparing the construction of $l(h)$ and the definition of $\varphi_{f,x,\delta,\epsilon}(h)$ in (2.2), we conclude that condition (i) in Lemma 2.1 is verified.

It remains to verify condition (ii) in the lemma. Take $y \in Y$, $\|y\| < \delta$, and $\gamma \in (0, 1)$ so that

$$\langle \xi, y \rangle > (M + 3\gamma)\|y\|. \tag{2.14}$$

Take a rational number ζ satisfying

$$0 < \zeta < \min\{\delta - \|y\|, \gamma\|y\|/(3\|\xi\|), \gamma\|y\|/(M + 1)\} \tag{2.15}$$

and put $U = \{h \in X : \|h - y\| < \zeta\}$. Now fix any $h \in U$ and any rational number $t \in (0, 1]$. Then $\|th\| \leq \|h\| < \delta$. Find $h_0 \in C$ so that $\|th - h_0\| < t\zeta$.

Consider any $a = (\alpha_i) \in \mathcal{A}$, $h_i \in X$, $\|h_i\| < \delta$, $i \in \mathrm{N}$, such that $\sum_{i=1}^{\infty} \alpha_i h_i = th$. As before, we may take $h_i = 0$ for all large $i \in \mathrm{N}$. Taking γ_n satisfying (2.12), we observe that (2.13) holds for all $n \in \mathrm{N}$ sufficiently large. Note also that $h_0 \in C_n$ for all large n. Thus putting $h_i^n := h_i + x - x_n$, $i, n \in \mathrm{N}$, and taking into account relationships (2.11)–(2.15) and the construction of Y (since $x \in Y$), we get the following chain of inequalities holding for all large $n \in \mathrm{N}$:

$$
\begin{aligned}
l(th) &:= \sum_{i=1}^{\infty} \alpha_i \big[f(x + h_i) + \epsilon\|h_i\| \big] = \sum_{i=1}^{\infty} \alpha_i \big[f(x_n + h_i^n) + \epsilon\|h_i\| \big] \\
&\geq \sum_{i=1}^{\infty} \alpha_i \big[f(x_n + h_i^n) + \epsilon\|h_i^n\| \big] - \epsilon\gamma_n
\end{aligned}
$$

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$$
\begin{aligned}
&\geq -\frac{1}{n} - \epsilon\gamma_n + \sum_{i=1}^{\infty} \alpha_i \big[f(x_n + g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n) \\
&\qquad\qquad + \epsilon \| g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n) \| \big] \\
&\quad \Big(\text{as } \|h_i^n\| < \delta,\ \Big\| \sum_{i=1}^{\infty} \alpha_i h_i^n - h_0 \Big\| \leq \|th - h_0\| + \|x_n - x\| < t\zeta + \gamma_n\Big) \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n + \sum_{i=1}^{\infty} \alpha_i \big[f\big(x + x_n - x + g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n)\big) \\
&\qquad\qquad + \epsilon \| x_n - x + g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n) \| \big] \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n + \Big\langle \xi, x_n - x + \sum_{i=1}^{\infty} \alpha_i g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n) \Big\rangle + f(x) \\
&\quad (\text{as } x_n - x + g_i(\cdots) \in Y \text{ and } \|x_n - x + g_i(\cdots)\| < \gamma_n + \delta < 2\delta) \\
&\geq -\frac{1}{n} - 2\epsilon\gamma_n + \langle \xi, h_0 \rangle - \|\xi\| \Big(\gamma_n + \Big\| \sum_{i=1}^{\infty} \alpha_i g_i(x_n, h_0, a, t\zeta + \gamma_n, \delta, n) - h_0 \Big\| \Big) + f(x) \\
&> -\frac{1}{n} - 2\epsilon\gamma_n + \langle \xi, ty \rangle - 2\|\xi\| t\zeta - \|\xi\| \big(\gamma_n + (t\zeta + \gamma_n) \big) + f(x) \\
&> -\frac{1}{n} - 2\epsilon\gamma_n + (M + 3\gamma) t \|y\| - 3\|\xi\| t\zeta - 2\|\xi\|\gamma_n + f(x).
\end{aligned}
$$

The latter yields, by passing to the limit as $n \to \infty$, that

$$
\begin{aligned}
l(th) - f(x) &\geq (M + 3\gamma) t \|y\| - 3\|\xi\| t\zeta \geq (M + 2\gamma) t \|y\| \\
&\geq (M + \gamma)(t\|h\| - t\zeta) + \gamma t \|y\| > (M + \gamma) t \|h\|
\end{aligned}
$$

by (2.15). Hence $l(th) \geq f(x) + (M + \gamma) t \|h\|$ for all $h \in U$ and all rational numbers $t \in (0, 1]$. If $t \in (0, 1]$ is irrational, then we change γ_n a little so that $t\zeta + \gamma_n$ is rational. This implies condition (ii) in Lemma 2.1 due to (2.2) and the above definition of $l(th)$. Thus we have (2.1) and complete the proof of the theorem. ∎

Corollary 2.4 *Let Y_0 be a separable subspace of a Banach space X, let $\Omega \subset X$ be a nonempty set, and let $\epsilon > 0$. Then there exists a separable subspace $Y \subset X$ such that $Y_0 \subset Y$ and one has*

$$
\tilde{N}_\epsilon(x; \Omega) \setminus M B_{X^*} \neq \varnothing \quad \textit{whenever} \quad x \in Y \quad \textit{and} \quad \tilde{N}_\epsilon(x; \Omega \cap Y) \setminus M B_{Y^*} \neq \varnothing
$$

for any $M > \epsilon$.

Proof This follows from Theorem 2.3 applied for the indicator function $f = \delta(\cdot\,; \Omega)$ due to (1.4). It can also be obtained directly from the separable reduction result of Corollary 2.2 using just the second half of the proof of Theorem 2.3. (Condition (i) in Lemma 2.1 is trivially fulfilled.) ∎

Remark 2.5 A careful reading of the proof of Theorem 2.3 yields that we may even assert that

$$\sup\{\|\eta\| : \eta \in \tilde{\partial}_\epsilon f_{|Y}(x)\} = \sup\{\|\xi\| : \xi \in \tilde{\partial}_\epsilon f(x)\}$$

provided that $x \in Y$ and $\tilde{\partial}_\epsilon(f_{|Y})(x) \neq \varnothing$. A similar remark applies to Corollary 2.4 as well as to Theorem 3.3 and Corollary 3.4 in the next section.

3 Separable Reduction for the Hat-Constructions

In this section we provide a separable reduction for the subdifferential constructions $\hat{\partial}_\epsilon$, $\epsilon \geq 0$, in the form

$$\hat{\partial}_\epsilon f(x) \setminus M\mathrm{B}_{X^*} \neq \varnothing \tag{3.1}$$

(with any given number $M > \epsilon$) which allows us to cover the case of $\hat{N}_\epsilon$ as well. It follows directly from the definitions that

$$\hat{\partial}_\epsilon f(x) = \bigcap_{\gamma>0} \tilde{\partial}_{\epsilon+\gamma} f(x) \tag{3.2}$$

for any $\epsilon \geq 0$, *i.e.*, the hat-constructions may be treated as a limiting case of the tilde-constructions with bigger ϵ. In particular, the crucial case of $\hat{\partial}_0 f$ in (3.2) corresponds to $\tilde{\partial}_\epsilon f$ with $\epsilon > 0$ going to 0. This allows us to use the methods and results of Section 2 to conduct the required separable reduction for the hat-constructions.

Let $\Delta := (\delta_i)_{i=1}^\infty$ be a sequence of positive numbers such that $\delta_1 > \delta_2 > \cdots > 0$ and $\delta_i \downarrow 0$ as $i \to \infty$. Given a proper function $f\colon X \to (-\infty, \infty]$, $x \in \operatorname{dom} f$, and $\epsilon \geq 0$, for $h \in X$ we define the function

$$\begin{aligned} \varphi_{f,x,\Delta,\epsilon}(h) = \inf\Big\{ &\sum_{i=1}^m \alpha_i \varphi_{f,x,\delta_i,\epsilon+1/i}(h_i) : m \in \mathrm{N}, h_i \in X, \\ &\alpha_i \geq 0, i = 1, \dots, m, \sum_{i=1}^m \alpha_i = 1, \sum_{i=1}^m \alpha_i h_i = h \Big\} \end{aligned} \tag{3.3}$$

where each $\varphi_{f,x,\delta_i,\epsilon+1/i}$, $i \in \mathrm{N}$, is constructed in (2.2). Note that $\varphi_{f,x,\Delta,\epsilon}\colon X \to [-\infty, \infty]$, that $\varphi_{f,x,\Delta,\epsilon}$ is convex if $\varphi_{f,x,\Delta,\epsilon} > -\infty$, and that $\varphi_{f,x,\Delta,\epsilon}(0) \leq f(x)$. It follows from the definitions that if $\hat{\partial}_\epsilon f(x) \neq \varnothing$, then $\varphi_{f,x,\Delta,\epsilon}(0) = f(x)$ and $\hat{\partial}_\epsilon f(x) \supset \partial\varphi_{f,x,\Delta,\epsilon}(0) \neq \varnothing$ for some Δ. Also

$$\partial\varphi_{f,x,\Delta,\epsilon}(0) \subset \hat{\partial}_\epsilon f(x) \quad \text{for any } \epsilon \geq 0 \tag{3.4}$$

if $\partial\varphi_{f,x,\Delta,\epsilon}(0) \neq \varnothing$ and $\varphi_{f,x,\Delta,\epsilon}(0) = f(x)$ for some Δ.

The next statement provides an equivalent translation of the basic assertion (3.1) into the language of the original space X.

Lemma 3.1 *Let $f\colon X \to (-\infty, \infty]$ be a function on a Banach space X, $x \in \operatorname{dom} f$, $\epsilon \geq 0$, and $M > \epsilon$. Then one has (3.1) if and only if there are numbers $\gamma > 0$ and $c \geq 0$, a sequence $\Delta = (\delta_i)_{i=1}^\infty \subset (0, \infty)$ with $\delta_i \downarrow 0$, and a nonempty open set $U \subset X$ such that*

(i) $\varphi_{f,x,\Delta,\epsilon}(h) \geq f(x) - c\|h\|$ *whenever* $h \in X$*, and*
(ii) $\varphi_{f,x,\Delta,\epsilon}(th) \geq f(x) + (M+\gamma)t\|h\|$ *whenever* $h \in U$ *and* $t \in (0,1]$.

Proof First let us show the necessity, that is, the existence of $\xi \in \hat{\partial}_\epsilon f(x) \setminus M B_{X^*}$ implies both conditions (i) and (ii) in the lemma. Using the definition of $\hat{\partial}_\epsilon f(x)$, for any $i \in N$ we find $\delta_i > 0$ such that $\delta_i < \min\{\frac{1}{i}, \delta_{i-1}\}$ (if $i > 1$) and

$$f(x+h) - f(x) \geq \langle \xi, h\rangle - \left(\epsilon + \frac{1}{i}\right)\|h\| \quad \text{whenever} \quad h \in X \quad \text{and} \quad \|h\| < \delta_i.$$

This yields by (2.2) that $\varphi_{f,x,\delta_i,\epsilon+1/i}(h) \geq \langle \xi, h\rangle$ for all $i \in N$. Now putting $\Delta := (\delta_i)_{i=1}^\infty$, we get $\varphi_{f,x,\Delta,\epsilon}(h) \geq \langle \xi, h\rangle$ for all $h \in X$. Hence (i) holds with $c := \|\xi\|$. Since $\|\xi\| > M$, we find $\gamma > 0$ and a nonempty open set $U \subset X$ so that $\langle \xi, h\rangle > (M+\gamma)\|h\|$ for every $h \in U$. This immediately implies (ii) with the sequence Δ chosen above, and thus we get the necessity part of the lemma.

The sufficiency part is almost identical with the proof of the sufficiency in Lemma 2.1. Indeed, in this way we find $\xi \in \partial\varphi_{f,x,\Delta,\epsilon}(0) \setminus M B_{X^*}$, and hence (3.4) finishes the proof. ■

Corollary 3.2 *Let* Ω *be a nonempty set in a Banach space* X*,* $x \in \Omega$*,* $\epsilon \geq 0$*, and* $M > \epsilon$*. Then*

$$\hat{N}_\epsilon(x;\Omega) \setminus M B_{X^*} \neq \varnothing$$

holds if and only if there exist a number $\gamma > 0$*, a sequence* $\Delta = (\delta_i)_{i=1}^\infty \subset (0,\infty)$ *with* $\delta_i \downarrow 0$*, and a nonempty open set* $U \subset X$ *such that*

$$\sum_{i=1}^m \alpha_i \sum_{l=1}^m \beta_{il}\left(\epsilon + \frac{1}{i}\right)\|h_{il}\| \geq (M+\gamma)\Big\|\sum_{i=1}^m \alpha_i \sum_{l=1}^m \beta_{il} h_{il}\Big\|$$

whenever $m \in N$*,* $h_{il} \in X$*,* $\|h_{il}\| < \delta_i$*,* $x + h_{il} \in \Omega$*,* $\alpha_i \geq 0$*,* $\beta_{il} \geq 0$*,* $l = 1,\dots,m$*,* $\sum_{l=1}^m \beta_{il} = 1$*,* $i = 1,\dots,m$*,* $\sum_{i=1}^m \alpha_i = 1$*, and* $\sum_{i=1}^m \alpha_i \sum_{l=1}^m \beta_{il} h_{il} \in (0,1]U$*.*

Proof This follows from Lemma 3.1 for the indicator function $f(x) := \delta(x;\Omega)$ due to (1.4), (3.3), and (2.2). Note that condition (i) of the lemma is automatically fulfilled in this case with $c = 0$. ■

Now we establish the main result of this section providing the desired separable reduction of the hat-contructions for any $\epsilon \geq 0$.

Theorem 3.3 *Let* Y_0 *be a separable subspace of an arbitrary Banach space* X*, let* $f\colon X \to (-\infty,\infty]$ *be a function locally bounded from below, and let* $\epsilon \geq 0$*. Then there exists a separable subspace* $Y \subset X$ *such that* $Y_0 \subset Y$ *and one has*

$$\hat{\partial}_\epsilon f(x) \setminus M B_{X^*} \neq \varnothing \quad \textit{whenever} \quad x \in Y \quad \textit{and} \quad \hat{\partial}_\epsilon(f_{|Y})(x) \setminus M B_{Y^*} \neq \varnothing \tag{3.5}$$

for any $M > \epsilon$*.*

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Proof To prove the theorem, we develop the same procedure as in the proof of Theorem 2.3, using Lemma 3.1 instead of Lemma 2.1. We need to construct a separable subspace $Y \subset X$ ensuring the fulfillment of (3.5) for any $M > \epsilon$. The construction of such a subspace presented below is more complicated in comparison with Theorem 2.1 since we need to take into account an additional sequential process generated by $(\delta_i)_{i=1}^\infty$ in (3.3).

Let $\mathcal{A}$ be the (countable) set of all sequences $(\alpha_i)_{i=1}^\infty$ with rational nonnegative entries satisfying $\alpha_i = 0$ for all large $i \in \mathbb{N}$ and $\sum_{i=1}^\infty \alpha_i = 1$. Let $\mathcal{B}$ be the (countable) set of all infinite matrices $(\beta_{il})_{i,l=1}^\infty$ with rational nonnegative entries satisfying $\beta_{il} > 0$ for only finitely many couples $(i,l) \in \mathbb{N}^2$ and $\sum_{l=1}^\infty \beta_{il} = 1$ for all $i \in \mathbb{N}$. Let $\mathcal{D}$ be the (countable) set of all sequences $(\delta_i)_{i=1}^\infty$ with rational entries satisfying $0 < \delta_1 \geq \delta_2 \geq \cdots \geq 0$ and $\delta_i = 0$ for all large $i \in \mathbb{N}$. Given $x \in X$, let $\rho(x) > 0$ be such that f is bounded from below on the ball around x with radius $\rho(x)$.

For $x \in X$, for $a = (\alpha_i) \in \mathcal{A}$, for $b = (\beta_{il}) \in \mathcal{B}$, for $\Delta = (\delta_i) \in \mathcal{D}$ satisfying $\delta_i > 0$ whenever $\alpha_i > 0$ and $\delta_1 < \rho(x)$, for a rational number $r > 0$, and for $k \in \mathbb{N}$ we find $u_{il}(x,a,b,r,\Delta,k) \in X$, $i,l \in \mathbb{N}$, such that $\|u_{il}(x,a,b,r,\Delta,k)\| < \delta_i$ if $\delta_i > 0$ and $u_{il}(x,a,b,r,\Delta,k) = 0$ otherwise for all $l \in \mathbb{N}$, that $\|\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} u_{il}(x,a,b,r,\Delta,k)\| < r$, and that

$$\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} \left[f\big(x + u_{il}(x,a,b,r,\Delta,k)\big) + \left(\epsilon + \frac{1}{i}\right) \|u_{il}(x,a,b,r,\Delta,k)\| \right] - \frac{1}{k} \leq \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} \left[f(x+h_{il}) + \left(\epsilon + \frac{1}{i}\right) \|h_{il}\| \right]$$

whenever $h_{il} \in X$, $\|h_{il}\| < \delta_i$ if $\delta_i > 0$ and $h_{il} = 0$ otherwise, and $\|\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} h_{il}\| < r$.

Further, for all a,b,r,Δ,k,x as above, and for all $h \in X$ with $\|h\| < \delta_1$ we find $g_{il}(x,h,a,b,r,\Delta,k) \in X$, $i,l \in \mathbb{N}$, such that $\|g_{il}(x,h,a,b,r,\Delta,k)\| < \delta_i$ if $\delta_i > 0$ and $g_{il}(x,h,a,b,r,\Delta,k) = 0$ otherwise for all $l \in \mathbb{N}$, that $\|\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} g_{il}(x,h,a,b,r,\Delta,k) - h\| < r$, and that

$$\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} \left[f\big(x + g_{il}(x,h,a,b,r,\Delta,k)\big) + \left(\epsilon + \frac{1}{i}\right) \|g_{il}(x,h,a,b,r,\Delta,)\| \right] - \frac{1}{k} \leq \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} \left[f(x+h_{il}) + \left(\epsilon + \frac{1}{i}\right) \|h_{il}\| \right]$$

whenever $\|\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} h_{il} - h\| < r$ with $h_{il} \in X$ satisfying $\|h_{il}\| < \delta_i$ if $\delta_i > 0$ and $h_{il} = 0$ otherwise.

Let us construct the desired separable subspace $Y \subset X$ by induction. If a separable subspace $Y_n \subset X$ is already constructed for some $n \in \mathbb{N} \cup \{0\}$ (Y_0 is given), let C_n be a countable and dense subset of Y_n. Then put

$$Y_{n+1} := \overline{\mathrm{sp}}\Big(C_n \cup \{u_{il}(x,a,b,r,\Delta,k) : x \in C_n, a \in \mathcal{A}, b \in \mathcal{B}, r > 0 \text{ rational}, \Delta \in \mathcal{D}, k,i,l \in \mathbb{N}\}$$

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$$\cup \{g_{il}(x,h,a,b,r,\Delta,k) : x,h \in C_n, \|h\| < \delta_1, a \in \mathcal{A}, b \in \mathcal{B},$$
$$r > 0 \text{ rational}, k,i,l \in \mathrm{N}\}\Big).$$

We obviously have $Y_0 \subset Y_1 \subset \cdots \subset X$. Finally let us define $Y = \overline{\bigcup\{Y_n : n \in \mathrm{N}\}}$ and $C = \bigcup\{C_n : n \in \mathrm{N}\}$. Then $\overline{C} = Y$ and Y is a separable subspace of X containing Y_0.

Let us prove that the constructed subspace Y ensures the fulfillment of (3.5) for any given numbers $\epsilon \geq 0$ and $M > \epsilon$. Taking $x \in Y$ with $\hat{\partial}_\epsilon(f_{|Y})(x) \setminus M\mathrm{B}_{Y^*}$, we need to show that (3.1) holds for this x. According to Lemma 3.1, the latter reduces to the verification of conditions (i) and (ii) therein.

Taking $\xi \in \hat{\partial}_\epsilon(f_{|Y})(x) \setminus M\mathrm{B}_{Y^*}$ and using the definition of $\hat{\partial}_\epsilon$, we find a sequence $\Delta := (\delta)_{i=1}^\infty$ of rational numbers δ_i such that $\delta_1 > \delta_2 > \cdots > 0$ and

$$f(x+y) + \left(\epsilon + \frac{1}{i}\right)\|y\| \geq f(x) + \langle \xi, y\rangle \quad \text{for all } y \in Y \quad \text{with} \quad \|y\| < 2\delta_i \tag{3.6}$$

and all $i = 1,2,\ldots$. We are going to prove that both conditions (i) and (ii) in Lemma 3.1 are satisfied along this Δ.

First let show that condition (i) in Lemma 3.1 holds with $c = \|\xi\|$ along the chosen sequence Δ. Similarly to the proof of Theorem 2.3, for any $n \in \mathrm{N}$ we find $x_n \in C_n$ and a rational number γ_n such that $\|x - x_n\| \to 0$ as $n \to \infty$ and

$$\|x - x_n\| \leq \gamma_n \leq 2\|x - x_n\|, \quad n \in \mathrm{N}.$$

Fix any $h \in X$ and assume that $\|h\| < \delta_1$; otherwise $\varphi_{f,x,\Delta,\epsilon}(h) = \infty$ by (3.3) and (2.2), and we are done. Further, take an arbitrary rational number $r \in (\|h\|, \delta_1)$ and consider any $a = (\alpha_i) \in \mathcal{A}$, any $b = (\beta_{il}) \in \mathcal{B}$, and any $h_{il} \in X$ with $\|h_{il}\| < \delta_i$, $i,l \in \mathrm{N}$, such that $\sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il} h_{il} = h$. Find $i_0 \in \mathrm{N}$ so large that $\alpha_i = 0$ for $i \geq i_0$ and $\beta_{il} = 0$ if either $i \geq i_0$ or $l \geq i_0$. Then we may take $h_{il} = 0$ for all such i and l. So we have

$$\|h\| + \gamma_n < r, \quad \|h_{il}\| + \gamma_n < \delta_i, \quad \text{and} \quad \delta_i + \gamma_n < 2\delta_i, \quad i \leq i_0, l \in \mathrm{N}, \tag{3.7}$$

for all sufficiently large $n \in \mathrm{N}$. Let us put $h_{il}^n := h_{il} + x - x_n$ for these n,i,l and consider the sequence $\Delta_0 := (\delta_1, \delta_2, \ldots, \delta_{i_0}, 0, 0, \ldots)$. Thus $\Delta_0 \in \mathcal{D}$. Taking into account the construction of $Y \ni x$ (the part involving u_{il}) and the above estimates, we get

$$\begin{aligned}
l(h) &:= \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il}\left[f(x+h_{il}) + \left(\epsilon + \frac{1}{i}\right)\|h_{il}\|\right] \\
&= \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il}\left[f(x_n+h_{il}^n) + \left(\epsilon + \frac{1}{i}\right)\|h_{il}\|\right] \\
&\geq \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il}\left[f(x_n+h_{il}^n) + \left(\epsilon + \frac{1}{i}\right)\|h_{il}^n\|\right] - \sum_{i=1}^\infty \alpha_i \left(\epsilon + \frac{1}{i}\right)\gamma_n \\
&\geq -\frac{1}{n} - \gamma_n + \sum_{i=1}^\infty \alpha_i \sum_{l=1}^\infty \beta_{il}\Big[f\big(x_n + u_{il}(x_n,a,b,r,\Delta_0,n)\big)
\end{aligned}$$

$$+\left(\epsilon+\frac{1}{i}\right)\|u_{il}(x_n,a,b,r,\Delta_0,n)\|\Bigg]$$

$$\text{(as } \|h^n_{il}\| \le \|h_{il}\| + \gamma_n < \delta_i \text{ for } i = 1,\dots,i_0$$

$$\text{and } \Big\|\sum_{i=1}^{\infty}\alpha_i\sum_{l=1}^{\infty}\beta_{il}h^n_{il}\Big\| \le \|h\| + \gamma_n < r)$$

$$\ge -\frac{1}{n} - 2\gamma_n + \sum_{i=1}^{\infty}\alpha_i\sum_{l=1}^{\infty}\beta_{il}\Bigg[f\big(x + u_{il}(x_n,a,b,r,\Delta_0,n) + x_n - x\big) + \left(\epsilon+\frac{1}{i}\right)\|u_{il}(x_n,a,b,r,\Delta_0,n) + x_n - x\|\Bigg]$$

$$\ge -\frac{1}{n} - 2\gamma_n + \Big\langle \xi, \sum_{i=1}^{\infty}\alpha_i\sum_{l=1}^{\infty}\beta_{il}u_{il}(x_n,a,b,r,\Delta_0,n) + x_n - x\Big\rangle + f(x)$$

$$\text{(as } u_{il}(x_n,a,b,r,\Delta_0,n) + x - x_n \in Y$$

$$\text{and } \|u_{il}(x_n,a,b,r,\Delta_0,n) + x_n - x\| < \delta_i + \gamma_n < 2\delta_i)$$

$$\ge -\frac{1}{n} - 2\gamma_n - \|\xi\|\gamma_n - \|\xi\|\,\Big\|\sum_{i=1}^{\infty}\alpha_i\sum_{l=1}^{\infty}\beta_{il}u_{il}(x_n,a,b,r,\Delta_0,n)\Big\| + f(x)$$

$$\ge -\frac{1}{n} - 2\gamma_n - \|\xi\|\gamma_n - \|\xi\|r + f(x).$$

Now letting first $n \to \infty$ and then $r \to \|h\|$, we arrive at

$$l(h) \ge -c\|h\| + f(x)$$

with $c := \|\xi\|$. Comparing the definition of $\varphi_{f,x,\Delta,\epsilon}$ in (3.3) with the above construction of l, we get condition (i) in Lemma 3.1 along the sequence Δ selected in (3.6).

To complete the proof of the theorem, it remains to verify condition (ii) in Lemma 3.1. Take $y \in Y$, $\|y\| < \delta$, and $\gamma \in (0,1)$ so that

$$\langle \xi, y\rangle > (M + 3\gamma)\|y\|.$$

Take a rational number ζ satisfying

$$0 < \zeta < \min\{\delta_1 - \|y\|, \gamma\|y\|/(3\|\xi\|), \gamma\|y\|/(M+1)\}$$

and put $U = \{h \in X : \|h - y\| < \zeta\}$. Now fix any $h \in U$ and any rational number $t \in (0,1]$. Then $\|th\| \le \|h\| < \delta$. Find $h_0 \in C$ so that $\|th - h_0\| < t\zeta$.

Then we repeat the arguments in the verification of condition (i) and find $i_0 \in \mathrm{N}$ such that (3.7) holds for the corresponding h_{il} when $n \in \mathrm{N}$ is sufficiently large. Also $h_0 \in C_n$ for all large n. Again putting $h^n_{il} := h_{il} + x - x_n$, $\Delta_0 := (\delta_1, \delta_2, \dots, \delta_{i_0}, 0, 0, \dots)$ and taking into account the construction of $Y \ni x$ (the part involving g_{il}) as well as the above estimates, we get

$$\begin{aligned}
l(th) &:= \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} \left[f(x+h_{il}) + \left(\epsilon + \frac{1}{i}\right) \|h_{il}\| \right] \\
&= \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} \left[f(x_n+h_{il}^n) + \left(\epsilon + \frac{1}{i}\right) \|h_{il}\| \right] \\
&\geq \sum \left[f(x_n+h_{il}^n) + \left(\epsilon + \frac{1}{i}\right) \|h_{il}^n\| \right] - \sum_{i=1}^{\infty} \alpha_i \left(\epsilon + \frac{1}{i}\right) \gamma_n \\
&\geq -\frac{1}{n} - \gamma_n(\epsilon+1) \\
&\qquad + \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} \Big[f(x_n + g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n) \\
&\qquad\qquad + \left(\epsilon + \frac{1}{i}\right) \|g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n)\| \Big] \\
&\qquad \text{(as } \|h_{il}^n\| < \delta_i \text{ for } i = 1, \dots, i_0, \\
&\qquad\quad \text{and } \Big\| \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} h_{il}^n - h_0 \Big\| \leq \|th - h_0\| + \|x_n - x\| < t\zeta + \gamma_n) \\
&\geq -\frac{1}{n} - 2\gamma_n(\epsilon+1) \\
&\qquad + \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} \Big[f\big(x + x_n - x + g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n)\big) \\
&\qquad\qquad + \left(\epsilon + \frac{1}{i}\right) \|x_n - x + g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n)\| \Big] \\
&\geq -\frac{1}{n} - 2\gamma_n(\epsilon+1) \\
&\qquad + \Big\langle \xi, x_n - x + \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n) \Big\rangle + f(x) \\
&\qquad \text{(as } x_n - x + g_{il}(\cdots) \in Y \text{ and } \|x_n - x + g_{il}(\cdots)\| < \gamma_n + \delta_i < 2\delta_i) \\
&\geq -\frac{1}{n} - 2\gamma_n(\epsilon+1) + \langle \xi, h_0 \rangle \\
&\qquad - \|\xi\| \Big(\gamma_n + \Big\| \sum_{i=1}^{\infty} \alpha_i \sum_{l=1}^{\infty} \beta_{il} g_{il}(x_n, h_0, a, b, t\zeta + \gamma_n, \Delta_0, n) - h_0 \Big\| \Big) + f(x) \\
&> -\frac{1}{n} - 2\gamma_n(\epsilon+1) + \langle \xi, ty \rangle - 2\|\xi\| t\zeta - \|\xi\| \big(\gamma_n + (t\zeta + \gamma_n) \big) + f(x)
\end{aligned}$$

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$$> -\frac{1}{n} - 2\gamma_n(\epsilon + 1) + (M + 3\gamma)t\|y\| - 3\|\xi\|t\zeta - 2\|\xi\|\gamma_n + f(x).$$

Passing there to the limit as $n \to \infty$, we get

$$l(th) \geq (M + 3\gamma)t\|y\| - 3\|\xi\|t\zeta + f(x).$$

Then arguing identically to the proof in the end of Theorem 2.3, we finally arrive at the estimate

$$l(th) \geq f(x) + (M + \gamma)t\|h\|.$$

This implies condition (ii) in Lemma 3.1 due to the above construction of $l(th)$ and the definition of $\varphi_{f,x,\Delta,\epsilon}$ in (3.3) along the sequence Δ selected in (3.6). This completes the proof of the theorem. ■

Corollary 3.4 *Let Y_0 be a separable subspace of a Banach space X, let $\Omega \subset X$ be a nonempty set, and let $\epsilon \geq 0$. Then there exists a separable subspace $Y \subset X$ such that $Y_0 \subset Y$ and one has*

$$\hat{N}_\epsilon(x;\Omega) \setminus MB_{X^*} \neq \emptyset \quad \textit{whenever } x \in Y \textit{ and } \hat{N}_\epsilon(x;\Omega \cap Y) \setminus MB_{Y^*} \neq \emptyset \tag{3.8}$$

for any $M > \epsilon$.

Proof This follows from Theorem 3.3 applied for the indicator function $f = \delta(\cdot;\Omega)$ due to (1.4). It can also be obtained directly from the separable reduction result of Corollary 3.2 using just the second half of the proof of Theorem 3.3. Note that condition (i) in Lemma 3.1 is trivially fulfilled in this case. ■

Remark 3.5 For $\epsilon = 0$ we know that $\hat{N}_0(x;\Omega)$ is a cone. Thus condition (3.8) can be read in this case as

$$\hat{N}_0(x;\Omega) \neq \{0\} \quad \text{whenever } x \in Y \text{ and } \hat{N}_0(x;\Omega \cap Y) \neq \{0\}. \tag{3.9}$$

Remark 3.6 The assumption about the local lower boundedness of the function f in Theorems 2.3 and 3.3 can be dropped by a further elaboration of the separable reduction method. Indeed, a rather simpler argument gives the following:

Given a separable subspace Y_0 of a Banach space X and a function $f: X \to (-\infty,\infty]$, there exists a separable subspace $Y \subset X$ such that $Y_0 \subset Y$ and f is bounded from below on a neighborhood of x whenever $x \in Y$ and $f_{|Y}$ is bounded from below on a neighborhood (in Y) of x.

Note that $f_{|Y}$ is locally bounded from below (in Y) if either $\tilde{\partial}_\epsilon(f_{|Y})(x) \neq \emptyset$ or $\hat{\partial}_\epsilon(f_{|Y})(x) \neq \emptyset$ for $\epsilon \geq 0$. Taking this into account and incorporating the proof of the latter statement into the proof of Theorem 2.3 and 3.3 respectively, we may drop the local lower boundedness assumption in these theorems. Of course, we do not need such an improvement for Corollaries 2.4 and 3.4.

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4 Characterizations of Asplund Spaces

In the concluding section of the paper we consider a remarkable subclass of Banach spaces called *Asplund spaces*. This class is well investigated in the geometric theory of Banach spaces. It is sufficiently broad and convenient for many applications; in particular, it contains reflexive Banach spaces. We refer the reader to the books [D], [P] for various properties and characterizations of Asplund spaces, and to [MS2] for recent applications in nonsmooth analysis and optimization. Note that this class includes every Banach space with Fréchet differentiable bump functions, being in general very much related to Fréchet type differentiability and subdifferentiability. On the other hand, there are Asplund spaces which fail to have even a Gâteaux differentiable renorm.

It is well known that the Asplund property is inherited by subspaces [P, Proposition 2.33]. Moreover, every separable Asplund space admits an equivalent norm Fréchet differentiable away from the origin; see [D, p. 118]. This ensures a peculiar efficiency of the separable reduction method in the framework of Asplund spaces.

In what follows we provide some applications of the separable reduction results obtained above to the case of Asplund spaces. In this way, we establish conditions in terms of $\tilde{N}_\epsilon$, $\epsilon > 0$, and $\hat{N}_\epsilon$, $\epsilon \geq 0$, which occur to be *characterizations* of Asplund spaces. First let us present new characterizations of the Asplund property formulated in terms of the ϵ-constructions $\tilde{N}_\epsilon$ and $\hat{N}_\epsilon$ with $\epsilon > 0$.

Theorem 4.1 *Let X be a Banach space. The following assertions are equivalent:*

(a) X is an Asplund space.

(b) For every proper closed subset Ω of X, every $\epsilon > 0$, and every $M > \epsilon$ the set

$$\{x \in \operatorname{bd}\Omega : \tilde{N}_\epsilon(x;\Omega) \setminus MB_{X^*} \neq \emptyset\}$$

is dense in the boundary of Ω.

(c) For every proper closed subset Ω of X, every $\epsilon > 0$, and every $M > \epsilon$ the set

$$\{x \in \operatorname{bd}\Omega : \hat{N}_\epsilon(x;\Omega) \setminus MB_{X^*} \neq \emptyset\}$$

is dense in the boundary of Ω.

(d) For every proper closed subset Ω of X, every $\epsilon > 0$, and every $M > \epsilon$ there exists $x \in \operatorname{bd}\Omega$ such that $\tilde{N}_\epsilon(x;\Omega) \setminus MB_{X^*} \neq \emptyset$.

(e) For every proper closed subset Ω of X, every $\epsilon > 0$, and every $M > \epsilon$ there exists $x \in \operatorname{bd}\Omega$ such that $\hat{N}_\epsilon(x;\Omega) \setminus MB_{X^*} \neq \emptyset$.

Proof Let us observe that

$$\tilde{N}_\epsilon(x;\Omega) \subset \hat{N}_\epsilon(x;\Omega) \subset \tilde{N}_{\epsilon+\gamma}(x;\Omega) \quad \text{for all } \epsilon > 0 \text{ and } \gamma > 0$$

whenever $x \in \Omega$. This implies that (b)⇔(c) and (d)⇔(e) in the theorem. Implication (c)⇒(e) is trivial. We need to prove that (a)⇒(c) and (e)⇒(a).

First we establish (a)⇒(c) using the separable reduction result from Corollary 3.4 with $\epsilon > 0$ and Ekeland's variational principle. So let Ω be a proper closed subset of an Asplund

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space X, let $\epsilon > 0$, and let $M > \epsilon$. Fix $\bar{x} \in \operatorname{bd}\Omega$ and $\gamma > 0$. We want to find $x \in \operatorname{bd}\Omega$ such that $\|x - \bar{x}\| < \gamma$ and

$$\hat{N}_\epsilon(x;\Omega) \setminus M B_{X^*} \neq \varnothing. \tag{4.1}$$

We can always assume that $\epsilon < \gamma$. Since $\bar{x}$ is a boundary point of Ω, there is $a \in X$ such that $\|a\| < \epsilon^2/(M+1)$ and $\bar{x} + a \notin \Omega$. Put $Y_0 := \operatorname{sp}\{\bar{x}, a\}$ and pick the separable subspace $Y \subset X$ from Corollary 3.4 found for the given Ω, Y_0, and ϵ. Note that $\Omega \cap Y$ is a proper closed subset of Y due to the choice of $Y_0 \subset Y$. Then we take an equivalent Fréchet smooth norm $|\cdot|$ on Y with $|\cdot| \leq \|\cdot\|$ and consider a lower semicontinuous function $f\colon Y \to (-\infty, \infty]$ defined by

$$f(y) := (M+1)|y - \bar{x} - a| + \delta(y;\Omega \cap Y), \quad y \in Y. \tag{4.2}$$

Note that $\inf f \geq 0$ and $f(\bar{x}) < \epsilon^2 \geq \inf f + \epsilon^2$. By Ekeland's variational principle (see [P, p. 45]), there is $y_0 \in Y$ such that $|y_0 - \bar{x}| < \epsilon < \gamma$ and

$$f(y_0) \leq f(y) + \epsilon|y - y_0| \quad \forall y \in Y.$$

Due to (4.2), this gives that y_0 lies in Ω and

$$(M+1)|y - \bar{x} - a| - (M+1)|y_0 - \bar{x} - a| \geq -\epsilon|y - y_0| \quad \forall y \in \Omega \cap Y. \tag{4.3}$$

Note that $y_0 - \bar{x} - a \neq 0$ since $\bar{x} + a \notin \Omega$. Denote by $\eta \in Y^*$ the Fréchet derivative of $|\cdot|$ at $y_0 - \bar{x} - a$. Then $|\eta| = 1$, and we get from (4.3) and the definition of the Fréchet derivative that

$$(M+1)\langle \eta, y - y_0\rangle + o(|y - y_0|) \geq -\epsilon|y - y_0| \quad \forall y \in \Omega \cap Y. \tag{4.4}$$

Now invoking the definition of $\hat{N}_\epsilon$, we conclude from (4.4) that $\xi \in \hat{N}_\epsilon(y_0;\Omega \cap Y)$ for $\xi := -(M+1)\eta$. Since $|\xi| = M+1$, we have $\hat{N}_\epsilon(y_0;\Omega\cap Y) \setminus M B_{Y^*} \neq \varnothing$. This implies (4.1) due to Corollary 3.4 and thus completes the proof of (a)$\Rightarrow$(c).

It remains to establish (e)$\Rightarrow$(a). We need to show the following: if X is not Asplund, then there exist a closed set $\Omega \subset X$ as well as numbers $\epsilon > 0$ and $M > \epsilon$ such that

$$\|x^*\| \leq M \quad \text{for all } x^* \in \hat{N}_\epsilon(x;\Omega) \quad \text{and all } x \in \operatorname{bd}\Omega. \tag{4.5}$$

Using arguments from the proof of [FM, Theorem 3.7], it is possible to get more: we construct a set $\Omega \subset X$ so that (4.5) holds with $M = K\epsilon$ for *every* $\epsilon > 0$ and some constant $K > 1$ independent of ϵ.

To furnish this, we take an arbitrary non-Asplund space X and represent it in the form $X = Z \times \mathrm{R}$ with the norm $\|(z,\mu)\| := \|z\| + |\mu|$ for $(z,\mu) \in X$. Then Z is non-Asplund as well, since the opposite implies the Asplund property of X. In this case there exist a number $c > 0$ and a norm $|\cdot|$ on Z, which is equivalent to the original norm $\|\cdot\|$, so that $|\cdot| \leq \|\cdot\|$ and

$$\limsup_{h\to 0} \frac{1}{\|h\|}\big[|z+h| + |z-h| - 2|z|\big] > c \quad \text{for all } z \in Z \tag{4.6}$$

(*cf.* [P, Corollary 2.35] and [FM, Theorem 2.1]). Based on the norm $|\cdot|$, we construct a set $\Omega \subset X$ in the epigraphical form

$$\Omega = \{(z,\mu) \in X : \mu \geq \varphi(z)\} \quad \text{with} \quad \varphi := -|\cdot|. \tag{4.7}$$

Clearly $\operatorname{bd}\Omega = \{(z,\varphi(z)) \in X : z \in Z\}$. Our goal is to show that there exists a constant $K > 1$ such that

$$\|(z^*,\lambda)\| \leq K\epsilon \quad \text{for all } (z^*,\lambda) \in \hat{N}_\epsilon\Big(\big(z,\varphi(z)\big);\Omega\Big), \quad z \in Z, \quad \text{and} \quad \epsilon > 0, \tag{4.8}$$

where $\|(z^*,\lambda)\| = \max\{\|z^*\|, |\lambda|\}$ is the dual norm to $\|(z,\mu)\| = \|z\| + |\mu|$.

To proceed, we fix arbitrary $\bar{z} \in Z$ and $(z^*,\lambda) \in \hat{N}_\epsilon\Big(\big(\bar{z},\varphi(\bar{z})\big);\Omega\Big)$. It follows directly from the definition of $\hat{N}_\epsilon$ that

$$\langle z^*, z - \bar{z}\rangle + \lambda\big(\mu - \varphi(\bar{z})\big) \leq 2\epsilon\big(\|z-\bar{z}\| + |\mu - \varphi(\bar{z})|\big) \tag{4.9}$$

for all z near $\bar{z}$ and all $\mu \geq \varphi(\bar{z})$ near $\varphi(\bar{z})$. Taking into account that $\varphi = -|\cdot|$ is Lipschitzian on $(Z, |\cdot|)$ with constant $L = 1$ and that $|\cdot| \leq \|\cdot\|$, we get from (4.9) the estimate

$$\langle z^*, z - \bar{z}\rangle + \lambda\big(\varphi(z) - \varphi(\bar{z})\big) \leq 4\epsilon\|z - \bar{z}\| \tag{4.10}$$

for all $z \in Z$ near $\bar{z}$, which further implies

$$\|z^*\| \leq 4\epsilon + |\lambda| \quad \text{for any } (z^*,\lambda) \in \hat{N}_\epsilon\Big(\big(\bar{z},\varphi(\bar{z})\big);\Omega\Big). \tag{4.11}$$

Taking $z = \bar{z}$ in (4.9), we get $\lambda \leq 2\epsilon$.

Now let us show that (4.11) ensures (4.8) with $K = \max\{6, 4 + 8/c\}$, where $c > 0$ is the fixed positive number from (4.6). We consider the two cases: $\lambda \geq 0$ and $\lambda < 0$.

If $\lambda \geq 0$, (4.11) gives $\|(z^*,\lambda)\| \leq 6\epsilon$, and we get (4.8) with $K = 6$. Let us consider the other case when $\lambda < 0$. Using (4.10) with $\varphi = -|\cdot|$, we conclude in this case that

$$|\bar{z} + (z - \bar{z})| + |\bar{z} - (z - \bar{z})| - 2|\bar{z}| \leq -\frac{8\epsilon}{\lambda}\|z - \bar{z}\|$$

which implies $|\lambda| < 8\epsilon/c$ by (4.6). Thus (4.11) gives $\|z^*\| \leq 4\epsilon + (8\epsilon/c)$ for $\lambda < 0$, and we arrive at (4.8) with $K = 4 + 8/c$. This ends the proof of the theorem. ∎

Remark 4.2 It follows from the above proof that to get characterizations of Asplund spaces, it is sufficient to have each of the equivalent properties (b)–(e) not for every closed set Ω but just for epigraphical sets of type (4.7) generated by norm functions. Furthermore, we can equivalently replace "every $\epsilon > 0$" with "exists $\epsilon > 0$".

Remark 4.3 The proof of Theorem 4.1 also justifies modified characterizations of Asplund spaces where M is replaced with $K\epsilon$, $K > 1$, in conditions (b)–(e). These modifications

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emphasize a *linear dependence* of M on ϵ, being more convenient for the limiting procedure as $\epsilon \downarrow 0$. In particular, condition (4.5) with $M = K\epsilon$ immediately yields that

$$N(x;\Omega) = \{0\} \quad \text{for all } x \in \operatorname{bd}\Omega$$

where

$$N(x;\Omega) := \limsup_{u\to x,\varepsilon\downarrow 0} \hat{N}_\varepsilon(u;\Omega)$$

is the normal cone used in [FM, Theorem 3.7] for characterizations of Asplund spaces. Here "limsup" denotes the sequential Painlevé-Kuratowski upper limit of multifunctions with respect to the norm topology in X and the weak-star topology in X^*.

Remark 4.4 One can observe that $(0, 2\epsilon) \in \hat{N}_\epsilon\big((0,0);\Omega\big)$ for all $\epsilon > 0$, where the set Ω is constructed in (4.7). Thus this set provides an example showing that $\hat{N}_\epsilon(x;\Omega)$ may contain elements with norm greater than ϵ, while $\hat{N}_0(x;\Omega) \subset N(x;\Omega) \equiv \{0\}$ for the limiting normal cone from the previous remark.

The concluding result of the paper provides characterizations of Asplund spaces through the normal cone $\hat{N}_0$. It contains a nonconvex extension of the Bishop-Phelps density theorem with a proof based on the separable reduction; see the comments in Introduction.

Theorem 4.5 *Let X be a Banach space. The following assertions are equivalent:*

(a) X is an Asplund space.

(b) For every proper closed subset Ω of X the set of points

$$x \in \operatorname{bd}\Omega \quad \textit{with } \hat{N}_0(x;\Omega) \neq \{0\}$$

is dense in the boundary of Ω.

(c) For every proper closed subset Ω of X there exists $x \in \operatorname{bd}\Omega$ such that $\hat{N}_0(x;\Omega) \neq \{0\}$.

Proof To justify (a)⇒(b), we proceed similarly to the proof of (a)⇒(c) in Theorem 4.1, just replacing Ekeland's principle by the Borwein-Preiss smooth variational principle for spaces with Fréchet smooth renorms; see [BP, Theorem 2.6 and its proof]. Using this result for the function f in (4.2) on the separable Asplund space Y found for the set Ω in question, for $\epsilon = 0$, and for $Y_0 = \operatorname{sp}\{\bar{x}, a\}$ in Corollary 3.4, we get a Fréchet smooth function $g\colon Y \to \mathrm{R}$ and a point $y_0 \in \Omega \cap Y$ such that $|y_0 - \bar{x}| < \epsilon$, $|g'(y_0)| < \epsilon$, and

(4.12)
$$(M+1)|y - \bar{x} - a| - (M+1)|y_0 - \bar{x} - a| + g(y) - g(y_0) \geq 0 \quad \forall y \in \Omega \cap Y.$$

Recall that $|\cdot|$ is Fréchet differentiable at $y_0 - \bar{x} - a \neq 0$ with the derivative $\eta \in Y^*$, $|\eta| = 1$. Employing this, the Fréchet differentiability of g at y_0, and the definition of $\hat{N}_0$, we derive from (4.12) that

$$-(M+1)\eta - g'(y_0) \in \hat{N}_0(y_0;\Omega \cap Y) \setminus MB_{Y^*} \tag{4.13}$$

if $\epsilon < 1$, which we can always assume. Since $\hat{N}_0$ is a cone and $M > 0$, (4.13) means that $\hat{N}_0(y_0;\Omega\cap Y) \neq \{0\}$. Finally, employing condition (3.9) (which is Corollary 3.4 as $\epsilon = 0$), we conclude that $\hat{N}_0(y_0;\Omega) \neq \{0\}$. Thus we have proved (a)⇒(b).

To complete the proof of the theorem, we observe that (b)⇒(c) is trivial, while (c)⇒(a) follows directly from (e)⇒(a) in Theorem 4.1. ∎

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Mathematical Institute
Czech Republic Academy of Sciences
Žitná 25
115 67 Prague 1
Czech Republic

Department of Mathematics
Wayne State University
Detroit, MI 48202
USA