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COVERINGS OF GROUPS BY ABELIAN SUBGROUPS

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Paul Erdös has suggested an investigation of infinite groups from the point of view of the partition relations of set theory. In particular, he suggested that given a group G, one considers the graph Γ with vertex set G whose edges are the pairs $\{g, h\}$ which do not commute. A subset $X \subseteq G$ is a *complete subgraph* of Γ if and only if no two elements of X commute, X is *independent* in Γ if and only if it is a commutative subset of G, and the *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the smallest number of abelian subgroups needed to cover G (we write $\chi(G)$ for $\chi(\Gamma)$).

In this setting, Erdös asked several natural questions. Let $P(G)$ be the smallest cardinal κ such that Γ has no complete subgraphs of cardinality κ. Is $P(G) \leqq \aleph_0$ if and only if $\chi(G) < \aleph_0$? We answer this affirmatively in Theorem 3. If κ is an infinite cardinal, does $P(G) \leqq \kappa^+$ imply that $\chi(G) \leqq \kappa$? With G.C.H. we answer this negatively in Example 1.

R. Baer has proved that $\chi(G) < \aleph_0$ if and only if $|G/Z(G)| < \aleph_0$, where $Z(G)$ is the center of G (see [**11**]). His proof uses a theorem of B. H. Neumann [**10**] which only works for $\aleph_0$. In Theorem 1, we show that $\chi(G) < \kappa$ if and only if $|G/Z(G)| < \kappa$ for all strong limit cardinals κ. In the corollary to Lemma 5 we show that $\chi(G) \leqq \kappa$ implies that

$$[G: Z(G)] \leqq 2^{2^{2^{\kappa}}} \quad \text{for all } \kappa.$$

Some of the results of this paper were announced in [**5**].

Notation. Let G be a group. If $S \subseteq G$, then $C(S) = C_G(S) = \{g \in G \mid gs = sg \text{ for all } s \in S\}$; $Z(G) = C_G(G)$; $\langle S \rangle$ is the group generated by S. G is FC if for all $g \in G$, $[G: C(g)] < \aleph_0$; if κ is a cardinal, G is κC if for all $g \in G$, $[G: C(G)] < \kappa$. (This is a change from the notation used in the first author's previous papers, where G was defined to be κC if for all $g \in C(g)] \leqq \kappa$. The present definition is the correct generalization of FC and is more workable.) If $g, h \in G$ let $g^h = h^{-1}gh$; if $S \subseteq G$, $S^g = \{s^g \mid s \in S\}$; let $[g, h] = g^{-1}h^{-1}gh = g^{-1}g^h$. Further notation can be found in [**15**].

A cardinal κ is *cofinal* with a cardinal λ if κ is the sum of λ smaller cardinals. The *cofinality* of κ, denoted by cf (κ), is the first cardinal cofinal with κ; κ is *singular* if cf $(\kappa) < \kappa$ and *regular* otherwise; κ is a *strong limit* if $\lambda < \kappa$ implies that $2^\lambda < \kappa$; κ is *strongly inaccessible* if it is a regular strong limit. We let *log* κ be the first cardinal λ such that $2^\lambda \geqq \kappa$. The cardinal successor of κ is denoted by κ^+. Some of the theorems and remarks below follow from the generalized

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continuum hypothesis, G.C.H., (for all infinite κ, $2^\kappa = \kappa^+$), but we state and prove them under an appropriate instance of the weaker assumption $2^{<\kappa} = \kappa$, where for any cardinal λ, $\lambda^{<\kappa} = \sum_{\gamma<\kappa} \lambda^\gamma$.

If X is a set, let $[X]^\kappa = \{Y \subseteq X \mid |Y| = \kappa\}$. Let $(\gamma_\alpha \mid \alpha < \lambda)$ be a collection of cardinals. We shall employ the arrow notation of Erdös and Rado to denote partition relations. We write $\kappa \to (\gamma_\alpha \mid \alpha < \lambda)^n$ if whenever $[\kappa]^n = \bigcup_{\alpha<\lambda} X_\alpha$, there exists an $\alpha < \lambda$ and $Y \in [\kappa]^{\gamma_\alpha}$ such that $[Y]^n \subseteq X_\alpha$. If $\gamma_\alpha = \gamma$ for all $\alpha < \lambda$, we write $\kappa \to (\gamma)_\lambda{}^n$. The partition relations used below, in addition to Ramsey's theorem [**14**], are the following cases of theorems of Erdös and Rado [**3**]. For all infinite cardinals κ,

(i) $(2^\kappa)^+ \to ((2^\kappa)^+, \kappa^+)^2$,
(ii) $(2^\kappa)^+ \to (\kappa^+)_\kappa{}^2$.
(iii) If $2^{<\kappa} = \kappa$, then $\kappa^+ \to (\kappa)_\gamma{}^2$, for all $\gamma < \operatorname{cf} \kappa$.

A collection $\mathscr{F}$ of sets forms a Δ-*system with kernel* H if $A \cap B = H$ for a $A \neq B \in \mathscr{F}$. The Erdös-Rado generalization [**4**] of Marczewski's theorem [**7**] states: if κ, λ are regular cardinals with $\kappa < \lambda$, if $\alpha^{<\kappa} < \lambda$ for all $\alpha < \lambda$, and if $\mathscr{G}$ is a family of sets such that $|A| < \kappa$ for each $A \in \mathscr{G}$ and $|\mathscr{G}| = \lambda$, then some $\mathscr{F} \subseteq \mathscr{G}$ with $|\mathscr{F}| = \lambda$ forms a Δ-system. (If $2^{<\sigma} = \sigma$, then $\lambda = \sigma^+$ and $\kappa = \operatorname{cf} \sigma$ satisfy these hypotheses.)

Unless otherwise indicated, all the cardinals in this paper are infinite.

LEMMA 1. *Let* $x, g, h \in G$. *If* $[g, h] = 1$ *and* $[gx, hx] = 1$, *then* $gC(x) = hC(x)$.

Proof. If $gh = hg$, then $gxhx = hxgx$ implies that $x = h^{-1}gxhg^{-1} = h^{-1}gxg^{-1}h$. Thus $g^{-1}h \in C(x)$, so $hC(x) = gC(x)$.

LEMMA 2. *If* $\lambda \to (\kappa)_2{}^2$ *and* $P(G) \leqq \kappa$, *then* G *is* λC.

Proof. Let $x \in G$ and $\{x_\alpha \mid \alpha < \lambda\} \subseteq G$. We want to find α, β, $\alpha \neq \beta$, such that $x_\alpha C(x) = x_\beta C(x)$. Partition $[\{x_\alpha : \alpha < \lambda\}]^2$ into two classes—the class of commutative pairs and the class of noncommutative pairs. By $\lambda \to (\kappa)_2{}^2$ and $P(G) \leqq \kappa$, there is an $S \in [\lambda]^\kappa$ such that $[x_\alpha, x_\beta] = 1$ for all $\alpha, \beta \in S$. Then $P(G) \leqq \kappa$ implies there are $\alpha, \beta \in S$, $\alpha \neq \beta$, such that $[x_\alpha x, x_\beta x] = 1$. By Lemma 1, $x_\alpha C(x) = x_\beta C(x)$.

Remark 1. If we assume $2^{<\kappa} = \kappa$, $P(\kappa)$ implies $\kappa^+ C$ (use $\kappa^+ \to (\kappa)_2{}^2$).

LEMMA 3. *If* $\chi(G) < \kappa$, *then* G *is* κC.

Proof. Let $x \in G$ and let $G = \bigcup_{\alpha<\lambda} A_\alpha$, with each A_α abelian and $\lambda < \kappa$. Suppose $G = \bigcup_{\beta\in T} x_\beta C(x)$. If $|T| \geqq \lambda^+$, then for some $\alpha < \lambda$ and $S \in [T]^{\lambda^+}$, $x_\beta \in A_\alpha$ for all $\beta \in S$. Consider $\{x_\beta x \mid \beta \in S\}$. There exists $R \in [S]^{\lambda^+}$ and $\gamma < \lambda$ such that $x_\beta x \in A_\gamma$ for every $\beta \in R$. Now by Lemma 1, $\{x_\beta \mid \beta \in T\}$ is not a set of distinct left coset representatives. It follows that $[G : C(x)] \leqq \lambda$.

LEMMA 4. *If* $\chi(G) \leqq \kappa$, *then there exists an abelian subgroup* A *such that* $[G : A] \leqq 2^{2^\kappa}$.

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Proof. By Lemma 3, G is κ^+C. Assume the lemma fails. We construct sequences $\{a_\alpha\}$, $\{b_\alpha\}$, $\{C_\alpha\}$ $(\alpha < (2^\kappa)^+)$ such that

(1) $[a_\alpha, b_\alpha] \neq 1$,

(2) $a_\beta, b_\beta \in C_\beta = C(\{a_\alpha, b_\alpha | \alpha < \beta\})$.

By (2), $[G: C_\beta] \leqq \prod_{\alpha<\beta} [G: C(a_\alpha)][G: C(b_\alpha)] \leqq (\kappa^+)^{|\beta|} \leqq 2^{2^\kappa}$, so no C_β can be abelian. Consider the products $\{a_\alpha b_\beta\}$ for $\alpha \neq \beta < (2^\kappa)^+$. Let $G = \bigcup_{\theta<\chi(G)} A_\theta$ with each A_θ abelian. Since $\chi(G) \leqq \kappa$, $(2^\kappa)^+ \to (3)_{\chi(G)}{}^2$. Thus there exists A_θ and α, β, γ such that $a_\alpha b_\beta$, $a_\beta b_\gamma$, $a_\alpha b_\gamma \in A_\theta$. But $a_\alpha b_\beta a_\beta b_\gamma = a_\beta b_\gamma a_\alpha b_\beta$ if and only if $b_\beta a_\beta = a_\beta b_\beta$, a contradiction.

Lemma 5. (i) *If G is κ^+C and has an abelian subgroup A such that $[G: A] \leqq \kappa$, then $[G: Z(G)] \leqq 2^\kappa$.*

(ii) *If κ is strongly inaccessible, if G is κC and if G has an abelian subgroup A such that $[G: A] < \kappa$, then $[G: Z(G)] < \kappa$.*

Proof. (i) Suppose $G = \bigcup_{\alpha<\kappa} x_\alpha A$. Then since

$$D = A \cap \left(\bigcap_{\alpha<\kappa} C(x_\alpha) \right) \subseteq Z(G),$$

it follows that

$$[G: Z(G)] \leqq [G: D] \leqq [G: A] \prod_{\alpha<\kappa} [G: C(x_\alpha)] \leqq \kappa\kappa^\kappa = 2^\kappa.$$

(ii) The proof is similar, so we omit it.

Corollary. *If $\chi(G) \leqq \kappa$, then $[G: Z(G)] \leqq 2^{2^{2^\kappa}}$.*

Proof. The proof is immediate from Lemmas 3, 4 and 5 (i).

Remark 2. When $\kappa = \aleph_0$ this solves a problem of B. H. Neumann [**12**].

There is room for strengthening of the bound $2^{2^{2^\kappa}}$—see Problem 2 below.

Lemma 6. *If $G = \bigcup_{\beta<\lambda} x_\beta H$, $[H: C_H(x_\beta)] \leqq \kappa_\beta$, $\sum_{\beta<\lambda} \kappa_\beta < \kappa$ and $\chi(H) < \kappa$, then $\chi(G) < \kappa$.*

Proof. Let $H = \bigcup_\alpha A_\alpha$, then $G = \bigcup_{\beta,\alpha} x_\beta A_\alpha$. Since $[A_\alpha: C(x_\beta) \cap A_\alpha] \leqq \kappa_\beta$, for each x_β there exists $\{y_{\gamma,\alpha,\beta}\} \in [A_\alpha]^{\kappa_\beta}$ such that $A_\alpha = \bigcup_{\gamma,\beta} y_{\gamma,\alpha,\beta}(C(x_\beta) \cap A_\alpha)$. Since G is covered by the abelian sets $x_\beta y_{\gamma,\alpha,\beta}(C(x_\beta) \cap A_\alpha)$, it follows that $\chi(G) \leqq \chi(H) \cdot \sum_{\beta<\lambda} \kappa_\beta < \kappa$.

Corollary. *If G is κC and has an abelian subgroup A such that $[G: A] <$ cf (κ), then $\chi(G) < \kappa$.*

Proof. The proof is immediate.

Lemma 7. *If $G = \bigcup_{\beta<\lambda} x_\beta H$, $[H: C_H(x_\beta)] \leqq \kappa_\beta$, $\sum_{\beta<\lambda} \kappa_\beta <$ cf (κ) and $P(H) \leqq \kappa$, then $P(G) \leqq \kappa$.*

Proof. If

$$H = \bigcup_{\substack{\beta<\lambda \\ \gamma<\kappa_\beta}} y_{\gamma,\beta}(C(x_\beta) \cap H),$$

then

$$G = \bigcup_{\substack{\beta<\lambda \\ \gamma<\kappa_\beta}} x_\beta y_{\gamma,\beta}(C(x_\beta) \cap H).$$

The number of sets in this union is $\sum_{\beta<\lambda} \kappa_\beta < \text{cf}(\kappa)$. Let $X \in [G]^\kappa$. Then there exists a $Y \in [X]^\kappa$ and β, γ such that $Y \subseteq x_\beta y_{\gamma,\beta}(C(x_\beta) \cap H)$. Now for $a, b \in C(x_\beta) \cap H$ the following equations are equivalent:

$$x_\beta y_{\gamma,\beta} a x_\beta y_{\gamma,\beta} b = x_\beta y_{\gamma,\beta} b x_\beta y_{\gamma,\beta} a,$$
$$x_\beta a y_{\gamma,\beta} b = a x_\beta y_{\gamma,\beta} b = b x_\beta y_{\gamma,\beta} a = x_\beta b y_{\gamma,\beta} a,$$
$$a y_{\gamma,\beta} b = b y_{\gamma,\beta} a,$$
$$y_{\gamma,\beta} a y_{\gamma,\beta} b = y_{\gamma,\beta} b y_{\gamma,\beta} a.$$

Since $y_{\gamma,\beta} a \in H$ if $a \in C(x_\beta) \cap H$, there must be two commuting elements in Y.

THEOREM 1. *If κ is a strong limit cardinal, the following statements are equivalent:*

(I) $\chi(G) < \kappa$;

(II) $|G/Z(G)| < \kappa$;

(III) *G is λC for some $\lambda < \kappa$ and has an abelian subgroup A such that* $[G: A] < \kappa$.

Proof. The case $\kappa = \aleph_0$ is proved in Theorem 3. That (I) implies (II) follows from the corollary to Lemma 5 (ii). That (II) implies (III) is obvious. That (III) implies (I) follows from Lemma 6.

THEOREM 2. *If κ is a strongly inaccessible cardinal, the following statements are equivalent:*

(I) $\chi(G) < \kappa$;

(II) $|G/Z(G)| < \kappa$;

(III) *G is λC for some $\lambda < \kappa$ and has an abelian subgroup A such that* $[G: A] < \kappa$;

(IV) *G is κC and has an abelian subgroup A such that* $[G: A] < \kappa$.

Proof. The case $\kappa = \aleph_0$ is proved in Theorem 3. By Theorem 1, (I), (II) and (III) are equivalent. Obviously, (III) implies (IV). That (IV) implies (I) follows from Lemma 6.

THEOREM 3. *The following statements are equivalent:*

(I) $\chi(G) < \aleph_0$;

(II) $|G/Z(G)| < \aleph_0$;

(III) *G is nC for some $n < \aleph_0$ and has an abelian subgroup of finite index;*

(IV) *G is FC and has an abelian subgroup of finite index;*

(V) $P(G) \leqq n$ *for some* $n < \aleph_0$;
(VI) $P(G) \leqq \aleph_0$.

Proof. That (IV) implies (II) follows from Lemma 5(ii). That (II) implies (III) is obvious. Lemma 6 yields (III) implies (I). It is obvious that (I) implies (V) and (V) implies (VI).

We show VI implies IV. Suppose $P(G) \leqq \aleph_0$. By Lemma 2 and Ramsey's Theorem, G is FC. Assuming that G does not satisfy (IV), we construct sequences $\{f_n\}$, $\{a_n\}$, $\{b_n\}$ with

(1) $f_i f_j \neq f_j f_i, \quad j \neq i$;

(2) $f_n \in \langle \{a_i \mid i \leqq n\} \cup \{b_i \mid i < n\} \rangle$;

(3) $b_n f_n \neq f_n b_n$.

Let $f_0 = a_0$ and b_0 be two non-commuting elements. Inductively, let $C = C(\{a_i \mid i \leqq n\} \cup \{b_i \mid i \leqq n\})$. Since G is FC, C has finite index in G and thus is non-abelian. Let a_{n+1} and b_{n+1} be two non-commuting elements in C and let $f_{n+1} = f_n b_n a_{n+1}$. Clearly (2) is satisfied. Suppose $b_{n+1} f_{n+1} = f_{n+1} b_{n+1}$. Then

$$f_n b_n b_{n+1} a_{n+1} = b_{n+1} f_n b_n a_{n+1} = f_n b_n a_{n+1} b_{n+1},$$

contradicting $[a_{n+1}, b_{n+1}] \neq 1$. Suppose $f_{n+1} f_n = f_n f_{n+1}$. Then

$$f_n b_n f_n a_{n+1} = f_n b_n a_{n+1} f_n = f_n f_n b_n a_{n+1},$$

contradicting (3). Suppose $f_{n+1} f_i = f_i f_{n+1}$ with $i < n$. Then

$$f_n f_i b_n a_{n+1} = f_n b_n a_{n+1} f_i = f_i f_n b_n a_{n+1},$$

contradicting (1). Thus (1), (2) and (3) are satisfied by f_{n+1}, a_{n+1} and b_{n+1}. The sequence $\{f_n\}$ contradicts $P(G) \leqq \aleph_0$.

Remark 3. As we mentioned in the introduction, the equivalence of (I) and (II) was shown by R. Baer by a different proof which does not generalize to strong limit cardinals. The equivalence of (II), (III) and (IV) was shown by B. H. Neumann by essentially the same proof (see [**9**].) We can show that (I), (V) and (VI) are equivalent for cancellation semigroups. B. H. Neumann has independently shown the equivalence of (IV) and (VI) in [**13**].

LEMMA 8. ($2^\kappa = \kappa^+$). *Let V be a κ^+ dimensional vector space over F_2 and let V_κ be a κ dimensional subspace. Let ρ_0 be an alternating bilinear function from $V_\kappa \times V_\kappa$ into the F_2 vector space W. Suppose also that for every $(\lambda, u, w) \in \kappa \times V \times V$, $S_\lambda(u, w)$ is a non-empty subset of W. Then there exists an alternating bilinear function ρ: $V \times V \rightarrow W$ satisfying*

(1) *$\rho = \rho_0$ on V_κ;*
(2) *for every $T \in [V]^\kappa$, $D(T) = \bigcup_{\lambda < \kappa} \{u \in V \mid \rho(w, u) \notin S_\lambda(w, u)\ \forall\ w \in T\}$ has cardinality at most κ.*

Proof. We assume that $\{v_\epsilon \mid \epsilon < \kappa^+\}$ is a basis for V such that $\{v_\epsilon \mid \epsilon < \kappa\}$ is a basis for V_κ. Let V_ϵ be the subspace of V spanned by $\{v_\alpha \mid \alpha < \epsilon)$. Well-order $[V]^\kappa$ with order type κ^+, $\{X_\epsilon \mid \epsilon < \kappa^+\}$, such that $X_\epsilon \subseteq V_\epsilon$. Let $\mathscr{R}_\epsilon = \{X_\tau \mid \tau \leqq \epsilon\}$. We suppose inductively that ρ has been defined on V_ϵ and we want to extend to $V_{\epsilon+1}$. Well-order $V_\epsilon \times \mathscr{R}_\epsilon \times \kappa$, $\{(u_\alpha, X_\alpha, \lambda_\alpha)\}_{\alpha<\kappa}$. Let $\{w_\alpha \mid \alpha < \kappa\}$ be an independent set of vectors such that $w_\alpha \in X_\alpha$ and extend this set to a basis $\mathscr{B}$ for V_ϵ. Then define $\rho(w_\alpha, v_\epsilon) + \rho(w_\alpha, u_\alpha) \in S_{\lambda_\alpha}(w_\alpha, u_\alpha + v_\epsilon)$ for each $\alpha < \kappa$. Complete the definition of ρ by defining $\rho(v_\epsilon, v_\epsilon) = 0$ and $\rho(b, v_\epsilon) = \rho(v_\epsilon, b)$ for all $b \in \mathscr{B}$. In this way ρ is extended to V. Now suppose $T \in [K]^\kappa$. Then $T = X_\eta$ for some η such that $X_\eta \subseteq V_\eta$. Let $u = \sum_{\lambda<\delta} a_\lambda v_\lambda + v_\delta$ with $\delta \geqq \eta$ such that $u \in D(T)$. We used $\mathscr{R}_\delta = \{X_\tau | \tau \leqq \delta\}$ going from V_δ to $V_{\delta+1}$. Since $u - v_\delta \in V_\delta$, for each $\lambda < \kappa$ there exists $\alpha < \kappa$ such that $(u - v_\delta, X_\eta, \lambda) = (u_\alpha, X_\alpha, \lambda_\alpha)$. Thus for every λ there exists $w = w_\alpha \in X_\eta$ such that

$$\rho(w, u) = \rho(w, v_\delta) + \rho(w, u - v_\delta) \in S_\lambda(w, u - v_\delta + v_\delta) = S_\lambda(w, u).$$

Hence $u \notin D(T)$. It follows that $D(T) \subseteq V_\eta$.

Remark 4. This construction was used in [**1**, p. 206] to show that under the assumption $2^\kappa = \kappa^+$ there is a 2-step nilpotent, *FC* group without equipotent abelian subgroups. Further details concerning this construction and those in the following three examples can be found in [**1**].

Example 1. ($2^\kappa = \kappa^+$) There exists a group G of cardinality κ^+ such that for each $X \in [G]^\kappa$ and $Y \in [G]^{\kappa^+}$ there exist (x, y), $(u, v) \in X \times Y$ such that $1 = [x, y] \neq [u, v]$.

Proof. Let $W = F_2$, $S_0(w, u) = \{0\}$, $S_1(w, u) = \{1\}$ in Lemma 8. Let γ: $V \times V \to F_2$ be any bilinear form such that $\rho(x, y) = \gamma(x, y) - \gamma(y, x)$. We form a group $G = V \gamma F_2$ on the set $V \times F_2$ with multiplication defined by $(x, a) \cdot (y, b) = (x + y, a + b + \gamma(x, y))$. Note that $[(x, a), (y, b)] = (0, \rho(x, y))$. If $X \in [G]^\kappa$, let

$$\bar{X} = \{x \in V \mid \exists\ a \in F_2\ (x, a) \in X\}.$$

Then $C(X) = \{y \in V \mid \rho(x, y) = 0\ \forall\ x \in \bar{X}\} \times F_2$, and $\{g \in G \mid [g, x] \neq 1\ \forall\ x \in X\} = \{y \in V \mid \rho(x, y) = 1\ \forall\ x \in \bar{X}\} \times F_2$. Both of these sets have cardinality at most κ by construction.

Example 2. ($2^\kappa = \kappa^+$) There exists a group G of cardinality κ^+ satisfying $P(G) \leqq \kappa^+$, which is not κ^+C. Consequently, $P(G \times G) > P(G)$.

Proof. In Lemma 8, let $W = V$, $S_0(w, u) = \{0\}$, $S_1(w, u) = \{w + u\}$. Let $T \in [F_2 \times V]^{\kappa^+}$. Suppose for all (a, x), $(b, y) \in T$, $\rho(x, y) \neq ay + bx$. There exists $S \in [T]^{\kappa^+}$ such that for every (a, x) and $(b, y) \in S$, $a = b$. Then, letting $\bar{S} = \{x \in V \mid \exists\ a \in F_2\ (x, a) \in S\}$,

$$D(\bar{S}) = \bigcup_{a \in F_2} \{u \in V \mid \rho(w, u) \notin S_a \forall w \in \bar{S}\} \supseteq \bar{S},$$

contradicting $|D(\bar{S})| \leqq \kappa$. Now let $U = F_2 \times V$ and define $\rho(1, v) = v$ for all $v \in V$ and extend ρ to an alternating bilinear function ρ: $U \times U \rightarrow V$. As in Example 1, let $G = U\gamma V$ where γ: $U \times U \rightarrow V$ satisfies $\rho(x, y) = \gamma(x, y) - \gamma(y, x)$ for all $x, y \in U$. Elements of G have the form $(a + x, v)$ with $a \in F_2$, x and $v \in V$; if $(a + x, v)$, $(b + y, w) \in G$,

$$[(a + x, v), (b + y, w)] = (0, \rho(a + x, b + y)) = (0, ay + bx + \rho(x, y)).$$

By construction, if $X \in [G]^{\kappa^+}$ there exists $(a + x, v)$, $(b + y, w) \in X$ such that $\rho(x, y) = ay + bx$. Thus

$$[(a + x, v), (b + y, w)] = (0, 0) = 1,$$

which implies that $P(G) \leqq \kappa^+$. The elements $(1, v)$ for $v \in V$ are all conjugates of $(1, 0)$, so G is not κ^+C. By Lemma 1, $P(G \times G) > \kappa^+$.

Example 3. ($2^\kappa = \kappa^+$) There exist groups G and H which are κC, $P(G) = P(H) = \kappa^+$, but $P(G \times H) > \kappa^+$.

Proof. We proceed as in Lemma 8 and Example 1. Let V be κ^+ dimensional over F_2 and W be κ dimensional over F_2. Let $\{V_\alpha: \alpha < \kappa^+\}$ be a basis for V. We construct alternating bilinear ρ_1, ρ_2 from $V \times V$ into W such that

(1) for every $\alpha < \beta < \kappa^+$, $\rho_1(v_\alpha, v_\beta) = 0$ implies that $\rho_2(v_\alpha, v_\beta) \neq 0$;

(2) for every $T \in [V]^\kappa$, $D_1(T) = \{u \in V \mid \rho_i(w, u) \neq 0 \ \forall\ w \in T\}$ has cardinality at most κ for $i = 1, 2$.

Let $V_\epsilon = \langle\{v_\alpha \mid \alpha < \epsilon\}\rangle$, the subspace of V spanned by $\{v_a \mid \alpha < \epsilon\}$. We suppose $[V]^\kappa = \{X_\epsilon \mid \kappa \leqq \epsilon < \kappa^+\}$ with $X_\epsilon \subseteq V_\epsilon$. First define ρ_1, ρ_2 on V_κ so that (1) is satisfied for $\alpha < \beta < \kappa$.

Now suppose ρ_1, ρ_2 have been defined on V_ϵ ($\kappa \leqq \epsilon < \kappa^+$) so that (1) holds for all $\alpha < \beta < \epsilon$. Well-order $\{v_\alpha \mid \alpha < \epsilon\}$ as $\{v_\tau' \mid \tau < \kappa\}$ and let $V_\tau' = \langle\{v_\gamma' \mid \gamma < \tau\}\rangle$. Well-order $V_\epsilon \times \{X_\tau \mid \kappa \leqq \tau \leqq \epsilon\}$ as $\{(u_\alpha, \bar{X}_\alpha)\}_{\alpha<\kappa}$. To extend ρ_1, ρ_2 to $V_{\epsilon+1}$, we make the following construction. Suppose $\beta = 3 \cdot \alpha + \gamma$, $0 \leqq \gamma < 3$ and ρ_1, ρ_2 have been defined on $Q_{\epsilon,\beta} \times \{v_\epsilon\}$ where $V_\alpha' \subseteq Q_{\epsilon,\beta}$ and $|Q_{\epsilon,\beta}| < \kappa$.

Case 1. $\gamma = 0$. Find $w \in \bar{X}_\alpha \backslash Q_{\epsilon,\beta}$. Put $\rho_1(w, v_\epsilon) = \rho_1(w, u_\alpha)$. Put $Q_{\epsilon,\beta+1} = \langle Q_{\epsilon,\beta} \cup \{w\}\rangle$. If there is no $v_\tau' \in Q_{\epsilon,\beta+1} \backslash Q_{\epsilon,\beta}$, let $\rho_2(w, v_\epsilon)$ be arbitrary. Otherwise, let $\{w + q_\mu\}_{\mu<\rho<\kappa}$ be all such v_τ'. Pick $x \in W \backslash \langle\{\rho_2(q_\mu, v_\epsilon) | \mu < \rho\}\rangle$ and put $\rho_2(w, v_\epsilon) = x$.

Case 2. $\gamma = 1$. Make the same construction as in case 1, but switch the roles of ρ_1 and ρ_2.

Case 3. $\gamma = 2$. If $v_\alpha' \in Q_{\epsilon,\beta}$, put $Q_{\epsilon,\beta+1} = Q_{\epsilon,\beta}$ and do nothing. If not, put $Q_{\epsilon,\beta+1} = \langle Q_{\epsilon,\beta} \cup \{v_\alpha'\}\rangle$ and define $\rho_i(v_\alpha', v_\epsilon)$ so that $\rho_i(v_\tau', v_\epsilon) \neq 0$ for all $v_\tau' \in Q_{\epsilon,\beta+1} \backslash Q_{\epsilon,\beta}$.

We leave it to the reader to verify that (1) and (2) are satisfied. Now let γ_i: $V \times V \rightarrow W$ be any bilinear maps such that $\rho_i(x, y) = \gamma_i(x, y) -$

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$\gamma_i(y, x)$. Put $G = V\gamma_1 W$ and $H = V\gamma_2 W$. Clearly, (1) implies that $P(G \times H) > \kappa^+$, but the proof in Example 1 shows that both G and H are κC and $P(G) = P(H) = \kappa^+$.

Remark 5. The existence of a group with the properties described in Example 1 for $\kappa = \aleph_0$ is independent of the usual axioms of set theory. Namely, it is a theorem of [**6**] that if $2^{\aleph_0} > \aleph_1$ and Martin's axiom [**8**] hold, then $\omega_1 \to (\omega_1, (\omega: \omega_1))^2$, that is, if $[\omega_1]^2 = A \cup B$ then either there is an $X \in [\omega_1]^{\omega_1}$ with $[X]^2 \subseteq A$ or there is a $Y \in [\omega_1]^\omega$ and an $X \in [\omega_1]^{\omega_1}$ such that $\{\{y, x\} \mid y \in Y$ and $x \in X\} \subseteq B$. This partition relation implies that there are no groups with the properties of Example 1, for $\kappa = \aleph_0$.

Example 4. For each cardinal λ, there is a group G_λ which has an abelian subgroup A_λ such that $[G_\lambda: A_\lambda] = 2$, but G_λ is not λC.

Proof. Let A_λ be an F_2 vector space of dimension λ with basis $\{v_\alpha, w_\alpha \mid \alpha < \lambda\}$. Let σ be the automorphism of A_λ defined by $\sigma(v_\alpha) = w_\alpha$, $\sigma(w_\alpha) = v_\alpha$. Let $G_\lambda = \langle\sigma\rangle A_\lambda$ be the split extension of A_λ by $\langle\sigma\rangle$. Then $[G: A_\lambda] = 2$, but $(\sigma, 0)^{(1, v_\alpha)} = (\sigma, w_\alpha + v_\alpha)$ for all $\alpha < \lambda$, so G is not λC.

Example 5. For every limit cardinal κ there is a group G which is κC, has $\chi(G) = \kappa$ and has an abelian subgroup A such that $[G: A] = \text{cf}(\kappa)$. If κ is regular, $P(G) = \kappa$. If κ is singular, $P(G) = \kappa^+$.

Proof. Let $G = \sum_{\lambda < \text{cf}(\kappa)} G_{\alpha_\lambda}$ where G_{α_λ}, of power α_λ, is the group in Example 3 and $\kappa = \lim_\lambda \alpha_\lambda$. Clearly $[G: \sum A_\lambda] = |\sum G_{\alpha_\lambda}/A_\lambda| = \text{cf}(\kappa)$ and G is κC. Since G is not γC for any $\gamma < \kappa$, $\chi(G) = \kappa$. Suppose κ is regular. Let $X \in [G]^\kappa$. For each $x \in X$, $A(x) = \{\alpha \mid x(\alpha) = 1\}$ is finite. By Marczewski's theorem there exists $Y = [X]^\kappa$ such that the sets $A(x)$ with $x \in Y$ form a Δ-system with kernel $H = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If $xy \neq yx$ for all $y \neq x \in Y$, there exists an $i \leqq n$ and a $Z \in [Y]^\kappa$ such that $xy(\alpha_i) \neq yx(\alpha_i)$ for all $x \neq y \in Z$, contradicting $|G_{\alpha_i}| < \kappa$. If κ is singular, $P(G) = \kappa^+$ follows directly from the next lemma and the fact that $|G| = \kappa$.

LEMMA 9. *Suppose κ is a singular cardinal and $\kappa = \lim_{\alpha < \text{cf}(\kappa)} \lambda_\alpha$. If G has a family of subsets X_α, $\alpha < \text{cf}(\kappa)$, with the properties*

(1) $[x, y] \neq 1$ *when* $x \neq y \in X_\alpha$;

(2) $|X_\alpha| = \lambda_\alpha$;

(3) $X_\beta \subseteq C(X_\alpha)$ *for* $\alpha < \beta$,

then $P(G) > \kappa$.

Proof. Since $\text{cf}(\kappa) < \kappa$, we may suppose $\lambda_0 = \text{cf}(\kappa)$. Well-order $X_0 = \{x_\alpha: \alpha < \text{cf}(\kappa)\}$. Consider the set $S = \{x_\alpha y_\alpha \mid y_\alpha \in X_\alpha\}$. If $y_\alpha, z_\alpha \in X_\alpha$ and $x_\alpha y_\alpha x_\alpha z_\alpha = x_\alpha z_\alpha x_\alpha y_\alpha$, then $y_\alpha z_\alpha = z_\alpha y_\alpha$, which implies $z_\alpha = y_\alpha$. If $\alpha \neq \beta$ and $x_\alpha y_\alpha x_\beta y_\beta = x_\beta y_\beta x_\alpha y_\alpha$, then $x_\alpha x_\beta = x_\beta x_\alpha$, a contradiction. Thus S is a set of κ pairwise non-commuting elements.

LEMMA 10. *Let κ be a strong limit cardinal. If G is a group which is κC and has $P(G) = \kappa$. then G has a subgroup H and a normal subgroup $K \subseteq H$ such that $[H: K] = \text{cf}(\kappa)$, $P(H) = \kappa$ and $P(K) < \kappa$.*

Proof. If κ is regular, let $H = G$ and $K = E$. Now assume that κ is singular. If G has a subgroup L such that $[G: L] < \kappa$ and $P(L) < \kappa$, then $K = \bigcap_{x \in G} L^x$ is normal in G and $[G: K] \leqq [G: L]^{[G:L]} < \kappa$. Let $\kappa = \lim_{\alpha < \text{cf}(\kappa)} \lambda_\alpha$ with $\lambda_0 = \text{cf}(\kappa)$. By Lemma 7, for each $\alpha < \text{cf}(\kappa)$ there exists $y_\alpha \in G \backslash K$ such that $[K: C_K(y_\alpha)] \geqq \lambda_\alpha$. Let $H = \langle \{y_\alpha | \alpha < \text{cf}(\kappa)\} \rangle K$. Clearly $[H: K] = |H/K| = \text{cf}(\kappa)$. If $P(H) = \gamma < \kappa$, then by Lemma 2, H is $(2^\gamma)^+C$. Since κ is a strong limit cardinal, $(2^\gamma)^+ < \kappa$ and there exists an α such that $\lambda_\alpha > 2^\gamma$. Since y_α has at least λ_α conjugates in H, we have a contradiction. Thus we may assume that every subgroup L such that $[G: L] < \kappa$ has $P(L) = \kappa$. We construct sets X_α, $\alpha < \text{cf}(\kappa)$, having the properties (1), (2) and (3) in Lemma 9 and the additional property

(4) $[G: C(x)] \leqq \kappa_\alpha < \kappa$ when $x \in X_\alpha$.

Let $C = C(X_\alpha | \alpha < \beta)$. Then

$$[G: C] \leqq \prod_{\alpha < \beta} \prod_{x \in X_\alpha} [G: C(x)] \leqq \prod_{\alpha < \beta} \kappa_\alpha^{\lambda_\alpha} < \kappa.$$

Since $P(C) = \kappa$, we can choose $X \subseteq C$, X a set of pairwise non-commuting elements and $|X|$ regular and at least λ_β. There exists $\kappa_\beta < \kappa$ and $X_\beta \in [X]^{\lambda_\beta}$ such that $x \in X_\beta$ implies that $[G: C(x)] \leqq \kappa_\beta$. Lemma 9 yields $P(G) > \kappa$, a contradiction.

THEOREM 4. *Suppose κ is a strong limit cardinal cofinal with ω. If G is κC, then $P(G) \neq \kappa$.*

Proof. By Lemma 10, if $P(G) = \kappa$, there exists an $H_0 \lhd G$ with $P(H_0) < \kappa$ and $[G: H_0] < \kappa$. Let $\kappa = \lim_{n < \omega} \lambda_n$ with $\lambda_n > \gamma = P(H_0)$. By Lemma 2, H_0 is $(2^\gamma)^+C$. By Lemma 7, if $K \subseteq H_0$ with $[H_0: K] < \kappa$, then for every n there exists a $y \in G$ such that $[K: C_K(y)] \geqq (2^{\lambda_n})^+$. Let $\mu_n = (2^{\lambda_n})^+$ and choose y_0 such that $[H_0: C_{H_0}(y_0)] \geqq \mu_0$. Then let $D_0 = \{h_{0,\alpha} | \alpha < \mu_0\}$ be a commutative subset of a transversal for $C_{H_0}(y_0)$ in H_0 (this is possible since $(2^{\lambda_0})^+ \to ((2^{\lambda_0})^+, \lambda_0^+)^2$). Suppose H_n, y_n and $D_n = \{h_{n,\alpha} | \alpha < \mu_n\}$ have been defined for all $n \leqq \gamma$ with $[H_n: H_{n+1}] < \kappa$. We let $H_{k+1} = C_{H_k}(y_k) \cap C(\{h_{n,\alpha} | \alpha < \mu_n, n \leqq k\})$. Then

$$[H_0: H_{k+1}] \leqq [H_0: H_k][H_k: C_{H_k}(y_k)] \prod_{n \leqq k} \prod_{\alpha < \mu_k} [H_0: C_{H_0}(h_{n,\alpha})]$$

$$\leqq [G: C_{H_k}(y_k)] \prod_{n \leqq k} (\mu_0)^{\mu_n} < \kappa.$$

Now let y_{k+1} be such that $[H_{k+1}: C_{H_{k+1}}(y_{k+1})] \geqq \mu_{k+1}$ and $D_{k+1} = \{h_{k+1,\alpha} | \alpha < \mu_{k+1}\}$ be a commutative subset of a set of coset representatives for $C_{H_{k+1}}(y_{k+1})$ in H_{k+1}. Thus we define H_n, y_n and D_n for all $n < \omega$. Let $C_n =$

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$C_{H_0}(y_n)$ for $n < \omega$. There is an equipotent subset E_0 of D_0 and an infinite set I_0 of positive integrers such that either

$$\text{(i) } y_0E_0 \cap \left(\bigcup_{i\in I_0} C_i\right) = \emptyset \quad \text{or} \quad \text{(ii) } y_0E_0 \subseteq \bigcap_{i\in I_0} C_i.$$

Namely to each $d \in D_0$ let $f_d\colon \omega \to 2$ satisfy $y_0d \in C_i$ if and only if $f(i) = 0$. Pick $E_0 \subseteq D_0$ with $|E_0| = |D_0|$ such that $e_1, e_2 \in E_0$ implies that $f_{e_1} = f_{e_2} = f$ (μ_0 is regular and greater than $2^{\aleph_0}$), and let I_0 be an infinite set on which f is constant. Suppose we have continued this construction and have found E_{n_i}, $i \leqq k$, and I_k, an infinite set of natural numbers, satisfying

(A) $n_0 = 0$ and $\{n_i | 0 \leqq i \leqq k\}$ is an initial segment of I_k;

(B) $E_{n_i} \subseteq D_{n_i}$;

(C) $|E_{n_i}| = \mu_{n_i}$;

(D) either (i) $y_{n_i}E_{n_i} \cap C_j = \emptyset \quad \forall\, j \in I_k,\ j > n_i$, or (ii) $y_{n_i}E_{n_i} \subseteq C_j$ $\forall\, j \in I_k, j > n_i$;

we find E_{n_k+1} and I_{k+1} by exactly the same method used to find E_0 and I_0.

Now consider the sets $J_1 = \{n_i | D(i) \text{ holds}\}$ and $J_2 = \{n_i | D(ii) \text{ holds}\}$. Consider further the elements $x_{n,\alpha} = y_nh_{n,\alpha}$ for all $n < \omega$ and $\alpha < \mu_n$. If $x_{n,\alpha}$ commutes with $x_{n,\beta}$, then $h_{n,\alpha}h_{n,\beta}^{-1} \in C_{Hn}(y_n)$. which by definition implies that $\alpha = \beta$. If $n < m$, since $h_{m,\beta}$ commutes with both y_n and $h_{n,\alpha}$, $x_{n,\alpha}$ commutes with $x_{m,\beta}$ if and only if $x_{n,\alpha} \in C(y_m)$. Suppose J_1 is infinite. Then

$$\{x_{n,\alpha}\colon n \in J_1 \text{ and } h_{n,\alpha} \in E_n\}$$

is a set of κ pairwise non-commuting elements, contradicting $P(G) \leqq \kappa$. On the other hand, if J_2 is infinite, let $X_n = \{x_{n,\alpha} | h_{n,\alpha} \in E_n\}$ for each $n \in J_2$. Since $[x_{n,\alpha}, x_{n,\beta}] \neq 1$ if $\alpha \neq \beta$, but $[x_{n,\alpha}, x_{m,\beta}] = 1$ if $n \neq m \in J_2$, Lemma 9 implies that $P(G) > \kappa$.

Definition. We denote by $\prod_{\alpha<\kappa}^{\gamma} G_\alpha$ the subgroup of $\prod_{\alpha<\kappa} G_\alpha$ consisting of all $x \in \prod_{\alpha<\kappa} G_\alpha$ such that $|\{\alpha | x(\alpha) \neq 1\}| < \gamma$.

THEOREM 5. *Let $G = \prod_{\alpha<\kappa}^{\gamma} G_\alpha$ with each G_α non-abelian. Let $\sigma = \sup_{\alpha<\kappa} \chi(G_\alpha)$ and let $\theta = \max\{\sigma, \log \kappa\}$. Assume $2^{<\theta} = \theta$. If $\gamma \leqq \operatorname{cf}\theta$, then $\chi(G) = \theta$.*

Proof. (i) $\chi(G) \geqq \theta$. Clearly $\chi(G) \geqq \chi(G_\alpha)$ for all α, so $\chi(G) \geqq \sigma$.

We claim that $\chi(G) \geqq \log \kappa$. Suppose on the contrary that G is a disjoint union of abelian sets A_θ, $\theta < \lambda$ for some cardinal λ with $2^\lambda < \kappa$. In each G_α choose two non-commuting elements x_α and y_α. For each $\alpha < \beta < \kappa$ consider the element $s_{\alpha\beta}$ of G defined by

$$s_{\alpha\beta}(\epsilon) = \begin{cases} 1 & \epsilon \neq \alpha, \beta \\ x_\alpha & \epsilon = \alpha \\ y_\beta & \epsilon = \beta. \end{cases}$$

Partition the pairs $\{\alpha, \beta\}$, $\alpha < \beta < \kappa$, into λ classes—put $\{\alpha, \beta\}$ in the θth class if $s_{\alpha\beta} \in A_\theta$. Since $\kappa \to (3)_\lambda^2$, there exists A_θ and α, β, γ such that $s_{\alpha\beta}, s_{\beta\gamma}, s_{\alpha\gamma} \in A_\theta$. However $[s_{\alpha\beta}, s_{\beta\gamma}](\beta) = [y_\beta, x_\beta] \neq 1$, a contradiction.

(ii) $\chi(G) \leqq \theta$. Consider the tree $T = (2)^{<\theta}$ of functions from ordinals $< \theta$ into 2, ordered by function extension. T has θ nodes and $2^\theta \geqq \kappa$ paths (a path corresponds to a function from θ into 2). We label κ of these paths by ordinals less than κ. We also suppose that for each $\alpha < \kappa$, $G_\alpha = \bigcup_{\beta<\theta} A_{\alpha,\beta}$ with each $A_{\alpha,\beta}$ abelian. For each function φ such that the domain of φ is a set of incomparable nodes of T with cardinality $< \gamma$ and the range of φ is a subset of θ, we form a set $C_\varphi \subseteq G$. For $f \in G, f \in C_\varphi$ if and only if

(a) there is a one to one correspondence ψ: dom $\varphi \to \{\alpha \mid f(\alpha) \neq 1\}$;

(b) for each node $a \in$ dom φ, a is on the path labeled $\psi(a)$;

(c) for each node $a \in$ dom $\varphi, f(\psi(a)) \in A_{\psi(a)\varphi(a)}$.

The number of C_φ's is $\theta^{<\gamma}$, which, since $\gamma \leqq \text{cf } \theta$ and $2^{<\theta} = \theta$, equals θ.

The theorem will be completed by showing that each C_φ is abelian and that $\bigcup_\varphi C_\varphi = G$. Suppose f, $g \in C_\varphi$. For each α such that $f(\alpha)$, $g(\alpha) \neq 1$, only one node $a \in$ dom φ can be on the path α, and so $f(\alpha)$, $g(\alpha) \in A_{\alpha\varphi(a)}$, and $[f(\alpha), g(\alpha)] = 1$. Thus $[f, g] = 1$. If $f \in G$, then since $|\{\alpha \mid f(\alpha) \neq 1\}| < \gamma \leqq$ cf θ, then there is a set B of incomparable nodes of T and a bijection ψ: $B \to \{\alpha \mid f(\alpha) \neq 1\}$ such that b is on the path labeled by $\psi(b)$ for each $b \in B$. Now for each $b \in B$ let $\varphi(b)$ satisfy $f(\psi(b)) \in A_{\psi(b)\varphi(b)}$. Then $f \in C_\varphi$.

THEOREM 6. *Let $G = \prod^\gamma_{\alpha<\kappa} G_\alpha$ and let $\sigma = \sup_{\alpha<\kappa} P(G_\alpha)$. Assume $2^{<\sigma} = \sigma$. If $\gamma \leqq$ cf σ, then $P(G) \leqq \sigma^+$.*

Proof. Suppose there were a set $X \in [G]^{\sigma^+}$ of pairwise non-commutative elements. Applying the Erdös-Rado generalization of Marczewski's theorem, there is a $Y \in [X]^{\sigma^+}$ such that the sets $A_y = \{\alpha | y(\alpha) \neq 1\}$, for $y \in Y$, form a Δ-system with kernel H. For $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there is an $\alpha \in H$ with $[y_1(\alpha), y_2(\alpha)] \neq 1$. Since $|H| <$ cf σ, we apply $\sigma^+ \to (\sigma)_{|H|}{}^2$ to obtain a $Z \in [Y]^\sigma$ and an $\alpha \in H$ with $[z_1(\alpha), z_2(\alpha)] \neq 1$ for all $z_1, z_2 \in Z$ with $z_1 \neq z_2$, contradicting $P(G_\alpha) \leqq \sigma$.

Remark 6. It is not hard to give examples where σ^+ is attained in the theorem. In addition to Examples 2 and 3, if G is the direct sum of free groups F_n on $\aleph_n$ generators for all $n < \omega$, $P(G) = \aleph_{\omega+1}$. However, if $P(G) = P(H) = \aleph_0$, then clearly from Theorem 3, $P(G \times H) = \aleph_0$. Can $P(G)$ be a singular cardinal?

Remark 7. It follows from Theorems 5 and 6 that if κ is an infinite cardinal and if $G = \sum_{\alpha<\kappa} G_\alpha$ with each G_α finite and non-abelian, then $\chi(G) = \log \kappa$ while $P(G) = \aleph_1$.

THEOREM 7. *Let G be a group of cardinality $(2^\kappa)^+$. Let $\gamma \leqq \kappa^+$. If for every collection of sets $(X_\alpha \mid \alpha < \gamma)$ with $X_\alpha \in [G]^{\kappa^+}$ there exists $\alpha \neq \beta$ such that $x_\alpha \in X_\alpha$ and $x_\beta \in X_\beta$ with $[x_\alpha, x_\beta] = 1$, then $P(G) \leqq \gamma$.*

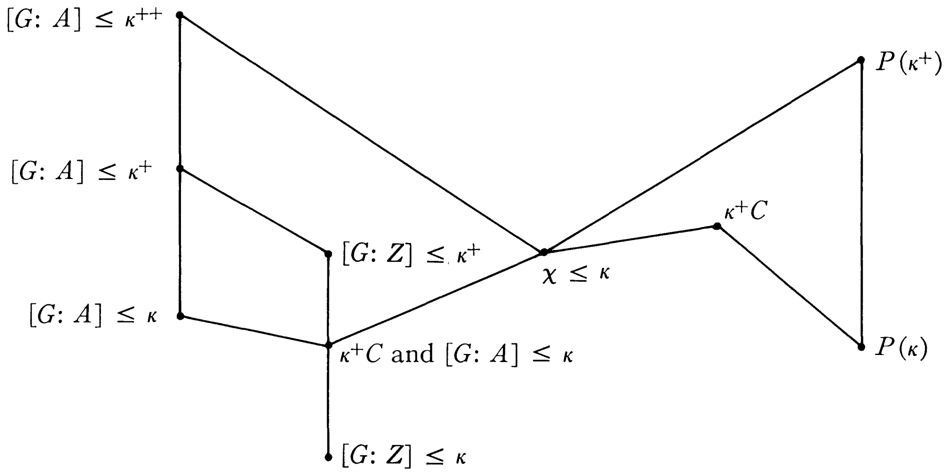
Proof. Suppose $P(G) > \gamma$. Let $X \in (G)^{\kappa^+}$. Write $X = \dot{\bigcup}_{\alpha<\alpha} X_\alpha$ with $|X_\alpha| = \kappa^+$. There exists $\alpha \neq \beta$ such that $x_\alpha \in X_\alpha$, $x_\beta \in X_\beta$ and $[x_\alpha, x_\beta] = 1$.

Thus $P(G) \leqq \kappa^+$. Let $X \in [G]^\gamma$ such that $xy \neq yx$ for each $x, y \in X$. Since

$$[G: C(X)] \leqq \prod_{x \in X} [G: C(x)] \leqq (2^\kappa)^\gamma = 2^\kappa,$$

we have $|C(X)| = (2^\kappa)^+$. Since $(2^\kappa)^+ \rightarrow (\kappa^+)_2{}^2$, we can choose an abelian subgroup $A \in [C(X)]^{\kappa^+}$. Consider $X_x = \{xa \mid a \in A\}$, $x \in X$. If $xayb = ybxa$, then $xyab = yxab$ and $xy = yx$, a contradiction.

Remark 8. (G.C.H.) If $G = \sum_{\alpha<\lambda^+} G_\alpha$ with each G_α a finite simple group, then $P(G) \leqq \aleph_1$, G is FC and $\chi(G) = \lambda$; on the other hand, $[G: Z] > \lambda$ and for every abelian A, $[G: A] > \lambda$. The group G in Example 2, has $P(G) = \kappa^+$ and $[G: Z] \leqq \kappa^+$; on the other hand, G is not κ^+C. The group G in Example 1 is FC and has $[G: Z] \leqq \kappa^+$, but has $\chi(G) = \kappa^+$, $P(G) = \kappa^+$ and for every abelian subgroup A of G, $[G: A] = \kappa^+$. If G is a free group on κ generators, $\chi(G) = \kappa$ and $[G: Z] = \kappa$, but G is κ^+C and $P(G) = \kappa^+$. The groups G_λ in Example 4 show that having an abelian subgroup of index 2 need not imply λC for any λ.



Class Inclusions (G.C.H.)

The figure illustrates the class inclusions under G.C.H. (A denotes an abelian subgroup for which $[G: A]$ is minimal.) All inclusions are proper if κ is a successor cardinal.

Problem 1. Does $\chi(G) \leqq \kappa$ imply that $[G: A] \leqq \kappa^+$?

Problem 2. Does $\chi(G) \leqq \kappa$ imply that $[G: Z] \leqq 2^{2^\kappa}$ or even $[G: Z] \leqq \kappa^+$?

Problem 3. If κ is a limit cardinal, does $P(G) \leqq \kappa$ imply that G is κC?

Problem 4. Does $|G| \leqq (2^\kappa)^+$ and $P(G) \leqq \kappa^+$ imply that $\chi(G) \leqq 2^\kappa$?

Problem 5. Can $P(G)$ be a singular cardinal?

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