One-sided C_p estimates via $M^\sharp$ function

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We recall that $w \in C_p^+$ if there exist $\varepsilon > 0$ and $C > 0$ such that for any $a < b < c$ with $c - b < b - a$ and any measurable set $E \subset (a, b)$, the following holds

$$\int_E w \leqslant C \left(\frac{|E|}{(c-b)} \right)^\varepsilon \int_{\mathbb{R}} \left(M^+ \chi_{(a,c)} \right)^p w < \infty.$$

This condition was introduced by Riveros and de la Torre [**33**] as a one-sided counterpart of the C_p condition studied first by Muckenhoupt and Sawyer [**30**, **34**]. In this paper we show that given $1 < p < q < \infty$ if $w \in C_q^+$ then

$$\|M^+ f\|_{L^p(w)} \lesssim \|M^{\sharp,+} f\|_{L^p(w)}$$

and conversely if such an inequality holds, then $w \in C_p^+$. This result is the one-sided counterpart of Yabuta's main result in [**37**]. Combining this estimate with known pointwise estimates for $M^{\sharp,+}$ in the literature we recover and extend the result for maximal one-sided singular integrals due to Riveros and de la Torre [**33**] obtaining counterparts a number of operators.

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1. Introduction and main result

One-sided theory of weights was begun by Sawyer in [**35**] where he provided the characterization of the two weighted inequalities for the one-sided maximal

functions

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f(y)|\, \mathrm{d}y \quad M^- f(x) = \sup_{h>0} \frac{1}{h} \int_{x-h}^{x} |f(y)|\, \mathrm{d}y.$$

In some sense it is a somehow curious fact that the results in [35] appeared more than a decade later than the characterization of the one weight inequalities for the maximal function due to Muckenhoupt [29] if one bears in mind that actually the maximal operators studied by Hardy and Littlewood [10] were M^+ and M^-.

Since Sawyer's work a number of papers such as [1, 7, 9, 20, 21, 23–26, 28, 31, 32] and even more that we will cite throughout this paper were devoted to develop the one-sided theory. However, at this point, we believe it is worth mentioning some papers which have expanded the field of one-sided estimates in the last years. Kinnunen and Saari [13, 14] studied parabolic Muckenhoupt conditions in connection with PDEs and more recently Hytönen and Rosén devoted their work [11] to causal sparse domination motivated by maximal regularity estimates for elliptic PDEs, obtaining results related to one-sided weighted estimates for singular integrals.

A well-known estimate in theory of weights that was settled by Coifman and Fefferman, says that if $w \in A_\infty$ then, for every $p \in (0, \infty)$ we have that

$$\|Tf\|_{L^p(w)} \leqslant c_{p,n,w} \|Mf\|_{L^p(w)} \tag{1.1}$$

where T stands for any Calderón–Zygmund operator.

Although $w \in A_\infty$ is sufficient for (1.1) to hold, it turns out to not to be necessary. Muckenhoupt [30] showed that if (1.1) holds for the Hilbert transform for $p > 1$ then there exist $c, \varepsilon > 0$ such that for every cube Q and every measurable subset $E \subset Q$,

$$w(E) \leqslant c \left(\frac{|E|}{|Q|} \right)^\varepsilon \int_{\mathbb{R}} (M\chi_Q)^p\, w,$$

namely $w \in C_p$. Later on, Sawyer [34] showed that if $1 < p < q < \infty$ and $w \in C_q$, then (1.1) holds.

Yabuta [37] provided a different approach to the question. He showed that if $1 < p < q < \infty$ and $w \in C_q$

$$\|Mf\|_{L^p(w)} \leqslant c_{p,n,w} \|M^\sharp f\|_{L^p(w)} \tag{1.2}$$

and also that if such an inequality holds then $w \in C_p$. An alternative proof of this estimate and a slight generalization of the C_p condition was studied by Lerner in [15].

In the last years some advances have been made in the study of this kind of questions. Lerner [16] fully characterized the weak type version of (1.1). Sawyer's result has been extended to the full range in [6] and also quantitative estimates in terms of a suitable C_p constant and further operators, such as rough singular integrals, have been explored in [4, 5].

In the one-sided setting we are aware of just one work in this direction in which Riveros and de la Torre [33] introduced the one-sided version of the C_p condition, which reads as follows. We say that $w \in C_p^+$ if there exist $\varepsilon > 0$ and $C > 0$ such

that for any $a < b < c$ with $c - b < b - a$ and any measurable set $E \subset (a,b)$, the following holds

$$\int_E w \leqslant C \left(\frac{|E|}{c-b}\right)^\varepsilon \int_{\mathbb{R}} \left(M^+ \chi_{(a,c)}\right)^p w. \qquad (1.3)$$

The main result in that work was the following one-sided counterpart of [**34**].

THEOREM 1.1 [**33**, theorem 1]. *Let $1 < p < q < \infty$. If $w \in C_q^+$ and $(T^+)^*$ is a maximal Calderón–Zygmund one-sided singular integral, then*

$$\int_{\mathbb{R}} |(T^+)^* f|^p w \leqslant C \int_{\mathbb{R}} (M^+ f)^p w. \qquad (1.4)$$

Observe that in [**33**], additionally the authors assume that the integral in the right-hand side of (1.3) is finite. Note that if the right-hand side of the condition C_q^+ is not finite, then the same happens to $\int_{\mathbb{R}} (M^+ f)^p w$ and hence the inequality is trivial.

Note that the C_p^+ class is defined in terms of the one-sided maximal operator M^+. We will review the definitions of M^+ and the remainder of the one-sided operators studied in this paper in § 2. We would like to observe as well that the assumption $c - b < b - a$ can be dropped. Assume that $b - a \leqslant c - b$. Let $\bar{a} < a$ such that $a - \bar{a} = c - a$. Note, that then, $b \in (\bar{a}, c)$, and $c - b < b - \bar{a}$. On the other hand observe that $M^+ \chi_{(\bar{a},c)} \simeq M^+ \chi_{(a,c)}$ and hence the C_p^+ condition would hold just with a larger constant C but without the restriction $c - b < b - a$.

The purpose of this paper is to provide a one-sided counterpart of Yabuta's characterization (1.2) and to derive a number of new results relying upon it. The precise statement of our theorem is the following.

THEOREM 1.2. *Let $1 < p < q < \infty$. If $f \in L^{p_0}(\mathbb{R})$ for some $1 < p_0 < \infty$ and $w \in C_q^+$ then*

$$\|M^+ f\|_{L^p(w)} \lesssim \|M^{\sharp,+} f\|_{L^p(w)},$$

provided the left-hand side of the estimate is finite. Conversely if the preceding estimate holds, then $w \in C_p^+$.

We would like to note that the corresponding counterpart for T^- operators holds as well. However, here and throughout the remainder of this paper we will just deal with the case of T^+ operators.

Exploiting the approach in [**6**] we shall derive a number of consequences of this result. Among them we will recover the result for one-sided Calderón–Zygmund singular integrals due to de la Torre and Riveros that we stated above. We present those results in § 3.

The remainder of the paper is organized as follows. We devote § 2 to gather some results and definitions that will be useful throughout the remainder of the work. In § 3 we present the applications of theorem 1.2, namely counterparts of (1.4) for some other one-sided operators, and even for one-sided Calderón–Zygmund singular integrals themselves. Section 4 is devoted to the proof of theorem 1.2. Additionally we provide an appendix settling a suitable Cotlar inequality that we have not been

able to find in the literature and that will be useful for us to recover and generalize [**33**, theorem 1].

2. Preliminaries and definitions

We recall that the one-sided maximal function M^+ is defined, as we noted in the introduction, as

$$M^+ f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} |f|$$

and the sharp maximal function $M^{\sharp,+}$, that was introduced in [**27**], as

$$M^{\sharp,+} f(x) = \sup_{h>0} \frac{1}{h} \int_x^{x+h} \left(f(y) - \frac{1}{h} \int_{x+h}^{x+2h} f \right)^+ \mathrm{d}y.$$

Another class of operators that we will be dealing with and that have already appeared in the previous section are one-sided singular integral operators that were introduced in [**1**]. We say that a function $K \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$ is a Calderón–Zygmund kernel if the following properties hold.

(1) There exists a finite constant B_1 such that

$$\left| \int_{\varepsilon < |x| < N} K(x) \, \mathrm{d}x \right| \leqslant B_1$$

for all $0 < \varepsilon < N$. Furthermore, $\lim_{\varepsilon \to 0^+} \int_{\varepsilon < |x| < N} K(x) \, \mathrm{d}x$ exists.

(2) There exists a constant B_2 such that

$$|K(x)| \leqslant \frac{B_2}{|x|}$$

for all $x \neq 0$.

(3) There exists a finite constant B_3 such that

$$|K(x-y) - K(x)| \leqslant B_3 \frac{|y|}{|x|^2} \tag{2.1}$$

for all x and y with $|x| > 2|y| > 0$.

We say that T^+ is a one-sided Calderón–Zygmund singular integral if

$$T^+ f(x) = \lim_{\varepsilon \to 0} \int_{x+\varepsilon}^{\infty} K(x-y) f(y) \, \mathrm{d}y \tag{2.2}$$

where K is a Calderón-Zygmund kernel with support in $\mathbb{R}^-$.

We would like to emphasize that this kind of operators are Calderón–Zygmund operators, and hence they have all the usual properties of operators in that class,

but with the extra feature that K is supported in $\mathbb{R}^-$. Examples of such operators are provided in [**1**].

Replacing (2.1) by some other smoothness conditions we obtain some more operators. For instance, we may assume that there exist numbers $c_r, C_r > 0$ such that for any $y \in \mathbb{R}$ and $R > c_r |y|$,

$$\sum_{m=1}^{\infty} 2^m R \left(\frac{1}{2^m R} \int_{2^m R < |x| \leqslant 2^{m+1} R} |K(x-y) - K(x)|^r \, \mathrm{d}x \right)^{\frac{1}{r}} \leqslant C_r \qquad (2.3)$$

if $1 \leqslant r < \infty$ and

$$\sum_{m=1}^{\infty} 2^m R \sup_{2^m R < |x| \leqslant 2^{m+1} R} |K(x-y) - K(x)| \leqslant C_\infty$$

if $r = \infty$. If K satisfies an L^r-Hörmander condition we say that $K \in \mathcal{H}_r$. This yields that we may define an operator T^+ exactly as we did in (2.2), but with K satisfying (2.3) instead of (2.1). We may go even further. Let us recall first the notion of Orlicz average. Let $A : [0, \infty) \to [0, \infty)$ a Young function, namely a convex function such that $A(0) = 0$, $A(1) = 1$ and $\lim_{t \to \infty} A(t) = \infty$. Given a measurable set E we define the average of f over E with respect to A as

$$\|f\|_{A,E} = \inf \left\{ \lambda > 0 \; : \; \frac{1}{|E|} \int_E A\left(\frac{|f(x)|}{\lambda} \right) \mathrm{d}x \leqslant 1 \right\}.$$

Relying upon that definition we may define the maximal function M_A^+ as follows

$$M_A^+ f(x) = \sup_{h > 0} \|f\|_{A,[x,x+h]}.$$

It is also worth mentioning that we can define a function associated to A, that we call $\overline{A}$, which turns out to be a Young function as well and satisfies the following inequalities

$$t \leqslant \overline{A}^{-1}(t) A^{-1}(t) \leqslant 2t.$$

Furthermore it can be shown that if $A_1, A_2, \ldots, A_n$ are Young functions such that

$$A_1^{-1}(t) \ldots A_n^{-1}(t) \lesssim t$$

then

$$\frac{1}{|E|} \int_E |f_1 \cdots f_n| \lesssim \|f_1\|_{A_1,E} \cdots \|f_n\|_{A_n,E}.$$

Coming back to the previous discussion, as we mentioned above, we may define a class of kernels generalizing the L^r-Hörmander condition. Given a Young function A we say that it satisfies an L^A-Hörmander condition if there exist $c_A, C_A \geqslant 1$ such that for any $x \in \mathbb{R}$ and $R > c_A |x|$

$$\sum_{m=1}^{\infty} 2^m R \|(K(x - \cdot) - K(-\cdot)) \chi_{2^m R < |\cdot| \leqslant 2^{m+1} R}\|_{A, B(0, 2^{m+1} R)} \leqslant C_A. \qquad (2.4)$$

If K satisfies this condition we say that $K \in \mathcal{H}_A$. In order to be able to deal with commutators we introduce another condition. We say that $K \in \mathcal{H}_{A,k}$ if there exist

$c_{A,k}, C_{A,k} \geqslant 1$ such that for any $x \in \mathbb{R}$ and $R > c_{A,k}|x|$

$$\sum_{m=1}^{\infty} 2^m R m^k \|(K(x - \cdot) - K(-\cdot))\chi_{2^m R < |\cdot| \leqslant 2^{m+1} R}\|_{A, B(0, 2^{m+1} R)} \leqslant C_{A,k}.$$

In both cases, whether $K \in \mathcal{H}_A$ or $K \in \mathcal{H}_{A,k}$, we may define a singular integral operator exactly as we did in (2.2). Those classes of kernels were introduced and studied in [**17**, **19**].

We would like to end recalling that we may define a maximal version of any of the singular integral operators that we have just presented in this section as follows

$$(T^+)^* f(x) = \sup_{\varepsilon > 0} \left| T_\varepsilon^+ f(x) \right| = \sup_{\varepsilon > 0} \left| \int_{\varepsilon + x}^{\infty} K(x - y) f(y) \, dy \right|.$$

3. Corollaries of the main theorem

As we announced in the previous section we will derive a number of applications of theorem 1.2. Our use of that theorem will rely upon the following lemma.

LEMMA 3.1. *Let $0 < p < \infty$. If $\delta \in (0, p)$ then, if $w \in C_\rho^+$ with $\rho > \frac{p}{\delta}$, we have that*

$$\|M_\delta^+ f\|_{L^p(w)} \lesssim \|M_\delta^{\sharp,+} f\|_{L^p(w)}$$

where $M_\delta^+(f) = (M^+(|f|^\delta))^{\frac{1}{\delta}}$ and $M_\delta^{\sharp,+}(f) = (M^{\sharp,+}(|f|^\delta))^{\frac{1}{\delta}}$.

Proof. Observe that, since $\delta \in (0, p)$ we have that $\frac{p}{\delta} > 1$. Taking that into account and the fact that $w \in C_\rho^+$ we have by theorem 1.2 that

$$\begin{aligned}
\|M_\delta^+ f\|_{L^p(w)}^p &= \int_{\mathbb{R}} M^+(|f|^\delta)^{\frac{p}{\delta}}(x) w(x) \, dx \\
&\lesssim \int_{\mathbb{R}} M^{\sharp,+}(|f|^\delta)^{\frac{p}{\delta}}(x) w(x) \, dx \\
&= \|M_\delta^{\sharp,+} f\|_{L^p(w)}^p
\end{aligned}$$

and we are done. $\qquad \square$

3.1. Singular integral operators, L^A-Hörmander operators and their commutators

In this section we present our results for singular integral operators, L^A-Hörmander operators and their commutators. We will provide some full arguments here, that we shall omit for the remainder of the corollaries since they will be analogous to the ones provided here.

We begin recalling that for one-sided Calderón–Zygmund singular integrals T^+ it was shown in [**22**, lemma 1] that for $0 < \delta < 1$,

$$M_\delta^{\sharp,+}(T^+ f) \lesssim M^+ f. \tag{3.1}$$

Using that estimate we can derive the following result.

THEOREM 3.2. *Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$. Then*

$$\|T^+ f\|_{L^p(w)} \lesssim \|M^+ f\|_{L^p(w)}.$$

Proof. Let $\delta \in (0,1)$ such that $1 < \frac{p}{\delta} < \max\{p,1\} + \varepsilon$. For that choice of δ, taking into account (3.1) and lemma 3.1 with $\rho = \max\{p,1\} + \varepsilon$, we have that

$$\|T^+ f\|_{L^p(w)} \lesssim \|M_\delta^+(T^+ f)\|_{L^p(w)} \lesssim \|M_\delta^{\sharp,+}(T^+ f)\|_{L^p(w)}$$
$$\leqslant \|M^+ f\|_{L^p(w)}$$

and we are done. $\qquad\square$

Recall that the commutator of a linear operator T and a locally integrable function b is defined as

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x).$$

The iterated commutator T_b^k consists precisely in iterating the commutator.

$$T_b^k f(x) = [b, T_b^{k-1}]f(x)$$

where $T_b^0 f(x) = Tf(x)$. For the commutator and the iterated commutator of $b \in BMO$ and a Calderón–Zygmund one-sided singular integral T^+ again in [**22**, lemma 1], it was shown that for $0 < \delta < \gamma < 1$, we have that

$$M_\delta^{\sharp,+}((T^+)_b^k f)(x) \lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^{k-j} M_\gamma^+((T^+)_b^j)f(x) + \|b\|_{BMO}^k (M^+)^{k+1} f(x). \quad (3.2)$$

Relying upon that pointwise estimate we have the following theorem.

THEOREM 3.3. *Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$. Then*

$$\|(T^+)_b^k f\|_{L^p(w)} \lesssim \|b\|_{BMO}^k \|(M^+)^{k+1} f\|_{L^p(w)}.$$

Proof. Observe that it suffices to show that for $\delta_1 \in (0,1)$ such that $1 < \frac{p}{\delta_1} < \max\{p,1\} + \varepsilon$ the following holds

$$\|M_{\delta_1}^+(T^+)_b^k f\|_{L^p(w)} \lesssim \|b\|_{BMO}^k \|(M^+)^{k+1} f\|_{L^p(w)}. \quad (3.3)$$

We proceed by induction. Assume first that $k = 1$. Let $0 < \delta_1 < \delta_2 < 1$ such that $1 < \frac{p}{\delta_i} < \max\{p,1\} + \varepsilon$. We have that, taking into account (3.2) and lemma 3.1,

$$\|M_{\delta_1}^+(T^+)_b^1 f\|_{L^p(w)} \lesssim \left\| M_{\delta_1}^{\sharp,+}\left((T^+)_b^1 f\right) \right\|_{L^p(w)}$$
$$\lesssim \|b\|_{BMO} \|M_{\delta_2}^+(T^+ f)\|_{L^p(w)} + \|b\|_{BMO} \|(M^+)^2 f\|_{L^p(w)}$$
$$\lesssim \|b\|_{BMO} \|M^+ f\|_{L^p(w)} + \|b\|_{BMO} \|(M^+)^2 f\|_{L^p(w)}$$
$$\lesssim \|b\|_{BMO} \|(M^+)^2 f\|_{L^p(w)}$$

where the estimate for $\|M_{\delta_2}(T^+ f)\|_{L^p(w)}$ follows by the same argument provided in the proof of theorem 3.2.

Assume now that (3.3) holds for $1, 2, \ldots k-1$. Let $0 < \delta_1 < \delta_2 < 1$ such that $1 < \frac{p}{\delta_i} < \max\{p, 1\} + \varepsilon$. Then, again by (3.2) and lemma 3.1,

$$\|M_{\delta_1}^+(T^+)_b^k f\|_{L^p(w)} \lesssim \left\|M_{\delta_1}^{\sharp,+}\left((T^+)_b^k f\right)\right\|_{L^p(w)}$$

$$\lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^{k-j} \|M_{\delta_2}^+((T^+)_b^j) f(x)\|_{L^p(w)}$$

$$+ \|b\|_{BMO}^k \|(M^+)^{k+1} f\|_{L^p(w)}.$$

$$\lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^k \|(M^+)^j f(x)\|_{L^p(w)} + \|b\|_{BMO}^k \|(M^+)^{k+1} f\|_{L^p(w)}$$

$$\lesssim \|b\|_{BMO}^k \|(M^+)^{k+1} f\|_{L^p(w)}$$

and we are done. □

With analogous arguments relying upon the corresponding pointwise sharp inequality we may settle the following result. We recall that if $\bar{A}$ is a Young function and T^+ is an operator associated to a kernel $K \in H_{\bar{A}}$ with support in $(-\infty, 0)$, then as it was established in [**17**, p. 505]

$$M_\delta^{\sharp,+}((T^+ f)(x)) \lesssim M_A^+ f(x)$$

and if, A and B are Young functions, $\overline{C}^{-1}(t) = e^{t^{\frac{1}{k}}}$ with k a positive integer such that $A^{-1}(t)B^{-1}(t)\overline{C}^{-1}(t) \leqslant t$ for $t \geqslant 1$ and $K \in H_B \cap H_{\bar{A},k}$ then, for $0 < \delta < \gamma < 1$,

$$M_\delta^{\sharp,+}((T^+)_b^k f)(x) \lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^{k-j} M_\gamma^+((T^+)_b^j) f(x) + \|b\|_{BMO}^k M_A^+ f(x). \qquad (3.4)$$

We remit the reader to [**19**, § 5.3].

Arguing as above, we have the following results.

THEOREM 3.4. *Let A be a Young function and assume that $K \in \mathcal{H}_{\bar{A}}$. Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$. Then*

$$\|T^+ f\|_{L^p(w)} \lesssim \|M_A^+ f\|_{L^p(w)}.$$

THEOREM 3.5. *Let k be a positive integer and assume that A and B are Young functions, and $A^{-1}(t)B^{-1}(t)\overline{C}^{-1}(t) \leqslant t$ for $t \geqslant 1$ where $\overline{C}^{-1}(t) = e^{t^{\frac{1}{k}}}$. Assume also that $K \in \mathcal{H}_B \cap \mathcal{H}_{\bar{A},k}$ and that $b \in BMO$. Then, if $0 < p < \infty$, $\varepsilon > 0$ and $w \in C^+_{\max\{p,1\}+\varepsilon}$, we have that*

$$\|(T^+)_b^k f\|_{L^p(w)} \lesssim \|b\|_{BMO}^k \|M_A^+ f\|_{L^p(w)}.$$

We would like to end the section providing a result for maximal singular integral operators.

THEOREM 3.6. *Let A be a Young function and assume that $K \in \mathcal{H}_{\overline{A}}$. Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$. Then*

$$\|(T^+)^* f\|_{L^p(w)} \lesssim \|M_A^+ f\|_{L^p(w)}.$$

Before settling this result, observe that if K satisfies (2.1) in particular $K \in \mathcal{H}_\infty$, and consequently, the preceding result recovers the main result in [**33**].

Proof of theorem 3.6. Observe that by the Cotlar type inequality in theorem A.1, we have that for any $\delta \in (0,1)$.

$$\|(T^+)^* f\|_{L^p(w)} \lesssim \|M_\delta^+ (T^+ f)\|_{L^p(w)} + \|M_A^+ f\|_{L^p(w)}$$

and hence it suffices to deal with the first term. An analogous argument to the one provided to settle theorem 3.2, choosing a suitable δ, shows that

$$\|M_\delta^+ (T^+ f)\|_{L^p(w)} \lesssim \|M_A^+ f\|_{L^p(w)}$$

and we are done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. The differential transform operator

Given $\{v_j\} \in \ell^\infty$ we define

$$T^+ f(x) = \sum_{j \in \mathbb{Z}} v_j (D_j f(x) - D_{j-1} f(x)), \quad D_j f(x) = \frac{1}{2^j} \int_x^{x+2^j} f(t) \, \mathrm{d}t.$$

As the authors point out in [**19**] this operator, that was previously studied in [**3**, **12**], arises when studying the rate of convergence of the averages $D_j f$. Note that $D_j f \to f$ a.e. when $j \to -\infty$ and that $D_j f \to 0$ when $j \to \infty$ for appropriate f.

Observe that T^+ is a one-sided singular integral since $T^+ f = K * f$ for K supported on $(-\infty, 0)$ and defined as

$$K(x) = \sum_{j \in \mathbb{Z}} v_j \left(\frac{1}{2^j} \chi_{(-2^j, 0)}(x) - \frac{1}{2^{j-1}} \chi_{(-2^{j-1}, 0)}(x) \right).$$

As it was stated in [**19**, remark 4.11], it is possible to show that $K \in \mathcal{H}_{A,k}$ with $A(t) = \exp\left(\frac{t^{\frac{1}{1+k}}}{(\log t)^{\frac{1+\varepsilon}{1+k}}} \right)$, and hence by (3.4) we have that if $b \in BMO$ and k is a non-negative integer, for $0 < \delta < \gamma < 1$,

$$M_\delta^{\sharp,+} ((T^+)_b^k f)(x) \lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^{k-j} M_\gamma^+ ((T^+)_b^j) f(x)$$

$$+ \|b\|_{BMO}^k M_{L(\log L)^{1+k}(\log\log L)^{1+\varepsilon}}^+ f(x).$$

where the first term is interpreted as 0 if $k = 0$. Then, arguing as in the preceding section we have the following result.

THEOREM 3.7. *Let k be a non-negative integer. Then, if $0 < p < \infty$, $\varepsilon > 0$ and $w \in C^+_{\max\{p,1\}+\varepsilon}$, we have that*

$$\|(T^+)^k_b f\|_{L^p(w)} \lesssim \|b\|^k_{BMO} \|M^+_{L(\log L)^{1+k}(\log \log L)^{1+\varepsilon}} f\|_{L^p(w)}.$$

3.3. The one-sided discrete square function and its commutator

We recall that the one-sided discrete square function is defined as follows. If f is locally integrable in $\mathbb{R}$ and $s > 0$ we consider the averages

$$A_s f(x) = \frac{1}{s} \int_x^{x+s} f(y)\, \mathrm{d}y.$$

Hence the one-sided discrete square function of f is given by

$$S^+ f(x) = \left(\sum_{n \in \mathbb{Z}} |A_{2^n} f(x) - A_{2^{n-1}} f(x)|^2 \right)^{\frac{1}{2}}.$$

This operator was studied in [**36**] and [**17**]. In [**18**] the authors deal with the following operator

$$\mathcal{O}^+ f(x) = \left(\sum_{n \in \mathbb{Z}} \sup_{s \in [2^n, 2^{n+1})} |A_{2^n} f(x) - A_s f(x)|^2 \right)^{\frac{1}{2}},$$

which dominates pointwise $S^+ f$, and show [**18**, Eq. (3.1) in p. 581] that if $0 < \delta < 1$

$$M^{\sharp,+}_\delta (\mathcal{O}^+ f)(x) \lesssim M^+_{L \log L} f(x).$$

This fact allows them to settle the corresponding Coifman–Fefferman estimate. Here, arguing as we did to settle theorem 3.2, we have the following result.

THEOREM 3.8. *Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$. Then*

$$\|Gf\|_{L^p(w)} \lesssim \|M^+_{L \log L} f\|_{L^p(w)}$$

where G stands either for S^+ or for $\mathcal{O}^+$.

Further assuming that $b \in BMO$ the authors also study the commutators associated to the operators above. In [**18**, lemma 4.6] it is shown that, for $0 < \delta < \gamma < 1$,

$$M^{\sharp,+}_\delta ((\mathcal{O}^+)^k_b f)(x) \lesssim \sum_{j=0}^{k-1} \|b\|^{k-j}_{BMO} M^+_\gamma ((\mathcal{O}^+)^j_b) f(x) + M^+_{L(\log L)^{1+k}} f(x).$$

As a consequence we can derive the following result.

THEOREM 3.9. *Let $0 < p < \infty$ and $\varepsilon > 0$ and assume that $w \in C^+_{\max\{p,1\}+\varepsilon}$ and that $b \in BMO$. If k is a positive integer, then*

$$\|G^k_b f\|_{L^p(w)} \lesssim \|b\|^k_{BMO} \|M^+_{L(\log L)^{1+k}} f\|_{L^p(w)}$$

where G stands either for S^+ or for $\mathcal{O}^+$.

3.4. Riemann–Liouville and Weyl fractional integral operators and their commutators

We recall that given $0 < \alpha < 1$ and locally integrable functions f and b, the Weyl fractional integral and its commutators are defined as

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\,\mathrm{d}y \quad \text{and}$$

$$(I_\alpha^+)_b^k f(x) = \int_x^\infty (b(x) - b(y))^k \frac{f(y)}{(y-x)^{1-\alpha}}\,\mathrm{d}y$$

respectively. Analogously we define the Riemann–Liouville fractional integral and its commutators as

$$I_\alpha^- f(x) = \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y \quad \text{and}$$

$$(I_\alpha^-)_b^k f(x) = \int_{-\infty}^x (b(x) - b(y))^k \frac{f(y)}{(x-y)^{1-\alpha}}\,\mathrm{d}y.$$

In [**2**, lemma 4.1] it was shown that for every non-negative k, if $b \in BMO$ and $0 < \delta < \gamma < 1$,

$$M_\delta^{\sharp,+}((I_\alpha^+)_b^k f)(x) \lesssim \sum_{j=0}^{k-1} \|b\|_{BMO}^{k-j} M_\gamma^+((I_\alpha^+)_b^j)f(x) + \|b\|_{BMO}^k M_{(\alpha),L(\log L)^k}^+ f(x)$$

where

$$M_{(\alpha),L(\log L)^k}^+ f(x) = \sup_{h>0} h^\alpha \|f\|_{L(\log L)^k,(x,x+h)}$$

and the first term in the right-hand side is interpreted as 0 if $k = 0$. Relying upon that $M_\delta^{\sharp,+}$ estimate and arguing as in the proofs of theorems 3.2 and 3.3 it is possible to settle the following result.

THEOREM 3.10. *Let k be a non-negative integer. Then, if $0 < p < \infty$, $\varepsilon > 0$ and $w \in C_{\max\{p,1\}+\varepsilon}^+$ we have that*

$$\|(I_\alpha^+)_b^k f\|_{L^p(w)} \lesssim \|b\|_{BMO}^k \|M_{(\alpha),L(\log L)^k}^+\|_{L^p(w)}.$$

4. Proof of theorem 1.2

4.1. Sufficiency

By a standard approximation argument it is enough to settle the result assuming that $f \in L_c^\infty$.

To settle the sufficiency part in theorem 1.2 we need to borrow two lemmas from [**33**]. The first one is the following.

LEMMA 4.1 [**33**, lemma 1]. *Assume that $w \in C_q^+$ with $1 < q < \infty$. Then, for any $\delta > 0$ there exists $c(\delta)$ such that for any disjoint family of intervals J_j contained in $I = (a, b)$ we have that*

$$\int_I \sum_j (M^+ \chi_{J_j})^q w \leqslant c(\delta) w(I) + \delta \int_{\mathbb{R}} (M^+ \chi_I)^q w$$

and

$$\int_{\mathbb{R}} \sum_j (M^+ \chi_{J_j})^q w \lesssim \int_{\mathbb{R}} (M^+ \chi_I)^q w.$$

To state the next lemma we need to define a new operator, $M_{p,q}^+$. Let f be an non-negative measurable function. Let us consider

$$\Omega_k = \left\{ x \in \mathbb{R} : f(x) > 2^k \right\} = \bigcup_i I_i^k$$

where I_i^k are the connected components of Ω_k. Then

$$(M_{p,q}^+ f(x))^p = \sum_{i,k} 2^{pk} (M^+ (\chi_{I_i^k})(x))^q.$$

Having this definition at our disposal we present the second lemma we borrow from [**33**].

LEMMA 4.2 [**33**, lemma 2]. *Let $1 < p < q < \infty$, $w \in C_q^+$ and f non-negative, bounded and of compact support. Then*

$$\int_{\mathbb{R}} M_{p,q}^+ (M^+ f) w \lesssim \int_{\mathbb{R}} (M^+ f)^p w.$$

Having those lemmas at our disposal we are in the position to settle theorem 1.2.

Proof of theorem 1.2. Let $\Omega_k = \{x : M^+ f(x) > 2 \cdot 2^k\} = \bigcup_j J_j^k$ where J_j^k are the connected components of Ω_k. Let us fix $(a, b) = J_j^k$. We partition (a, b) as follows. Let $x_0 = a$ and choose x_{i+1} such that $x_{i+1} - x_i = b - x_{i+1}$ and let $I_i^k = (x_i, x_{i+1})$. By the good-λ inequality established in [**27**, theorem 4], we have that

$$|E_i^k| = \left| \left\{ x \in I_i^k : M^+ f(x) > 2^{k+1}, M^{+,\sharp} f(x) \leqslant \gamma 2^k \right\} \right| \leqslant C\gamma |I_{i+1}^k| \quad 0 < \gamma < 1.$$

From the C_q^+ condition it follows that

$$w(E_i^k) \leqslant C\gamma^\varepsilon \int_{\mathbb{R}} \left(M^+ \chi_{I_i^k \cup I_{i+1}^k} \right)^q w.$$

Summing on i and taking into account lemma 4.1 we have that

$$w \left(\left\{ x \in J_j^k : M^+ f(x) > 2^{k+1}, M^{\sharp,+} f(x) \leqslant \gamma 2^k \right\} \right)$$

$$\leqslant C\gamma^\varepsilon \sum_i \int_{\mathbb{R}} \left(M^+ \chi_{I_i^k \cup I_{i+1}^k} \right)^q w \lesssim C\gamma^\varepsilon \int_{\mathbb{R}} \left(M^+ \chi_{J_j^k} \right)^q w.$$

Now summing over all j,

$$w\left(\{x \in \Omega_k \,:\, M^+ f(x) > 2^{k+1}, M^{+,\sharp} f(x) \leqslant \gamma 2^k\}\right)$$
$$\leqslant C\gamma^\varepsilon \sum_j \int_{\mathbb{R}} \left(M^+ \chi_{J_j^k}\right)^q w.$$

Having those estimates at our disposal we can argue as follows.

$$\int_{\mathbb{R}} (M^+ f(x))^p w(x)\,\mathrm{d}x \lesssim \sum_{k \in \mathbb{Z}} 2^{kp} w(\Omega_k)$$

$$\leqslant \sum_{k \in \mathbb{Z}} 2^{kp} w\left(\{x \in \Omega_k \,:\, M^+ f(x) > 2^{k+1}, M^{+,\sharp} f(x) \leqslant \gamma 2^k\}\right)$$

$$+ \sum_{k \in \mathbb{Z}} 2^{kp} w\left(\{x \in \mathbb{R} \,:\, M^{+,\sharp} f(x) > \gamma 2^k\}\right)$$

$$\leqslant C\gamma^\varepsilon \sum_{k \in \mathbb{Z}} 2^{kp} \sum_j \int_{\mathbb{R}} \left(M^+ \chi_{J_j^k}\right)^q w + \sum_{k \in \mathbb{Z}} 2^{kp} w\left(\{x \in \mathbb{R} \,:\, M^{+,\sharp} f(x) > \gamma 2^k\}\right)$$

$$\leqslant C\gamma^\varepsilon \int_{\mathbb{R}} M_{p,q}^+ (M^+ f)(x) w(x)\,\mathrm{d}x + c_\gamma \int_{\mathbb{R}} (M^{+,\sharp} f(x))^p w(x)\,\mathrm{d}x$$

$$\leqslant C\gamma^\varepsilon \int_{\mathbb{R}} (M^+ f(x))^p w(x)\,\mathrm{d}x + c_\gamma \int_{\mathbb{R}} (M^{+,\sharp} f(x))^p w(x)\,\mathrm{d}x$$

where in the last step we have used lemma 4.2.

Observe that if $\int_{\mathbb{R}} (M^+ f(x))^p w(x)\,\mathrm{d}x < \infty$, choosing γ small enough the desired estimate follows. We end the proof observing that for $f \in L_c^\infty$

$$\int_{\mathbb{R}} \left(M^{+,\sharp} f(x)\right)^p w(x)\,\mathrm{d}x < \infty$$

implies

$$\int_{\mathbb{R}} \left(M^+ f(x)\right)^p w(x)\,\mathrm{d}x < \infty.$$

Indeed, note that since $f \in L_c^\infty$ we may assume that $\operatorname{supp} f \subset [a,b]$. For $x > b$ we have that $M^{+,\sharp} f(x) = M^+ f(x) = 0$, and for $x \to -\infty$, $M^+ f(x) \simeq M^{+,\sharp} f(x) \simeq \frac{1}{|x|}$. $\qquad\square$

4.2. Necessity

The proof of the necessity will rely upon the following lemma, which is a one-sided version of some of the results in [**30**], namely theorem 4.1, lemma 5.1 and the proof of theorem 1.2 and of [**34**, lemma 1].

LEMMA 4.3. *If for every $a < b < c$ with $c - b < b - a$ and $E \subset (a,b)$*

$$w(E) \lesssim \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \int_{\mathbb{R}} M^+(\chi_{(a,c)})^p w(x)\,\mathrm{d}x \tag{4.1}$$

then $w \in C_p^+$.

Before settling the lemma we show how to derive from it the necessity in theorem 1.2.

Proof of the necessity in theorem 1.2. Assume that for a certain $1 < p < \infty$ and a weight w,

$$\|M^+ f\|_{L^p(w)} \lesssim \|M^{\sharp,+} f\|_{L^p(w)}.$$

Let $I = (a, c)$ an interval. Let $a < b < c$. Assume that E is a measurable set contained in (a, b). Let us define

$$f(x) = \log^+ \left(\frac{|(b,c)|}{|E|} M^-(\chi_E)(x) \right) + \chi_I(x)$$

$$= g(x) + \chi_I(x).$$

Analogously as in [**34**, theorem A] and [**37**] we have that

$$\frac{1}{|(a,c)|} \int_{(a,c)} \log^+ \left(\frac{|(b,c)|}{|E|} M^-(\chi_E) \right) \lesssim 1 \tag{4.2}$$

$$\|f\|_{BMO^+} \lesssim 1 \tag{4.3}$$

$$f(x) = \log^+ \left(\frac{|(b,c)|}{|E|} \right) + 1 \quad \text{a.e.} \quad x \in E. \tag{4.4}$$

where $\|f\|_{BMO^+} = \|M^{\sharp,+} f\|_{L^\infty}$. Note (4.4) readily follows from the definition of f. Assume by now that (4.2) and (4.3) hold as well. We shall show that this is the case at the end of this proof. We continue as follows.

Observe that $M^-(\chi_E) \leqslant \frac{|E|}{\text{dist}(x,E)}$ for $x \notin E$. In particular, if $x \geqslant c$ we have that

$$M^-(\chi_E)(x) \leqslant \frac{|E|}{\text{dist}(x,E)} \leqslant \frac{|E|}{x-b} \leqslant \frac{|E|}{c-b}.$$

This yields $g(x) = 0$ if $x \geqslant c$. On the other hand, if $x \leqslant a$ then also $M^-(\chi_E)(x) = 0$. And consequently $g(x) = 0$. Hence supp $f \subset (a,c)$. Now we observe that if $x > c$ we have that $M^+ f(x) = 0$ and $M^{\sharp,+} f(x) = 0$. If $x < c$ we have two cases. If $x \in (a-|I|, c)$ then, by (4.3)

$$M^{\sharp,+} f(x) \lesssim \|f\|_{BMO^+} \lesssim M^+(\chi_{(a,c)})(x).$$

If $x < a - |I|$ then $M^+ \chi_{(a,c)}(x) = \frac{c-a}{c-x}$ and $c - x \geqslant 2|I|$, from which it follows that $a - x = c - x - |I| \geqslant \frac{c-x}{2}$. Hence, we have that by (4.2)

$$M^{\sharp,+} f(x) \lesssim M^+ f(x) \leqslant M^+ g(x) + M^+(\chi_{(a,c)})(x)$$

$$\leqslant \frac{\int_a^c g(y)\,dy}{a-x} + M^+(\chi_{(a,c)})(x)$$

$$\lesssim \frac{c-a}{c-x} \frac{1}{c-a} \int_a^c g(y)\,dy + M^+(\chi_{(a,c)})(x)$$

$$\lesssim \frac{c-a}{c-x} + M^+(\chi_{(a,c)})(x) \lesssim M^+(\chi_{(a,c)})(x).$$

Gathering the estimates above we have that

$$M^{\sharp,+} f(x) \lesssim M^+(\chi_{(a,c)})(x) \quad x \in \mathbb{R}.$$

Taking into account the preceding estimate and (4.4), we have that

$$
\begin{aligned}
w(E) &= \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p \int_E w(x)\,\mathrm{d}x \\
&= \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \int_E |f(x)|^p w(x)\,\mathrm{d}x \\
&\leqslant \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \int_{\mathbb{R}} (M^+ f(x))^p w(x)\,\mathrm{d}x \\
&\lesssim \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \int_{\mathbb{R}} |M^{\sharp,+} f(x)|^p w(x)\,\mathrm{d}x \\
&\lesssim \frac{1}{\left[1 + \log^+\left(\frac{|(b,c)|}{|E|}\right)\right]^p} \int_{\mathbb{R}} (M^+ \chi_{(a,c)})^p w(x)\,\mathrm{d}x
\end{aligned}
$$

and we are done.

As we mentioned above, we are left with settling (4.2) and (4.3).

To establish (4.2), note that

$$\frac{1}{|(a,c)|} \int_{(a,c)} \log^+\left(\frac{|(b,c)|}{|E|} M^-(\chi_E)\right) = \frac{1}{|(a,c)|} \int_{(a,c)\cap F} \log\left(\frac{|(b,c)|}{|E|} M^-(\chi_E)\right)$$

where $F = \left\{\frac{|(b,c)|}{|E|} M^-(\chi_E) \geqslant 1\right\}$. If $|(a,c) \cap F| = 0$ then there is nothing to show. Hence we may assume that $|(a,c) \cap F| \neq 0$. Taking that into account we argue as follows. Let $\delta \in (0,1)$. Then

$$
\begin{aligned}
&\frac{1}{|(a,c)|} \int_{(a,c)\cap F} \log\left(\frac{|(b,c)|}{|E|} M^-(\chi_E)\right) \\
&= \frac{1}{\delta} \frac{|(a,c)\cap F|}{|(a,c)|} \frac{1}{|(a,c)\cap F|} \int_{(a,c)\cap F} \log\left(\left(\frac{|(b,c)|}{|E|}\right)^{\delta} M^-(\chi_E)^{\delta}\right) \\
&\leqslant \frac{1}{\delta} \frac{|(a,c)\cap F|}{|(a,c)|} \log\left(\frac{1}{|(a,c)\cap F|} \int_{(a,c)\cap F} \left(\frac{|(b,c)|}{|E|}\right)^{\delta} M^-(\chi_E)^{\delta}\right) \\
&= \frac{|(a,c)\cap F|}{|(a,c)|} \log\left(\frac{|(b,c)|}{|E|}\left(\frac{1}{|(a,c)\cap F|} \int_{(a,c)\cap F} M^-(\chi_E)^{\delta}\right)^{\frac{1}{\delta}}\right)
\end{aligned}
$$

$$\leqslant \frac{|(a,c) \cap F|}{|(a,c)|} \log \left(\frac{|(b,c)|}{|E|} c_\delta \frac{|E|}{|(a,c) \cap F|} \right)$$

$$= \frac{|(a,c) \cap F|}{|(a,c)|} \log \left(\frac{|(b,c)|}{|(a,c) \cap F|} \right)$$

$$\leqslant \frac{|(a,c) \cap F|}{|(a,c)|} \log \left(\frac{|(a,c)|}{|(a,c) \cap F|} \right) \leqslant 1$$

where in the third line we used Jensen's inequality, in the fifth Kolmogorov's inequality and in the last one that $\log(t) \leqslant t$ for every $t \geqslant 1$.

Now we focus on (4.3). First we observe that

$$\|f\|_{BMO^+} \leqslant \|g\|_{BMO^+} + \|\chi_I\|_{BMO^+} \leqslant \|g\|_{BMO^+} + 3$$

so it suffices to provide a bound for $\|g\|_{BMO^+}$. It is not hard to check that

$$\| \max \{h_1, h_2\} \|_{BMO^+} \lesssim \max \{\|h_1\|_{BMO^+}, \|h_2\|_{BMO^+}\}.$$

Hence

$$\|g\|_{BMO^+} = \left\| \max \left\{ \log \left(\frac{|(b,c)|}{|E|} M^-(\chi_E) \right), 0 \right\} \right\|_{BMO^+}$$

$$\lesssim \left\| \log \left(\frac{|(b,c)|}{|E|} M^-(\chi_E) \right) \right\|_{BMO^+}$$

Now we observe that

$$\left\| \log \left(\frac{|(b,c)|}{|E|} M^-(\chi_E) \right) \right\|_{BMO^+} = \left\| \log \left(\frac{|(b,c)|}{|E|} \right) + \log \left(M^-(\chi_E) \right) \right\|_{BMO^+}$$

$$= \left\| \log \left(M^-(\chi_E) \right) \right\|_{BMO^+}$$

$$= 2 \left\| \log \left(M^-(\chi_E)^{\frac{1}{2}} \right) \right\|_{BMO^+}$$

and since for every $\delta \in (0,1)$ we have that $M^-(\chi_E)(\cdot)^\delta \in A_1^+$ with constant depending just on δ, then using results in [**27**, § 2],

$$\left\| \log \left(M^-(\chi_E)^{\frac{1}{2}} \right) \right\|_{BMO^+} \lesssim 1.$$

Combining the estimates above yields (4.3). This ends the proof. $\qquad \square$

We devote the remainder of the section to settle lemma 4.3. First we will need the following lemma.

LEMMA 4.4. *Let $1 < p < \infty$. Assume that w is a weight such that*

$$\int_E w \lesssim \frac{1}{\left(1 + \log^+ \left(\frac{c-b}{|E|} \right) \right)^p} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)} \right)^p w$$

for every $a < b < c$ with $c - b < b - a$ and where E is any measurable set contained in (a,b). Then, for every family $\{I_k\}_{k=1}^n$ of disjoint subintervals of an interval I, if

we denote

$$\Delta(x) = \sum_{k=1}^{n} M^+(\chi_{I_k})^p(x),$$

then

$$\int_{\mathbb{R}} \Delta w \lesssim \frac{1}{\left(1 + \log\left(\frac{|I|}{\sum_{k=1}^{n} |I_k|}\right)\right)^{p-1}} \int_{\mathbb{R}} \left(M^+ \chi_I\right)^p w. \tag{4.5}$$

Proof. Let $I = (a, c)$ be an interval and $\{I_k\}$ a family of disjoint subintervals of I. First we note that

$$M^+(\chi_{I_k})(x) = 0$$

for each $x > c$. Then we have that

$$\int_{\mathbb{R}} \Delta w = \int_{-\infty}^{c} \Delta w = \int_{a'}^{c} \Delta w + \int_{-\infty}^{a'} \Delta w.$$

where $a' = a - |I|$. First we deal with the second term. Observe that if d_k is the right endpoint of I_k, then

$$\int_{-\infty}^{a'} \Delta w = \int_{-\infty}^{a'} \sum_{k=1}^{n} \left(M^+ \chi_{I_k}\right)^p (x) w(x)\, \mathrm{d}x \leqslant \int_{-\infty}^{a'} \sum_{k=1}^{n} \left(\frac{|I_k|}{d_k - x}\right)^p w(x)\, \mathrm{d}x$$

$$\leqslant \int_{-\infty}^{a'} \left(\sum_{k=1}^{n} \frac{|I_k|}{d_k - x}\right)^p w(x)\, \mathrm{d}x \lesssim \int_{-\infty}^{a'} \left(\sum_{k=1}^{n} \frac{|I_k|}{c - x}\right)^p w(x)\, \mathrm{d}x$$

$$\leqslant \int_{-\infty}^{a} \left(\frac{|I|}{c - x}\right)^p w(x)\, \mathrm{d}x \lesssim \int_{-\infty}^{c} (M^+ \chi_I)^p w(x)\, \mathrm{d}x.$$

Now we deal with the first term. Let j be the least integer such that

$$\log\left(\frac{\sum_{k=1}^{n} |I_k|}{|I|}\right) \leqslant j$$

and J the least integer such that for some $D \in (0, 1)$ to be chosen later,

$$\log\left(\frac{1}{D} \log\left(\frac{e|I|}{\sum_{k=1}^{n} |I_k|}\right)\right) \leqslant J.$$

Note that since $j \leqslant 0$ and $J > 0$ we have that $j < J$. Let

$$Q = \{\Delta(x) \leqslant e^j\}$$
$$S = \{e^j < \Delta(x) \leqslant e^J\}$$
$$T = \{\Delta(x) > e^J\}$$

For Q we have that

$$\int_{(a',c)\cap Q} \Delta(x) w(x)\, \mathrm{d}x \leqslant \int_{(a',c)\cap Q} e^j w(x)\, \mathrm{d}x$$

$$= e \int_{(a',c)\cap Q} e^{j-1} w(x)\, \mathrm{d}x$$

$$\leqslant e \int_{(a',c)\cap Q} e^{\log\left(\frac{\sum_{k=1}^n |I_k|}{|I|}\right)} w(x)\, \mathrm{d}x$$

$$\leqslant e \frac{\sum_{k=1}^n |I_k|}{|I|} \int_{(a',c)} w(x)\, \mathrm{d}x$$

and the right-hand side is bounded by the right-hand side of (4.5). We continue with S.

$$\int_{(a',c)\cap S} \Delta(x) w(x)\, \mathrm{d}x \leqslant \sum_{k=j}^{J-1} e^{k+1} \int_{\{\Delta(x)>e^k\}\cap(a',c)} w(x)\, \mathrm{d}x$$

$$= \sum_{k=j}^{J-1} e^{k+1} w(E_{e^k})$$

where

$$E_\lambda = \{\Delta(x) > \lambda\} \cap (a',c).$$

To continue with the argument, we borrow ideas from [**33**, p. 406]. We begin noting that there exists $B > 1$ and that we can choose the $D \in (0,1)$ above in such a way that

$$|E_\lambda| \leqslant B\, e^{-D\lambda} |(a',c)|.$$

At this point we use our hypothesis on the weight w. For that purpose we split (a',c) as follows. Let $x_0 = a'$ and let us define recursively $x_i - x_{i-1} = c - x_i$. Associated to the collection of intervals (x_i, x_{i+1}) we consider the sets

$$E_\lambda^i = E_\lambda \cap (x_i, x_{i+1}).$$

Observe that for each i we may assume that the elements that we consider in the sum $\Delta(x)$ are contained in (x_i, c) (the remaining terms are zero). Hence we have that

$$|E_\lambda^i| \leqslant B\, e^{-D\lambda} |(x_i, c)| = 4B\, e^{-D\lambda}(x_{i+2} - x_{i+1}).$$

Now we use the hypothesis for x_i, x_{i+1}, x_{i+2} and we have that

$$w(E_\lambda^i) \lesssim \frac{1}{\left(1 + \log^+\left(\frac{x_{i+2}-x_{i+1}}{|E_\lambda^i|}\right)\right)^p} \int_{-\infty}^{\infty} (M^+ \chi_{(x_i,x_{i+2})})^p w$$

$$\lesssim \frac{1}{\left(1 + \log^+\left(\frac{1}{4B\, e^{-D\lambda}}\right)\right)^p} \int_{-\infty}^{\infty} (M^+ \chi_{(x_i,x_{i+2})})^p w.$$

Summing in i we have that it follows from the definition of the partition x_i, which leads to a geometric series, that

$$\sum_i \int_{-\infty}^{\infty} \left(M^+(\chi_{(x_i, x_{i+2})})\right)^p w \leqslant C \int_{-\infty}^{\infty} \left(M^+ \chi_{(a',c)}\right)^p w.$$

Hence,

$$w(E_\lambda) \lesssim \frac{1}{\left(1 + \log^+\left(\frac{1}{4Be^{-D\lambda}}\right)\right)^p} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a',c)}\right)^p w$$

$$\lesssim \frac{1}{\left(1 + \log^+\left(\frac{1}{4B\,e^{-D\lambda}}\right)\right)^p} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)}\right)^p w$$

and we have that

$$\int_{(a',c)\cap S} \Delta(x) w(x)\,\mathrm{d}x \leqslant \sum_{k=j}^{J} e^{k+1} w(E_{e^k})$$

$$\lesssim \sum_{k=j}^{J} \frac{1}{\left(1 + \log^+\left(\frac{e^{De^k}}{4B}\right)\right)^p} e^{k+1} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)}\right)^p w$$

and it suffices to estimate the sum. We proceed as follows.

$$\sum_{k=j}^{J} \frac{1}{\left(1 + \log^+\left(\frac{e^{De^k}}{4B}\right)\right)^p} e^{k+1}$$

$$\leqslant \sum_{k=j}^{\lfloor \log\left(\frac{1}{D}\log(4B)\right)\rfloor} e^{k+1} + \sum_{k=\lfloor \log\left(\frac{1}{D}\log(4B)\right)\rfloor+1}^{J} \frac{1}{(1 + De^k - \log(4B))^p} e^{k+1}$$

$$\lesssim e^j \frac{\left(1 - e^{\log\left(\frac{1}{D}\log(4B)\right)}\right)}{1-e} + \sum_{k=\lfloor \log\left(\frac{1}{D}\log(4B)\right)\rfloor+1}^{J} \frac{1}{\left(De^k + 1 - De^{\log\left(\frac{1}{D}\log(4B)\right)}\right)^p} e^{k+1}$$

$$\lesssim e^j \frac{\frac{1}{D}\log(4B) - 1}{e-1} + \sum_{k=\lfloor \log\left(\frac{1}{D}\log(4B)\right)\rfloor+1}^{J} \frac{e}{c^p D^p e^{k(p-1)}}$$

$$\lesssim e^{j-1} \frac{1}{D}\log(4B) + \kappa e^{-J(p-1)}$$

$$\lesssim \frac{\sum_{k=1}^{n} |I_k|}{|I|} + \kappa \frac{1}{\left[\frac{1}{D}\log\left(\frac{e|I|}{\sum_{k=1}^{n}|I_k|}\right)\right]^{p-1}}$$

and again this term is bounded by the right-hand side of (4.5).

Finally, for T we have that

$$
\int_{T \cap (a',c)} \Delta(x) w(x) \, dx \leqslant \sum_{i=J}^{\infty} e^{i+1} \int_{\{\Delta(x) > e^i\} \cap (a',c)} w(x) \, dx
$$

$$
\lesssim \sum_{i=J}^{\infty} e^{i+1} \frac{1}{\left(1 + \log^+ \left(\frac{1}{4Be^{-De^i}}\right)\right)^p} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a',c)}\right)^p w
$$

$$
\lesssim \sum_{i=J}^{\infty} \frac{1}{e^{i(p-1)}} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)}\right)^p w
$$

$$
\lesssim \frac{1}{e^{J(p-1)}} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)}\right)^p w
$$

$$
\lesssim \frac{1}{\left[\log \left(\frac{e|I|}{\sum_{k=1}^{n} |I_k|}\right)\right]^{p-1}} \int_{-\infty}^{\infty} \left(M^+ \chi_{(a,c)}\right)^p w.
$$

$\square$

Armed with the preceding lemma we can finally settle lemma 4.3.

Proof of lemma 4.3. Let $I = (a,c)$ be an interval. Let $\delta > 0$ such that if $\sum |I_k| \leqslant 2\delta |I|$ then

$$
\int_{\mathbb{R}} \Delta w \leqslant \frac{1}{2} \int_{\mathbb{R}} (M^+ \chi_I)^p w. \tag{4.6}
$$

Now assume that $a < b < c$ where $c - b < b - a$ and let $E \subset (a,b)$ be a measurable set. Let n be the least integer such that $\delta^n |(a,b)| < |E|$. Now we let $E_j = \{x : M^+(\chi_E)(x) > \delta^j\}$ for $1 \leqslant j \leqslant n$. Let J_i^j be the component intervals of E_j and $\Delta_j(x) = \sum_i M^+(\chi_{J_i^j})(x)^p$. We claim that for $2 \leqslant j \leqslant n$

$$
\int_{\mathbb{R}} \Delta_{j-1}(x) w(x) \, dx \leqslant \frac{1}{2} \int_{\mathbb{R}} \Delta_j(x) w(x) \, dx.
$$

Assume by now that the claim holds. Note that since $\chi_E(x) \leqslant \Delta_1(x)$ then

$$
w(E) \leqslant \int_{\mathbb{R}} \Delta_1(x) w(x) \, dx
$$

and iterating the preceding inequality,

$$
\int_{\mathbb{R}} \Delta_1(x) w(x) \, dx \leqslant \frac{1}{2^{n-1}} \int_{\mathbb{R}} \Delta_n(x) w(x) \, dx
$$

and consequently

$$
w(E) \leqslant 2 \frac{1}{2^n} \int_{\mathbb{R}} \Delta_n(x) w(x) \, dx.
$$

Note that by the definition of n, and taking into account that $b - c < b - a$,

$$\delta^n |(a,b)| < |E| \iff \frac{1}{2^{n(-\log_2 \delta)}} < \frac{|E|}{|(a,b)|}$$

$$\iff \frac{1}{2^n} < \left(\frac{|E|}{|(a,b)|} \right)^{\frac{1}{-\log_2 \delta}} \leqslant \left(\frac{|E|}{|(b,c)|} \right)^{\frac{1}{-\log_2 \delta}}$$

so if we can show that

$$\int_{\mathbb{R}} \Delta_n(x) w(x) \, \mathrm{d}x \lesssim \int_{\mathbb{R}} (M^+ \chi_{(a,c)}(x))^p w(x) \, \mathrm{d}x \tag{4.7}$$

then

$$w(E) \lesssim 2 \left(\frac{|E|}{|(b,c)|} \right)^{\frac{1}{-\log_2 \delta}} \int_{\mathbb{R}} (M^+ \chi_{(a,c)})^p w$$

and we would be done.

Let us settle (4.7). Observe that since

$$\frac{|E \cap (a,b)|}{|(a,b)|} = \frac{|E|}{|(a,b)|} > \delta^n,$$

we have that $a \in J_i^n$ for some component J_i^n of E_n. Let us call $J_i^n = (a', b')$. Observe that since $a \in J_i^n$ then $a' < a$. Observe that $M^+ \chi_E(a') = \delta^n$ and since (a', b') is a component, for some $\varepsilon > 0$, we have that if $x \in [a' - \varepsilon, a')$, then $M^+ \chi_E(x) \leqslant \delta^n$. Note that since $E \subset (a,b)$ and $a > a'$ this yields that for every $x < a' - \varepsilon$

$$M^+ \chi_E(x) \leqslant M^+ \chi_E(a' - \varepsilon) \leqslant \delta^n.$$

This yields that all the connected components are contained in the interval (a', b''), for some $b'' \leqslant b$, since $M^+ \chi_E(x) = 0$ for every $x > b$. Now, that by the weak type $(1,1)$ of M^+, combined with the definition of n,

$$\sum_i |J_i^n| = \left| \left\{ x \in \mathbb{R} \, : \, M^+ \chi_E(x) > \delta^n \right\} \right| \leqslant \frac{1}{\delta^n} |E|$$

$$= \frac{1}{\delta} \frac{1}{\delta^{n-1}} \frac{|E|}{|(a,b)|} |(a,b)| \leqslant \frac{1}{\delta} |(a,b)|.$$

Observe that since (a', b') is some component J_i^n in E_n then

$$a - a' \leqslant b' - a' \leqslant \sum_i |J_i^n| \leqslant \frac{1}{\delta} |(a,b)|.$$

Consequently

$$b - a' = b - a + a - a'$$

$$\leqslant b - a + \frac{1}{\delta}(b - a) = \left(1 + \frac{1}{\delta} \right)(b - a).$$

Since all the intervals J_i^n are contained in (a', b), by (4.5) we have that

$$\int_{\mathbb{R}} \Delta w \lesssim \frac{1}{\left(1 + \log\left(\frac{|(a',b)|}{\sum_i |J_i^n|}\right)\right)^{p-1}} \int_{\mathbb{R}} M^+(\chi_{(a',b)})^p w.$$

$$\leqslant \int_{\mathbb{R}} (M^+ \chi_{(a,b)})^p w \leqslant \int_{\mathbb{R}} (M^+ \chi_{(a,c)})^p w$$

We end the proof of (4.7) just noting that

$$M^+(\chi_{(a'',b)}) \lesssim M^+(\chi_{(a,c)})$$

since taking into account that $|(a'', b)| \leqslant \frac{1}{\delta}(\frac{1}{\delta} + 1)|(a, b)|$ yields that $M^+(\chi_{(a'',b)}) \simeq M^+(\chi_{(a,b)})$.

Since as we have just shown (4.7) holds, we are left with settling the claim. We argue as follows. Let H be a component of E_j. Note that, since H is a component

$$\frac{|H \cap E|}{|H|} = \delta^j. \tag{4.8}$$

Our next step is to show that

$$H \cap E_{j-1} = \{M^+ \chi_{H \cap E} > \delta^{j-1}\}. \tag{4.9}$$

First, we observe that the components of E_{j-1} are contained in the components of E_j. Hence

$$H \cap E_{j-1} = \bigcup I_k^{j-1}$$

where the I_k^{j-1} are the components of E_{j-1} contained in H. Then we have that if $x \in H \cap E_{j-1}$ then $x \in I_k^{j-1} = (\alpha, \beta)$ for some k. Since $x \in E_{j-1}$, then $M^+(\chi_E)(x) > \delta^{j-1}$ and

$$\frac{1}{|(x, \beta)|} \int_x^\beta \chi_E > \delta^{j-1}.$$

Observe that $(x, \beta) \subset I_k^{j-1} \subset H$, so

$$\frac{1}{|(x, \beta)|} \int_x^\beta \chi_{E \cap H} > \delta^{j-1}.$$

Consequently

$$M^+ \chi_{H \cap E}(x) > \delta^{j-1}$$

and this yields $\{M^+ \chi_{H \cap E} > \delta^{j-1}\} \supset H \cap E_{j-1}$.

Now we prove the converse inclusion. Observe that if $M^+ \chi_{H \cap E}(x) > \delta^{j-1}$ then

$$M^+ \chi_E(x) \geqslant M^+ \chi_{H \cap E}(x) > \delta^{j-1}$$

and consequently $x \in E_{j-1}$. Now we have to see that $x \in H$. Let us call $H = (d, e)$. Observe that $M^+ \chi_{H \cap E}(x) = 0$ for every $x > e$, and hence $x \notin \{M^+ \chi_{H \cap E} > \delta^{j-1}\}$.

On the other hand, observe that since H is a component of E_j there exist some $\varepsilon < 0$ such that if $x \in [d - \varepsilon, d]$ then $M^+ \chi_E(x) \leqslant \delta^j$. Relying upon this fact, note that if $x < d$ then we have that if $x \in [d - \varepsilon, d]$, then $M^+ \chi_{H \cap E}(x) \leqslant M^+ \chi_E(x) \leqslant \delta^j$ and consequently $x \notin \{M^+ \chi_{H \cap E} > \delta^{j-1}\}$ and if $x < d - \varepsilon$ then, $M^+ \chi_{H \cap E}(x) \leqslant M^+ \chi_{H \cap E}(d - \varepsilon) \leqslant \delta^j$ and also $x \notin \{M^+ \chi_{H \cap E} > \delta^{j-1}\}$. Hence

$$\{M^+ \chi_{H \cap E} > \delta^{j-1}\} \subset H$$

and we are done.

The weak type $(1,1)$ of M^+ combined with (4.9) yields

$$|H \cap E_{j-1}| \leqslant \delta^{1-j} |E \cap H|$$

and combining this with (4.8) we have that

$$|H \cap E_{j-1}| \leqslant \delta |H|.$$

If we denote

$$\Delta_{\mathcal{H}}(x) = \sum_{I_k \in \mathcal{H}} (M^+ \chi_{I_k}(x))^p$$

where $\mathcal{H}$ is the set of component intervals in $H \cap E_{j-1}$, we have that

$$\sum_{I_k \in \mathcal{H}} |I_k| \leqslant |H \cap E_{j-1}| \leqslant \delta |H|.$$

By the definition of δ and (4.6) we have that

$$\int_{-\infty}^{\infty} \Delta_{\mathcal{H}}(x) w(x)\, \mathrm{d}x \leqslant \frac{1}{2} \int_{-\infty}^{\infty} M^+(\chi_H)^p w(x)\, \mathrm{d}x.$$

Adding those inequalities for all the components H of E_j gives

$$\int_{\mathbb{R}} \Delta_{j-1}(x) w(x) \leqslant \frac{1}{2} \int_{\mathbb{R}} \Delta_j(x) w(x)\, \mathrm{d}x$$

for $2 \leqslant j \leqslant n$ as we wanted to show. This ends the proof of the theorem. $\qquad \square$

Appendix A. Cotlar type inequalities

Since we have not been able to find the following Cotlar type inequality in the one-sided setting in the literature, we provide here a proof for reader's convenience.

THEOREM A.1. *Let A be a Young function. Let T^+ be a one-sided singular integral operator with associated kernel $K \in \mathcal{H}_{\bar{A}}$. Then*

$$(T^+)^* f(x) \lesssim M_\delta^+(T^+ f)(x) + M_A^+ f(x) \quad \delta \in (0, 1).$$

Observe that if T^+ is a one-sided Calderón–Zygmund operator its associated kernel is in particular a $\mathcal{H}_\infty$ kernel and hence this result covers that case, with M_A^+ replaced by M^+.

Proof of theorem A.1. Observe that it suffices to show that for every $\varepsilon > 0$

$$|T_\varepsilon^+ f(0)| \lesssim M_A^+ f(0) + M_\delta^+(T^+ f)(0)$$

where

$$T_\varepsilon^+ f(0) = \int_\varepsilon^\infty K(0 - y) f(y) \, \mathrm{d}y.$$

Observe that for $x > 0$ we can write

$$T_\varepsilon^+ f(0) = T_\varepsilon^+ f(0) - T^+ f(x) + T^+ f(x)$$
$$= T_\varepsilon^+ f(0) - T^+ f_2(x) - T^+ f_1(x) + T^+ f(x)$$

where $f_1(x) = f(x)\chi_{(0,\varepsilon)}(x)$ and $f_2(x) = f(x)\chi_{(\varepsilon,\infty)}(x)$. Then we have that

$$\left|T_\varepsilon^+ f(0)\right| \leqslant \left|T^+ f_1(x)\right| + \left|T^+ f(x)\right| + \left|T_\varepsilon^+ f(0) - T^+ f_2(x)\right|$$

and δ-averaging over x, if we call $h = \frac{\varepsilon}{2c_A}$

$$\left|T_\varepsilon^+ f(0)\right|$$
$$\lesssim \left(\frac{1}{h} \int_0^h \left|T^+ f_1(x)\right|^\delta \, \mathrm{d}x\right)^{\frac{1}{\delta}} + \left(\frac{1}{h} \int_0^h \left|T^+ f(x)\right|^\delta \, \mathrm{d}x\right)^{\frac{1}{\delta}}$$
$$+ \left(\frac{1}{h} \int_0^h \left|T_\varepsilon^+ f(0) - T^+ f_2(x)\right|^\delta\right)^{\frac{1}{\delta}}$$
$$:= I + II + III.$$

For I, we observe that any Hörmander condition implies that $K \in \mathcal{H}_1$ and hence by [8], we have that T^+ is of weak type $(1, 1)$. Then, by Kolmogorov inequality,

$$I \lesssim \frac{1}{h} \int_\mathbb{R} |f_1(x)| \, \mathrm{d}x = \frac{2c_A}{\varepsilon} \int_0^\varepsilon |f(x)| \, \mathrm{d}x \leqslant 2c_A M^+ f(0).$$

For II

$$II \leqslant M_\delta^+(T^+ f)(0),$$

and it remains to deal with *III*. Observe that for $x \in \left(0, \frac{\varepsilon}{2c_A}\right)$

$$
\begin{aligned}
T^+ f_2(x) &= \lim_{\delta \to 0} \int_{\delta+x}^{\infty} K(x-y) f_2(y) \, \mathrm{d}y \\
&= \int_{\varepsilon}^{\infty} K(x-y) f(y) \, \mathrm{d}y = T_{\varepsilon}^+ f(x).
\end{aligned}
$$

Bearing that in mind and since for every $x \in \left(0, \frac{\varepsilon}{2c_A}\right)$ we have that $x < \frac{\varepsilon}{2c_A}$ or equivalently $xc_A < \frac{\varepsilon}{2}$, we have that

$$
\begin{aligned}
&\left| T_{\varepsilon}^+ f(0) - T^+ f_2(x) \right| \\
&= \left| T_{\varepsilon}^+ f(0) - T_{\varepsilon}^+ f(x) \right| = \left| \int_{\varepsilon}^{\infty} \left(K(-y) - K(x-y) \right) f(y) \, \mathrm{d}y \right| \\
&\leqslant \int_{\varepsilon}^{\infty} |K(-y) - K(x-y)| \, |f(y)| \, \mathrm{d}y \\
&\leqslant \sum_{m=1}^{\infty} \left(2^m \frac{\varepsilon}{2} \right) \frac{1}{\left(2^m \frac{\varepsilon}{2} \right)} \int_{2^m \frac{\varepsilon}{2} < y \leqslant 2^{m+1} \frac{\varepsilon}{2}}^{\infty} |K(-y) - K(x-y)| \, |f(y)| \mathrm{d}y \\
&\lesssim \sum_{m=1}^{\infty} \left(2^m \frac{\varepsilon}{2} \right) \left\| (K(-\cdot) - K(x-\cdot)) \chi_{2^m \frac{\varepsilon}{2} < y \leqslant 2^{m+1} \frac{\varepsilon}{2}} \right\|_{\bar{A}, \left(0, 2^{m+1} \frac{\varepsilon}{2} \right)} \|f\|_{A, \left(0, 2^{m+1} \frac{\varepsilon}{2} \right)} \\
&\lesssim C_A M_A^+ f(0),
\end{aligned}
$$

and consequently

$$
III \lesssim C_A M_A^+ f(0).
$$

This ends the proof. $\qquad\qquad\square$

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