HOMOTOPY OF NATURAL TRANSFORMATIONS

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1. Introduction. Let $\mathbf{C}$ be a full subcategory of $\mathbf{T}$, the category of based topological spaces and based maps, and let $\mathbf{C}^n$ be the corresponding category of n-tuples. Let $S, T: \mathbf{T}^n \to \mathbf{T}$ be covariant functors which respect homotopy classes and let $u, v: S \to T$ be natural transformations. u and v are *homotopic in* $\mathbf{C}$, denoted $u \simeq v$ ($\mathbf{C}$), if $uX \simeq vX: SX \to TX$ ($X \in \mathbf{C}^n$), that is to say, for every $X \in \mathbf{C}$, uX and vX are homotopic (all homotopies are required to respect base points). u and v are *naturally homotopic in* $\mathbf{C}$, denoted $u \simeq_{\mathrm{n}} v$ ($\mathbf{C}$), if there exist morphisms

$$u_t X: SX \to TX \qquad (t \in I, X \in \mathbf{C})$$

such that, for every $X \in \mathbf{C}$, $u_t X$ is a homotopy from uX to vX and such that, for every $t \in I$, $u_t: S \to T$ is a natural transformation. As examples, let $C, C': \mathbf{T} \to \mathbf{T}$ be the reduced, unreduced cone functors respectively, and, for any S, T, let $c: S \to T$ denote the constant natural transformation (i.e. $cX = *$, the constant map $SX \to TX$, for each $X \in \mathbf{T}$). Then we certainly have

$$i_C \simeq_{\mathrm{n}} c\ (\mathbf{T}),$$

where i_C denotes the identity natural transformation $C \to C$. Since any point of a CW-complex is non-degenerate, it follows [**7**, p. 333, E (proposition)] that

$$i_{C'} \simeq c\ (\mathbf{CW}),$$

where $\mathbf{CW}$ is the full subcategory of based CW-complexes. However, the assertion $i_{C'} \simeq_{\mathrm{n}} c$ ($\mathbf{C}$) is false unless $\mathbf{C}$ contains only one-point spaces. For let X have more than one point and let $*: X \to X$ be the constant map. Then it is easy to see that no null homotopy of the identity map $C'X \to C'X$ can commute with $C'*$.

One may ask the question: Does any fixed object X of $\mathbf{T}^n$ have the property that $uX \simeq cX$ implies $u \simeq c$ ($\mathbf{C}$)? An answer is possible if one also restricts the class of functors to which S and T belong. Let P be a based 0-sphere and let $P \in \mathbf{T}^n$ also denote the n-tuple each of whose components is P. In this paper we shall consider the case $X = P \in \mathbf{T}^n$ and restrict S and T to the class of cellular P-functors which we define in § 2. Let $\mathbf{W}$ be the full subcategory of countable CW-complexes. We shall prove the following result.

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THEOREM 1.1. *If S, T: $\mathbf{T}^n \to \mathbf{T}$ are cellular P-functors, if TP is* 1*-connected and if $u: S \to T$ is a natural transformation such that $uP \simeq cP: SP \to TP$, then $u \simeq c$ ($\mathbf{W}$). Moreover, TY is* 1*-connected ($Y \in \mathbf{W}^n$).*

It would be very satisfactory to be able to replace c in Theorem 1.1 by an arbitrary natural transformation $v: S \to T$ satisfying $uP \simeq vP: SP \to TP$ but in the general case, I have not been able to achieve the desired extension. However, if S is of the form $\Sigma S'$, where $\Sigma: \mathbf{T} \to \mathbf{T}$ denotes the reduced suspension functor, then $u - v: S \to T$ is defined and Theorem 1.1 implies that $u - v \simeq c$ ($\mathbf{W}$) which in turn yields $u \simeq v$ ($\mathbf{W}$). The stronger result may also be obtained if instead there is a weak homotopy equivalence $S \to \Sigma S'$, but this lies outside the scope of the present work. The objective envisaged is a method of extending identities known to hold for "ordinary" homotopy operations to "generalized" homotopy operations. In particular, I hope to give (elsewhere) a proof along these lines of the Jacobi identity for generalized higher-order Whitehead products.

For an application of Theorem 1.1 as it stands, let†

$$W(i) \in [\Sigma^2 \wedge X, T_1\Sigma X] \qquad (X = (X_1, X_2, X_3))$$

be the universal example for the third-order generalized Whitehead product [**6**]. Here $\wedge$ and T_1 denote the smash and the fat wedge functors. Let $p: T_1 \to T_1/T_2$ be the projection which shrinks the thin wedge T_2. We have the following result.

THEOREM 1.2. $p_* W(i) = 0$ ($X_i \in \mathbf{W}$, $i = 1, 2, 3$).

Proof. Let

$$JX = (CX_1 \times CX_2 \times X_3) \cup (CX_1 \times X_2 \times CX_3) \cup (X_1 \times CX_2 \times CX_3).$$

Then there is a homotopy equivalent transformation $\theta: J \to \Sigma^2 \wedge$ and a natural transformation $\mu: J \to T_1\Sigma$ such that $W(i) = \{\mu \cdot \theta^{-1}\}$. Let $u = p \cdot \mu: J \to T_1\Sigma/T_2\Sigma$. We shall prove that $u \simeq c$ ($\mathbf{W}$). We have $JP \cong S^2$ and $(T_1\Sigma/T_2\Sigma)P \cong S^2 \vee S^2 \vee S^2$, which is 1-connected. Since J and $T_1\Sigma/T_2\Sigma$ are cellular P-functors, the required result will follow from Theorem 1.1 if we can prove that $\{uP\} = 0$. $\{uP\}$ is in effect an element of $\pi_2(S^2 \vee S^2 \vee S^2)$ and, since $\pi_2(S^2 \vee S^2 \vee S^2) \approx Z + Z + Z$, we need only observe that the projection of $\{uP\}$ on to one of the copies of S^2 is zero. This is so since the projection is a class which can be factored through $\{J(i_P, i_P, *)\} = 0 \in [JP, JP]$. We remark that use of Theorem 1.1 is an essential feature of the foregoing proof, for whereas it can be argued similarly that $(T_1\Sigma/T_2\Sigma)X \cong \Sigma^2(X_1 \wedge X_2) \vee \Sigma^2(X_1 \wedge X_3) \vee \Sigma^2(X_2 \wedge X_3)$ and similarly that the projections of $\{uX\}$ onto $\Sigma^2(X_1 \wedge X_2)$, $\Sigma^2(X_1 \wedge X_3)$, and $\Sigma^2(X_2 \wedge X_3)$ are trivial, this is not by itself sufficient to ensure that $\{uX\} = 0$

†$[A, B]$ denotes the set of based homotopy classes of based maps from A to B.

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since by the Hilton-Milnor theorem (see, e.g., [**4**, p. 13, Theorem 4]), $[JX, (T_1\Sigma/T_2\Sigma)X]$ contains, in addition to the summand

$$[JX, \Sigma^2(X_1 \wedge X_2)] + [JX, \Sigma^2(X_1 \wedge X_3)] + [JX, \Sigma^2(X_2 \wedge X_3)],$$

summands of the form $[JX, \Sigma^3(X_1 \wedge X_2 \wedge X_2 \wedge X_3)]$ which in general are non-trivial. The special case of Theorem 1.2 in which each X_i is a suspension was proved by Porter [**5**, p. 43, Theorem 14.1].

I am grateful to Dr. Porter for sending me a copy of the relevant pages of his dissertation.

2. P-functors. In this section we recall the definition and principal properties of P-functors [**1**]. If $X, Y \in \mathbf{T}$, let $|X, Y|$ denote their morphism set. Let $\Pi: \mathbf{T}^n \to \mathbf{T}$ be the topological product functor (if $n = 1$, then Π is the identity functor) and let $\pi_j \in |\Pi X, \Pi X|$ denote the projection which leaves unaltered all the coordinates of $x \in \Pi X$ except the jth which it replaces by $*$. Let $W \in \mathbf{T}$ and let the n-tuple $\phi = (\phi_1, \phi_2, \ldots, \phi_n)$ be such that

$$\phi_j \in |W, W|, \phi_j \cdot \phi_j = \phi_j \ (1 \leqq j \leqq n), \phi_i \cdot \phi_j = \phi_j \cdot \phi_i \ (i \neq j).$$

Associated with the pair (W, ϕ) is a covariant functor $\Phi: \mathbf{T}^n \to \mathbf{T}$ described as follows. If $Y \in \mathbf{T}^n$, then ΦY is the space obtained from (the based topological product) $\Pi Y \times W$ by performing the identification

$$(2.1) \qquad (\pi_j y, w) = (y, \phi_j w) \qquad (Y \in \Pi Y, w \in W, 1 \leqq j \leqq n).$$

Let $\phi Y: Y \times W \to \Phi Y$ denote the identification map. Then if $f \in |Y, Z|$,

$$\Phi f = \phi Z \cdot (\Pi f \times i_W) \cdot (\phi Y)^{-1},$$

which is base-point preserving, single-valued, and hence continuous. $S: \mathbf{T}^n \to \mathbf{T}$ is a *P-functor* in $\mathbf{C}$ if $P \in \mathbf{C}$ and if, for some (W, ϕ), the restrictions of S and Φ to C^n are naturally equivalent. Examples of P-functors were given in [**1**]. We remark here that they include the various wedges, joins, suspensions, and their composites; however, Lemma 2.1 (below) yields a test for whether a given $S: \mathbf{T}^n \to \mathbf{T}$ is a P-functor or not.

If $Y \in T^n$, then a point of ΠY may be regarded as an element of $|P, Y|$. Consequently, for every $S: \mathbf{T}^n \to \mathbf{T}$ and every $Y \in \mathbf{T}^n$, we may define a function $\psi_S Y: \Pi Y \times SP \to SY$ by the rule

$$\psi_S Y(y, x) = Sy(x) \qquad (y \in \Pi Y, x \in SP).$$

$\psi_S Y$ certainly respects base-points. S is *valuable in* $\mathbf{C}$ if $\psi_S Y$ is continuous for every $Y \in \mathbf{C}^n$.

Let (W, ϕ), (W', ϕ') be pairs. A *pair map* $\bar{u}: (W, \phi) \to (W', \phi')$ is a morphism $\bar{u} \in |W, W'|$ such that $u \cdot \phi_j = \phi_j' \cdot \bar{u}$ $(1 \leqq j \leqq n)$. Associated with any functor $S: \mathbf{T}^n \to \mathbf{T}$ is a pair (SP, ϕ_S), where

$$(\phi_S)_j = S(g^j) \qquad (1 \leqq j \leqq n)$$

and where $g^j \in |P, P|$ is the n-tuple such that

$$(g^j)_k = \begin{cases} i_P & (k \neq j), \\ * & (k = j). \end{cases}$$

The following result may be found in [**1**, Lemma 1.4].

LEMMA 2.1. *If Φ is the functor associated with the pair (W, ϕ), then Φ is valuable in* $\mathbf{T}$ *and there exists a pair equivalence v: $(\Phi P, \phi_\Phi) \to (W, \phi)$ such that $\psi_\Phi = \phi \cdot (i_\Pi \times v)$.*

It follows from Lemma 2.1 that to test whether S: $\mathbf{T}^n \to \mathbf{T}$ is a P-functor, we need only set $(W, \phi) = (SP, \phi_S)$ and examine whether or not S and the resulting Φ are naturally equivalent. The reader may find it helpful to perform the test, for example, on the reduced, unreduced cone functors C, C': $\mathbf{T} \to \mathbf{T}$.

The proof of Theorem 1.1 proceeds by induction on n. The following result, which is that of [**1**, Lemma 1.5], will be required in § 3 in the proof of the case $n = 1$ of Theorem 1.1.

LEMMA 2.2. *If S is a P-functor in* $\mathbf{C}$, *if T is valuable in* $\mathbf{C}$, *and if $\bar{u}$: $(SP, \phi_S) \to (TP, \phi_T)$, is a pair map, then there exists a transformation u: $S \to T$ natural in* $\mathbf{C}^n$ *such that $uP = \bar{u}$.*

If $Y \in \mathbf{C}^n$, we remark that uY is the unique map which completes the following diagram:

$$\begin{array}{ccc} \Pi Y \times SP & \xrightarrow{i_{\Pi Y} \times \bar{u}} & \Pi Y \times TP \\ \downarrow \psi_S Y & & \downarrow \psi_S Y \\ SY & \overset{uY}{\cdots\cdots\rightarrow} & TY \end{array}$$

Moreover, if $\bar{u}_t$: $SP \to TP$ is a homotopy which is a pair map for each $t \in I$, then $u_t Y$: $SY \to TY$ is a homotopy.

Let $Y' = (Y_1, Y_2, \ldots, Y_{n-1})$ be a fixed object of $\mathbf{C}^{n-1}$. We define $(Y'|S)$: $\mathbf{T} \to \mathbf{T}$ to be such that

$$(Y'|S) Y_n = S(Y', Y_n) = S(Y_1, Y_2, \ldots, Y_n) \qquad (Y_n \in \mathbf{T}),$$
$$(Y'|S)f = S(i_{Y'}, f) \qquad (f \in |Y_n, Z|).$$

Similarly, if $X \in \mathbf{C}$ is fixed, we define $(S|X)$: $\mathbf{T}^{n-1} \to \mathbf{T}$ to be such that

$$(S|X) Y' = S(Y', X) \qquad (Y' \in \mathbf{T}^{n-1}),$$
$$(S|X) f' = S(f', i_X) \qquad (f' \in |Y', Z'|).$$

$(Y'|S)$ and $(S|X)$ are partial functors of S which arise naturally in certain inductive arguments. The following lemma is [**1**, Lemma 2.6], to which the condition $SP \in \mathbf{D}$ should be added to the hypothesis.

LEMMA 2.3. *If* $\mathbf{C}$ *is closed with respect to finite topological products, if* $f \times i_W$ *is an identification for every* $W \in \mathbf{C}$ *and every identification* $f \in \mathbf{C}$, *if* $SP \in \mathbf{C}$ *and if* S *is a* P*-functor in* $\mathbf{C}$, *then* $(Y'|S)$ *and* $(S|X)$ *are* P*-functors in* $\mathbf{C}$.

A pair (W, ϕ) is *cellular* if $W \in \mathbf{W}$ and if ϕ_j is a cellular map whose image $\phi_j(W)$ is a subcomplex of W $(1 \leqq j \leqq n)$. S is a *cellular* P*-functor* if (SP, ϕ_S) is a cellular pair and if S is a P-functor in $\mathbf{W}$. Let S be a cellular P-functor and let $Y, Z \in \mathbf{W}^n$. We shall require the following lemma.

LEMMA 2.4. (i) $SY \in \mathbf{W}$; (ii) *if* $f \in |Y, Z|$ *is an* n*-tuple of cellular maps, then* Sf *is cellular;* (iii) *if* $X \in \mathbf{W}$ *and* $Y' \in \mathbf{W}^{n-1}$, *then* $(Y'|S)$ *and* $(S|X)$ *are cellular* P*-functors.*

The proof of part (i) has essentially been given in [**1**, pp. 26, 27] and (ii) is a corollary of the proof of (i). Finally, Lemma 2.3, together with (ii), yields (iii).

The following lemma concerning adjunction spaces assembles mostly well-known results.

LEMMA 2.5. *If* A *is a closed subset of* X, *if the injection* $A \to X$ *is a cofibration, if* f: $(X, A) \to (Y, B)$ *is an identification map which is a relative homeomorphism and if* $f(A) = B$, *then the injection* $B \to Y$ *is a cofibration and* f *induces relative homology isomorphisms. If, further,* B *and* (X, A) *are* 1*-connected, then* Y *is* 1*-connected.*

Proof. It follows from [**8**, *Satz* 1 and Definition 2] that there exist maps ψ: $X \times I \to X$, v: $X \to I$, and w: $X \to I$ such that $\psi(x, 0) = x$ $(x \in X)$, $\psi(x, 1) \in A$ if $v(x) < 1$, $\psi(a, t) = a$ $(a \in A, t \in I)$, $v(A) = 0$, $A = w^{-1}(0)$. Then if we define $\psi'(y, t) = f\psi(f^{-1}y, t)$, $v'y = vf^{-1}y$, $w'y = wf^{-1}y$ $(y \in Y, t \in I)$, we obtain single-valued and hence continuous maps ψ': $Y \times I \to Y$, v': $Y \to I$, and w': $Y \to I$ satisfying the conditions $\psi'(y, 0) = y$ $(y \in Y)$, $\psi'(y, 1) \in B$ if $v'(y) < 1$, $\psi'(b, t) = b$ $(b \in B, t \in I)$, $v'(B) = 0$, $B = w'^{-1}(0)$. Hence $B \to Y$ is a cofibration. An application of [**2**, p. 122, Corollary] now proves that f induces relative homology isomorphisms. Finally, let $U = v^{-1}([0, 1))$ and let $U' = v'^{-1}([0, 1))$. Then (X, U) and U' are 1-connected since there are deformation retractions of (X, U) onto (X, A) and of U' onto B. Since $Y = U' \cup (Y - B)$, every path in Y is the sum of a finite number of paths each of which is entirely contained in U' or entirely contained in $Y - B$. Hence it will suffice to prove that every path

$$\lambda': I, \dot{I} \to Y - B, (Y - B) \cap U',$$

is homotopic relative to $\dot{I}$ to a path entirely contained in U'. In view of the relative homeomorphism, there is a unique path λ: $I \to X$ such that $f \cdot \lambda = \lambda'$, and we have $\lambda(\dot{I}) \subseteq U$. Since (X, U) is 1-connected, λ is homotopic relative to $\dot{I}$ to a path μ in U. Such a homotopy composed with f yields a homotopy relative to $\dot{I}$ of λ' to a path μ' completely contained in U', which completes the proof of Lemma 2.5.

We conclude this section with the inductive argument which reduces the proof of Theorem 1.1 to the case $n = 1$. Suppose that the assertions of Theorem 1.1 hold whenever $n \leqq m - 1$ $(m \geqq 2)$. Let $Y \in \mathbf{W}^m$, let $Y' = (Y_1, \ldots, Y_{m-1}) \in \mathbf{W}^{m-1}$ and let $(Y'|u)\colon (Y'|S) \to (Y'|T)$, be the natural transformation such that

$$(Y'|u)Y_m = uY \in |SY, TY| = |(Y'|S)Y_m, (Y'|T)Y_m| \qquad (Y_m \in \mathbf{W}).$$

Similarly, if $X \in \mathbf{W}$, let $(u|X)\colon (S|X) \to (T|X)$ be the natural transformation such that

$$(u|X)Y' = u(Y_1, Y_2, \ldots, Y_{m-1}, X) \qquad (Y' \in \mathbf{W}^{m-1}).$$

Then $(u|P)P = uP \simeq cP$, $(S|P)$ and $(T|P)$ are cellular P-functors and $(T|P)P = TP$ is 1-connected. Hence the inductive hypothesis implies that $(u|P) \simeq c$ $(\mathbf{W})$ and that $(T|P)Y'$ is 1-connected $(Y' \in \mathbf{W}^{m-1})$. Thus $(Y'|u)P = (u|P)Y' \simeq cY'$, for each $Y' \in \mathbf{W}^{m-1}$. By Lemma 2.4, $(Y'|S)$ and $(Y'|T)$ are cellular P-functors and since $(Y'|T)P = (T|P)Y'$ is 1-connected, a second application of the inductive hypothesis yields $(Y'|u) \simeq c$ $(\mathbf{W})$ and $(Y'|T)Z$ 1-connected $(Z \in \mathbf{W})$, for each $Y' \in \mathbf{W}^{m-1}$. It follows that $u \simeq c$ $(\mathbf{W})$ and that TY is 1-connected $(Y \in \mathbf{W}^m)$.

3. The case of $n = 1$. For the proof of the case $n = 1$ of Theorem 1.1 we shall require the notion of a singular homotopy equivalence of functors. Let $u\colon S \to T$ be a natural transformation. u is a *singular homotopy equivalence* (SHE) in $\mathbf{C}$ if, for each $X \in \mathbf{C}^n$, uX is an SHE. We recall that this means that uX induces a one-to-one correspondence between the path components of SX and TX and that, for every $x \in SX$,

$$(uX)_*\colon \pi_q(SX, x) \to \pi_q(TX, (uX)x) \qquad (q > 0)$$

are isomorphisms.

For the remainder of this section we shall assume that $n = 1$. If (W, ϕ) is a pair, let $W_0 = \phi_1(W)$, let $j\colon W_0 \to W$ be the injection, and let $\phi\colon W \to W_0$ also denote the map which agrees with ϕ_1. (W, ϕ) is *cofibrant* if j is a cofibration. Thus every cellular pair is cofibrant. (W, ϕ) is *fibrant* if ϕ is a Hurewicz fibration and *bifibrant* if it is both fibrant and cofibrant. We shall prove the following lemma.

Lemma 3.1. *If (W, ϕ) is a cellular pair, then there exists a bifibrant pair (E, p) and a pair map $v\colon (W, \phi) \to (E, p)$ such that $v_0 = v|W_0\colon W_0 \to E_0$ is a homeomorphism and $v\colon W \to E$ is a homotopy equivalence.*

Using Lemma 3.1 we shall prove the following result.

Lemma 3.2. *If T is a cellular P-functor (with $n = 1$), then there exists a P-functor R in $\mathbf{W}$, and a natural transformation $v\colon T \to R$ such that (RP, ϕ_R) is fibrant and such that v is an SHE in $\mathbf{W}$.*

One further basic lemma will be needed, the proof of which we postpone.

LEMMA 3.3. *Let* $u: (W, \phi) \to (V, \psi)$ *be a pair map, where* (W, ϕ) *is cofibrant and* (V, ψ) *is fibrant. If* $u: W \to V$ *is null-homotopic, then there exists a homotopy* $u_t: W \to V$ *with* $u_0 = u$ *and* $u_1 = *$ *such that, for each* $t \in I$, $u_t: (W, \phi) \to (V, \psi)$ *is a pair map.*

Proof of Theorem 1.1 ($n = 1$). Let $v: T \to R$ be as in Lemma 3.2. Then $vP \cdot uP \simeq vP \cdot cP = *: SP \to RP$. Applying Lemma 3.3 with

$$(W, \phi) = (SP, \phi_S), \qquad (V, \psi) = (RP, \phi_R), \qquad \text{and} \qquad u = v \cdot u,$$

it follows that there exists a homotopy $\bar{w}_t$ with $\bar{w}_0 = vP \cdot uP$ and $\bar{w}_1 = *$ such that $\bar{w}_t: (SP, \phi_S) \to (RP, \phi_R)$ is a pair map, for each $t \in I$. By the remarks following Lemma 2.2 we have, for every $Y \in \mathbf{W}$,

$$vY \cdot uY \simeq *: SY \to RY.$$

But SY is a CW-complex and vY is a singular homotopy equivalence, hence there is no obstruction to defining a homotopy $uY \simeq *: SY \to TY$ and we may conclude that $u \simeq c$ ($\mathbf{W}$). Since T is a P-functor in $\mathbf{W}$, we see that

$$\psi_T Y: Y \times TP, (Y \times T*) \cup (* \times TP) \to TY, T*$$

is a relative homeomorphism and an identification map (here $*$ denotes a space with just one point), for every $Y \in \mathbf{W}$. $T*$ is 1-connected, since it is a retract of TP, and hence, in view of Lemma 2.5, it will be sufficient to prove that (X, A) is 1-connected, where $X = Y \times TP$ and

$$A = (Y \times T*) \cup (* \times TP).$$

We recall that the pair (X, A) is 1-connected if every point of X can be joined by a path to some point of A, and if every map $(I, \dot{I}) \to (X, A)$ is homotopic relative to $\dot{I}$ to some map of I into A. Since TP is 1-connected, the first condition is certainly satisfied. For the second condition we can assume without loss of generality that Y is arcwise-connected. In that case A is arcwise-connected, and therefore any path beginning and ending in A is homotopic to a path in A followed by a loop in X based at $(*, *)$ followed by a path in A. Since TP is 1-connected, the loop in X is homotopic to a loop in $Y \times (*)$. Hence the original path is homotopic relative to $\dot{I}$ to a path in A, as required.

In the proof of Lemma 3.3 we shall need certain results concerning separation elements. Let $H: W \times I \to V$ be a homotopy. The *reverse homotopy* $rH: W \times I \to V$ is such that

$$rH(w, t) = H(w, 1 - t) \qquad (w \in W, t \in I).$$

H is a null homotopy if $H(W \times (1)) = * \in V$. If H, H' are homotopies such that $H|W \times (1) = H|W \times (0)$, then their *conjunction* is the homotopy $H \oplus H'$ such that

$$H \oplus H'(w, t) = \begin{cases} H(w, 2t) & (0 \leqq t \leqq \frac{1}{2}) \\ H'(w, 2t-1) & (\frac{1}{2} \leqq t \leqq 1) \end{cases} \quad (w \in W).$$

We recall that the *reduced suspension* of W is the space $\Sigma W = (W \times I)/A$, where

$$A = (W \times (0)) \cup (W \times (1)) \cup ((*) \times I).$$

Let H, H': $W \times I \to V$ be null homotopies such that

$$H|W \times (0) = H'|W \times (1).$$

Then their *separation element* is the class

$$d(H, H') = \{(rH \oplus H') \cdot q^{-1}\} \in [\Sigma W, V],$$

where q: $W \times I \to \Sigma W$ is the identification. We recall that the *track addition* in $[\Sigma W, V]$ is given by the rule

$$\{H \cdot q^{-1}\} + \{H' \cdot q^{-1}\} = \{(H \oplus H') \cdot q^{-1}\},$$

where $H(A) = H'(A) = *$. The proofs of the following two lemmas are straightforward and will be omitted.

LEMMA 3.4. *If $d(H, H')$ is defined and $H''(A) = *$, then $d(H, H) = 0$, $d(H', H) = -d(H, H')$, and $d(H, H' \oplus H'') = d(H, H') + \{H'' \cdot q^{-1}\}$.*

LEMMA 3.5. *$d(H, H') = 0$ if and only if there exists a homotopy θ: $W \times I \times I \to V$ such that $\theta(w, 1, t) = *$, $\theta(w, 0, t) = H(w, 0) = H'(w, 0)$, $\theta(w, t, 0) = H(w, t)$, $\theta(w, t, 1) = H'(w, t)$ for all $w \in W$ and all $t \in I$.*

Proof of Lemma 3.3. Let F: $u \simeq *$: $W \times I \to V$ be a homotopy. We first show that F can be replaced by a homotopy F': $u \simeq *$: $W \times I \to V$ such that $F'(W_0 \times I) \subseteq V_0$. Let $F_0 = F|W_0 \times I$. Then, since u is a pair map, $F_0|W_0 \times (0) = \psi_1 \cdot F_0|W_0 \times (0)$ and if we set

$$H = F \oplus ((rF_0 \oplus \psi_1 \cdot F_0) \cdot (\phi \times i_I)): W \times I \to V$$

and $H_0 = H|W_0 \times I$ we find, after expanding by means of Lemma 3.4, that

$$d(H_0, \psi_1 \cdot H_0) = d(\psi_1 \cdot F_0, \psi_1 \cdot F_0) = 0 \in [\Sigma W_0, V].$$

In consequence of Lemma 3.5, there exists θ: $W_0 \times I \times I \to V$ such that $\theta(w, 1, t) = *$, $\theta(w, 0, t) = u(w)$, $\theta(w, t, 0) = H_0(w, t)$, and $\theta(w, t, 1) = \psi \cdot H_0(w, t)$ $(w \in W_0, t \in I)$. Let

$$B = (W \times (0)) \cup (W_0 \times I) \cup (W \times (1))$$

and let θ be extended to a map θ: $B \times I \to V$ by defining $\theta(w, 1, t) = *$, $\theta(w, 0, t) = u(w)$ $(w \in W, t \in I)$. Since the injection $B \to W \times I$ is a cofibration, θ may be extended further to a map θ: $W \times I \times I \to V$. If we now set

$$F'(w, t) = \theta(w, t, 1) \qquad (w \in W, t \in I),$$

we obtain a homotopy F': $u \simeq *$ with the desired property.

The next stage of the proof is to replace F' by a homotopy

$$G: u \simeq *: W \times I \to V$$

such that $G(W_0 \times I) \subseteq V_0$ and for which there exists a homotopy

$$\mu: \psi \cdot G \simeq G \cdot (\phi \times i_I): W \times I \times I \to V_0$$

such that $\mu(w, 0, t) = \psi \cdot u(w)$ and $\mu(w, 1, t) = *$ $(w \in W, t \in I)$. Let $H = rF' \cdot (\phi \times i_I) \oplus \psi \cdot F'$: $W \times I \to V_0$. Then

$$\{H_0 \cdot q_0^{-1}\} = d(F_0', F_0') = 0 \in [\Sigma W_0, V_0].$$

Since the injection $A \to W \times I$ is a cofibration, there exists a homotopy relative to A from H to H', where H': $W \times I \to V_0$ is such that $H'(W_0 \times I) = *$. Moreover, it follows that

$$\{H' \cdot q^{-1}\} = \{H \cdot q^{-1}\} = d(F' \cdot (\phi \times i_I), \psi \cdot F') \in [\Sigma W, V_0].$$

Now let $G = F' \oplus rH'$. Then

$$\begin{aligned} d(G \cdot (\phi \times i_I), \psi \cdot G) &= d(F' \cdot (\phi \times i_I) \oplus rH' \cdot (\phi \times i_I), \psi \cdot F' \oplus r\psi \cdot H') \\ &= d(F' \cdot (\phi \times i_I) \oplus rH' \cdot (\phi \times i_I), \psi \cdot F') + \{rH' \cdot q^{-1}\} \\ &= -d(\psi \cdot F', F' \cdot (\phi \times i_I)) - \{rH' \cdot (\phi \times i_I) \cdot q^{-1}\} - \{H' \cdot q^{-1}\} \\ &= \{H' \cdot q^{-1}\} - \{* \cdot q^{-1}\} - \{H' \cdot q^{-1}\} = 0, \end{aligned}$$

so that a homotopy μ with the required properties does exist. Finally, we recall that ψ: $V \to V_0$ is a fibration and that $B \to W \times I$ is a cofibration. By a lifting homotopy extension property [**10**, Theorem 4] it follows that we may lift μ to a homotopy ν: $W \times I \times I \to V$ such that

$$\psi \cdot \nu = \mu, \ \nu(w, t, 0) = G(w, t), \ \nu(x, t) = GX \qquad (w \in W, t \in I, x \in B).$$

If we now set

$$u_t(w) = \nu(w, t, 1) \qquad (w \in W, t \in I),$$

then we obtain a homotopy with the desired properties, for we have

$$u_t \cdot \phi_1 w = \nu(\phi w, t, 1) = G(\phi w, t) = \mu(w, t, 1) = \psi \cdot \nu(w, t, 1) = \psi_1 \cdot u_t w,$$

as required.

Proof of Lemma 3.1. If $x \in W_0$, let $\gamma_x \in W_0{}^I$ denote the constant path at x, let $E = \{(w, \lambda) \in W \times W_0{}^I | \lambda(0) = \phi w\}$ and let

$$p_1(w, \lambda) = (\lambda(1), \gamma_{\lambda(1)}), \quad vw = (w, \gamma_{\phi w}) \qquad (w \in W, \lambda \in W_0{}^I).$$

Then v is a pair map, v_0 is certainly a homeomorphism, and it is well known

[**9**, p. 99, Theorem 9], that v is a homotopy equivalence and p is a fibration. It remains to prove that the injection $E_0 \to E$ is a cofibration. Since (W, ϕ) is cellular, $W_0 \to W$ is a cofibration. But (W, W_0) and (E, E_0) are homotopy equivalent pairs and therefore, by [**8**, p. 85, Corollary 1], (E, E_0) has the weak homotopy extension property. By [**8**, p. 85, Corollary 3], it is sufficient to demonstrate the existence of a continuous function $f: E \to I$ such that $f^{-1}(0) = E_0$. Now W_0 is a countable CW-complex, and hence is an $\aleph_0$-space. By [**3**, p. 984, property (J)], W_0^I is an $\aleph_0$-space, hence paracompact and hence normal. Hence [**3**, p. 983, property (D)] implies that W_0^I is perfectly normal. Therefore W_0', the subspace of constant paths, is a closed G_δ in W_0^I. It follows that there exists a continuous function $h: W_0^I \to I$ such that $h^{-1}(0) = W_0'$. If we now set $f(w, \lambda) = \frac{1}{2}(gw + h\lambda)$, where $g: W \to I$ is such that $W_0 = g^{-1}(0)$, we obtain the desired function, which completes the proof of Lemma 3.1.

Proof of Lemma 3.2. Applying Lemma 3.1 to the cellular pair (TP, ϕ_T), let R be the functor corresponding to the associated pair (E, p). Then (RP, ϕ_R) is fibrant, being equivalent to the pair (E, p). Moreover, the pair map v determines a unique natural transformation $v: T \to R$. It remains to prove that v is an SHE in **W**. An application of Lemma 2.5 shows that

$$\psi_T Y: Y \times TP, (Y \times T*) \cup (* \times TP) \to TY, T*,$$

and

$$\psi_R Y: Y \times RP, (Y \times R*) \cup (* \times RP) \to RY, R*$$

induce relative homology isomorphisms ($Y \in \mathbf{W}$). TY and similarly RY are 1-connected, as proved earlier; hence we may argue as in [**1**, p. 27, proof of Theorem 4.1, case $n = 1$] that v is an SHE. This completes the proof of Lemma 3.2 and Theorem 1.1.

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