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AN EXTRAPOLATION THEOREM FOR POSITIVE OPERATORS

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Introduction. Denote by S and M respectively the complex vector spaces of simple and measurable complex valued functions defined on the finite measure space X. Let T be a positive linear map from S to M such that for each p, $1 < p < \infty$, sup $\{||Tf||_p : f \in S, ||f||_p \leqq 1\}$ is finite. T then has an extension to a bounded transformation of every $L_p(X)$, $1 < p < \infty$, and these extensions are "consistent". The norm of T as a transformation of L_p is denoted $||T||_p$. The aim of this note is to prove the following theorem.

THEOREM. *Suppose* (i) $||T||_p = 1$ *for three values* p_0, r, p_1 *of* p, *where* $1 < p_0 < r < p_1 < \infty$ *: and* (ii) *there is a* p, $p_0 < p < p_1$, *and* $f > 0$ a.e. *such that* $||Tf||_p = ||f||_p = 1$. *Then* $T1 = T^*1 = 1$, *where* 1 *is the function whose value is* 1 *at all points of* X (*and consequently* T *is a contraction on all* $L_p(X)$ *spaces*, $1 \leqq p \leqq \infty$).

The main step in proving this result is to show that if T satisfies (i), and if (ii)′ there is an $f \geqq 0$ a.e. such that $||Tf||_p = ||f||_p = 1$, then $T\chi_A = \chi_B$, where $A = \{f > 0\}$ and $B = \{Tf > 0\}$. The hypotheses (i) and (ii)′ imply $T^*(Tf)^{p-1} = f^{p-1}$, so T^* satisfies (i) and (ii)′ for p_0^*, r^*, p_1^*, p^* and $(Tf)^{p-1}$. Noting $\{(Tf)^{p-1} > 0\} = \{Tf > 0\}$ and $\{f^{p-1} > 0\} = \{f > 0\}$, we see $T^*\chi_B = \chi_A$. The theorem follows, since $\mu(A) = \mu(B)$ and if $f > 0$ a.e., then $\chi_A = 1$ (§ 4).

The idea of the proof of this supporting result is to examine the function $F : (0, 1) \rightarrow [0, \infty)$ given by

$$F(t) = \int T(f^{pt})g^{q(1-t)},$$

where $g = (Tf)^{p-1}$. This is logarithmically convex (cf. § 1, Prop. 1). F is also dominated by the $1/t$-norm of T. It follows from (i) and (ii)′ that $F(t) = 1$ if $t \in [1/p_1, 1/p_0]$. By (i) and the condition for equality in Hölder's Inequality, we conclude that for each $t \in [1/p_1, 1/p_0]$,

$$(*) \quad T(f^{pt}) = (Tf)^{pt} \quad \text{a.e.}$$

(cf. § 3, Prop. 3).

In the case of a discrete measure on a finite set, T is a positive matrix, and both sides of (*) are analytic on **C**. Since they agree on $[1/p_1, 1/p_0]$, they

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agree everywhere, so putting $t = 0$ gives the desired result. In the general case, an appropriate analyticity holds for the strip $\{z : 0 < \operatorname{Re} z < 1\}$ (cf. § 2, Prop. 2). In § 4 we use this to obtain the conclusion in the general case, and give the details of the proof of the main theorem.

In § 5 we include remarks relating to the results given here.

1. Lemmas 1 and 2 are needed for Proposition 1, the main result of this section. Lemma 1 is known (cf. [**1**, p. 17, Prop. 2.1.2] for a similar result).

LEMMA 1. *Let* $1 < r < \infty$, $0 \leqq f$, $f \in L_r$, $0 \leqq f_n$, $f_n \in L_r$, $n = 1, 2, \ldots$.
(1) *If* $f_n \uparrow f$, *then* $Tf_n \uparrow Tf$ a.e.
(2) *If* $f_n \downarrow f$, *then* $Tf_n \downarrow Tf$ a.e.

Proof. (1) For each $n \geqq 1$, there is a set A, $\mu(A) = 0$, such that $0 \leqq Tf_n(x) \leqq Tf_{n+1}(x) \leqq Tf(x) < \infty$ if $x \in X - A$. This follows from the positivity of T and the L_r boundedness of T. The union of the sets A has measure zero, and on its complement, $g = \lim_n Tf_n$ exists. Put $g = 0$ at all other points. Then g is finite everywhere and $g \leqq Tf$. If $n \geqq 1$,

$$\int (Tf - g)^r \leqq \int (Tf - Tf_n)^r \leqq ||T||_r^r \int (f - f_n)^r.$$

Since $f_n \to f$ in L_r (by Lebesgue's Theorem of Dominated Convergence) we conclude $Tf = g$ a.e.

(2) This follows from (1) using the sequence $f_1 - f_n \uparrow f_1 - f$.

LEMMA 2. *If* $h \geqq 0$, $h \in L_1$, $0 < x < y < 1$, $0 < a < 1$, *then* $T(h^w) \leqq [T(h^x)]^{1-a} \cdot [T(h^y)]^a$ a.e., *where* $w = (1 - a)x + ay$.

Proof. If h is simple, this follows from Hölder's inequality. If h is not simple, let s be simple and $0 \leqq s \leqq h$. Then by the positivity of T,

$$T(s^w) \leqq [T(h^x)]^{1-a} \cdot [T(h^y)]^a \quad \text{a.e.}$$

The conclusion follows from Lemma 1-(1).

PROPOSITION 1. *Let* h *and* $k \in L_1$, $h \geqq 0$, $k \geqq 0$. *If* $G : (0, 1) \to [0, \infty)$ *is given by*

$$G(t) = \int T(h^t)k^{1-t},$$

then G *is logarithmically convex.*

Proof. We must show that if $0 < t, s, a < 1$ then $G((1 - a)t + as) \leqq G(t)^{1-a} \cdot G(s)^a$.

Using Lemma 2, we see that

$$G((1 - a)t + as) \leqq \int [T(h^t) \cdot k^{1-t}]^{1-a}[T(h^s)k^{1-s}]^a,$$

so the proposition follows from Hölder's inequality for the conjugate indices $1/1 - a$ and $1/a$.

2.

LEMMA 3. *Let* $a > b > 0$, x, y *real*, $d \geqq x \geqq c > 0$. *Let* $z = x + iy$.
(1) *If* $b \geqq 1$, *then* $|a^z - b^z| \leqq |z|(a^d - b^d)/d$.
(2) *If* $a \leqq 1$, *then* $|a^z - b^z| \leqq |z|(a^c - b^c)/c$.

Proof. We have

$$a^z - b^z = z \int_b^a t^{z-1} dt.$$

Thus,

$$|a^z - b^z| \leqq |z| \int_b^a t^{x-1} dt.$$

For (1), since $b \geqq 1$, if $t \geqq b$, $t^{x-1} \leqq t^{d-1}$. Thus,

$$|a^z - b^z| \leqq |z| \int_b^a t^{d-1} dt = \frac{|z|}{d}(a^d - b^d).$$

For (2), since $a \leqq 1$, if $t \leqq a$, then $t^{x-c} \leqq 1$, so $t^{x-1} \leqq t^{c-1}$.
Thus,

$$|a^z - b^z| \leqq |z|(a^c - b^c)/c.$$

PROPOSITION 2. *If* $1 < r < \infty$, $g \in L_r$, $g \geqq 0$ *and* $h \in S$, *then*

$$H(z) = \int T(g^{rz})h$$

is analytic on the strip $0 < \operatorname{Re} z < 1$.

Proof. If g is simple, say $g = \sum_1^n c_i \chi(A_i)$, where $c_i > 0$, A_i are disjoint, $i = 1, 2, \ldots, n$, then for each z,

$$g^{rz} = \sum_1^n c_i^{rz} \chi(A_i) \quad \text{a.e.,}$$

so

$$H(z) = \int T(g^{rz})h = \sum_1^n c_i^{rz} \int T\chi(A_i)h.$$

Clearly H is analytic (for all z). If g is not simple, let $g_n \in L_r$, $g_n \geqq 0$, $n = 1, 2, \ldots$, be simple functions such that $g_n \uparrow g$ and such that $\{g_1 \geqq 1\} = \{g \geqq 1\}$. Let

$$H_n(z) = \int T(g_n^{rz})h.$$

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To show H is analytic, it is sufficient to show $H_n \to H$ uniformly on compact subsets of $0 < \operatorname{Re} z < 1$. Note

$$|H(z) - H_n(z)| \leqq \int T|g^{rz} - g_n^{rz}| \cdot |h|.$$

If $A = \{g_1 \geqq 1\}$ and $B = \{g < 1\}$, $A \cap B$ has measure zero and $A \cup B = X$. Thus $|H(z) - H_n(z)| \leqq \int_A + \int_B$. Let $\epsilon > 0$ and let K be a compact subset of $0 < \operatorname{Re} z < 1$. There are $R, d, c > 0$ such that $z \in K \Rightarrow |z| < R$, $1 > d > \operatorname{Re} z > c$. We have then

$$|H(z) - H_n(z)| \leqq \frac{R}{d} \int_A T(g^{rd} - g_n^{rd}) \cdot |h| + \frac{R}{c} \int_B T(g^{rc} - g_n^{rc}) \cdot |h|.$$

Let $0 < a < 1$. By Lemma 2-(1), $T(g^{ra} - g_n^{ra}) \cdot |h| \to 0$ as $n \to \infty$. Also, this function is dominated a.e. by $T(g^{ra}) \cdot |h|$, which has finite integral since the $1/a$-norm of T is finite. By Lebesgue's Theorem of Dominated Convergence, if $\delta > 0$, there is N such that if $n \geqq N$

$$\int T(g^{ra} - g_n^{ra}) \cdot |h| < \delta/2.$$

From this, if $\delta = \epsilon c/R$ it is clear that there is an N such that if $n \geqq N$, then $|H(z) - H_n(z)| < \epsilon$. This gives the result of the proposition.

3.

PROPOSITION 3. *Suppose T satisfies the hypotheses* (i), (ii)′ *stated in the Introduction. If $t \in (0, 1)$, and f is the function mentioned in* (ii)′, *then*

$$T(f^{pt}) = (Tf)^{pt} \quad \text{a.e.}$$

Proof. Let $g = (Tf)^{p-1}$. Then $||g||_q = 1$, where $1/p + 1/q = 1$. Consider F, where

$$F(t) = \int T(f^{pt}) g^{q(1-t)}, \quad t \in (0, 1).$$

By Proposition 1, F is a log-convex function. Since f^{pt} and $g^{q(1-t)}$ have respectively $1/t$ and $1/(1-t)$ norm equal to 1, we see $F(t) \leqq 1/t$-norm of T. Thus, $F(1/p_1) \leqq 1$, $F(1/p_0) \leqq 1$. Since $F(1/p) = 1$, (by (ii)′), we conclude from the log-convexity that $F(t) = 1$ if $t \in [1/p_1, 1/p_0]$. By the Riesz convexity theorem, and (i), the $1/t$-norm of T is 1 for $t \in [1/p_1, 1/p_0]$. Using $F(t) = 1$ and Hölder's inequality for the conjugate indices $r = 1/t$ and $s = 1/(1-t)$, we have if $r \in [p_0, p_1]$ that

$$1 \leqq ||T(f^{pt})||_r \, ||g^{q(1-t)}||_s = ||T(f^{pt})||_r \leqq ||T||_r = 1.$$

By the condition for equality in Hölder's theorem, there are positive constants A, B not both zero such that

$$A(T(f^{pt}))^r = B(g^{q(1-t)})^s \quad \text{a.e.}$$

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Integrating and using $||T(f^{pt})||_r = 1$, we see $A = B \neq 0$. Thus,

$$T(f^{pt}) = g^{qt} \quad \text{a.e.,}$$

if $t \in [1/p_1, 1/p_0]$. This equality can be extended to all $t \in (0, 1)$, since for any simple function h, on $\{0 < \text{Re}\, z < 1\}$ the function K,

$$K(z) = \int [T(f^{pz}) - g^{qz}] \cdot h,$$

is analytic by Proposition 2. Since K vanishes on $[1/p_1, 1/p_0]$ it must vanish identically. Since h was arbitrary, we conclude $T(f^{pz}) = g^{qz}$ a.e., for each $z \in \{0 < \text{Re}\, z < 1\}$.

4. We may assume $f < \infty$. Let $A = \{f > 0\}$, $B = \{Tf > 0\}$. Put $f_0 = \chi\{0 < f < 1\} \cdot f$, $f_1 = \chi\{1 \leqq f\} \cdot f$, $f_{0n} = \exp\{(p/n) \log f_0\}$, $f_{1n} = \exp\{(p/n) \log f_1\}$, $f_n = \exp\{(p/n) \log f\}$. Then $f_{0n} \uparrow \chi\{0 < f < 1\}$, $f_{1n} \downarrow \chi\{1 \leqq f\}$, and $f_n = f_{0n} + f_{1n}$. Noting $\chi\{0 < f < 1\} \in L_2$, say, and $f_{12} \in L_2$, we have by Lemma 2, $Tf_{0n} \to T\chi\{0 < f < 1\}$ and $Tf_{1n} \to T\chi\{1 \leqq f\}$. This gives $Tf_n \to T\chi\{0 < f\}$. It is clear that $\exp\{(p/n) \log Tf\} \to \chi\{(Tf) > 0\}$. Thus, by Proposition 3,

$$T\chi\{f > 0\} = \chi\{(Tf) > 0\},$$

i.e. $T\chi(A) = \chi(B)$. To show $T^*\chi(B) = \chi(A)$, we note that T^* satisfies the hypothesis (i) of the theorem with indices conjugate to those given for T. Also, if $q = p^*$, $||g||_q = 1$, and by Hölder's inequality,

$$1 = \int fT^*g \leqq ||f||_p||T^*g||_q = ||T^*g||_q \leqq ||T^*||_q = 1.$$

We conclude $||T^*g||_q = 1$. Hence, hypothesis (ii)′ of the theorem is satisfied for T^*, g, p^*, and we can conclude as above that

$$T^*\chi\{g > 0\} = \chi\{T^*g > 0\}.$$

Now $\{g > 0\} = B$; we must show $\{T^*g > 0\} = A$. By the condition for equality in Holder's theorem, there are constants $A, B \geqq 0$, not both zero, such that

$$Af^p = B(T^*g)^q.$$

Integrating gives $A = B$. Thus, $f^p = (T^*g)^q$, so $\chi\{T^*g > 0\} = \chi\{f > 0\}$. This concludes the proof of the supporting result.

To obtain the Main Theorem, we note that in general

$$\int T\chi_A\chi_B = \int \chi_B = \mu(B)$$

and

$$\int T\chi_A\chi_B = \int \chi_A T^*\chi_B = \int \chi_A = \mu(A).$$

Thus, $\mu(A) = \mu(B)$. If T satisfies (ii), then $\chi_A = 1$, and from the above we conclude $\chi_B = 1$ a.e. Thus, $T1 = 1$ and $T^*1 = 1$, as desired. That T is a contraction on all $L_p(X)$ follows from the Riesz Convexity Theorem since $||T||_\infty = 1$, and $||T^*||_\infty = 1$ so that $||T||_1 = ||T^*||_\infty = 1$.

5. The function F in Proposition 1 was suggested by the proof of the Riesz Convexity Theorem (see e.g. [**2**, pp. 69–70]).

If (ii) is not assumed, the conclusion of the Theorem is in general false. An example is provided by the direct sum of two finite measure spaces X_1 and X_2. Let T_1 and T_2 be bounded transformations of $L_p(X_1)$ and $L_p(X_2)$ respectively for each $1 \leqq p \leqq \infty$. Assume $||T_1||_p = 1$ for all $1 \leqq p \leqq \infty$ (e.g., $T = I$) and let T_2 be such that $||T_2||_1 > 1$, $||T_2||_\infty > 1$, $||T_2||_2 < 1$. By the Riesz Convexity Theorem, the set of p such that $||T_2||_p < 1$ is an open interval (p_0, p_1) where $1 < p_0 < p_1 < \infty$. If $T = T_1 \oplus T_2$, then $||T||_p = \max\{||T_1||_p, ||T_2||_p\}$. Clearly (i) holds. Since every f measurable on X can be written $f = f_1 + f_2$ where $\operatorname{supp} f_i \subset X_i$, $i = 1, 2$, if $p \in (p_0, p_1)$, then

$$||Tf||_p^p = ||T_1f_1||_p^p + ||T_2f_2||_p^p \leqq ||f_1||_p^p + ||f_2||_p^p$$

with strict inequality if $f_2 > 0$ on a set of positive measure. Hence (ii) cannot hold. It is also clear that the conclusion of the theorem fails.

We note finally that the hypothesis (ii)$'$ always holds when T is compact, as pointed out by P. Rosenthal. If $Tf(x) = \int K(x, y)f(y)dy$, where K is measurable in the product σ-algebra, then T is compact provided $\int (\int K(x, y)^q dy)^{p/q} dx < \infty$ (see [**3**, pp. 277–278]).

References

1. A. Garsia, *Topics in almost everywhere convergence* (Markham, Chicago).
2. W. Rudin, *Real and complex analysis* (McGraw-Hill, 1974).
3. E. M. Stein, *Topics in harmonic analysis* (Princeton, 1970).
4. K. Yosida, *Functional analysis*, 4th Ed. (Springer, 1974).

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