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Projective Plane Bundles Over an Elliptic Curve

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Abstract. We calculate the dimension of cohomology groups for the holomorphic tangent bundles of each isomorphism class of the projective plane bundle over an elliptic curve. As an application, we construct the families of projective plane bundles, and prove that the families are effectively parametrized and complete.

1 Introduction

Let B be an elliptic curve defined over the field $\mathbb{C}$ of complex numbers, and let Σ be a ruled surface over B. In [4], the dimension of the cohomology group $H^i(\Theta_\Sigma)$ is calculated for $i = 0, 1, 2$, where Θ_Σ is the holomorphic tangent sheaf of Σ. In particular, it is obtained that $H^2(\Theta_\Sigma) = 0$ holds for any isomorphism class of a ruled surface. Consequently, by using the result of [2, 3], we have that there exists a family $\xi: \mathcal{X} \to \mathcal{M}$ with the following properties:

(i) $\dim \mathcal{M} = \dim H^1(\Theta_\Sigma)$;
(ii) $\xi^{-1}(0) \cong \Sigma$ for some point $0 \in \mathcal{M}$;
(iii) the Kodaira–Spencer map at 0 is bijective.

From (iii) this family is effectively parametrized and complete at 0.

In [4], the family as above is concretely constructed for every isomorphism class Σ of a ruled surface. It is constructed by parametrizing the transition function of the ruled surface as a $\mathbb{P}^1$-bundle.

In this paper, we expand the arguments of [4] to projective plane bundles over an elliptic curve, namely, $\mathbb{P}^2$-bundles over an elliptic curve.

If W is a $\mathbb{P}^2$-bundle over an elliptic curve B, then $W \cong \mathbb{P}(E)$ holds for some vector bundle E of rank 3. In Section 2, we summarize the results obtained in [1, 4].

In Section 3, we classify the isomorphism classes of $\mathbb{P}^2$-bundles over B. We have to consider it by dividing the argument into the following 3 cases:

(a) E is isomorphic to a direct sum of 3 line bundles;
(b) E is isomorphic to a direct sum of a line bundle and an indecomposable vector bundle of rank 2;
(c) E is indecomposable.

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In case (c), there exist only three types of isomorphism classes. In case (b), an isomorphism class depends only on the degree of the direct sum components of the defining vector bundle. In case (a), the classification is a bit more complicated than in cases (b) and (c).

In Section 4, we calculate the dimension of the cohomology group $H^i(\Theta_W)$ $(i = 0, 1, 2)$ for every $\mathbb{P}^2$-bundle W, where Θ_W is the holomorphic tangent sheaf of W. In order to do it, we write the patching data, namely, the transition function as a $\mathbb{P}^2$-bundle for every isomorphism classes.

As an application, we construct the families of $\mathbb{P}^2$-bundles that are similar to the cases of ruled surfaces. We set some $\mathbb{P}^2$-bundle W, and construct the family $\xi\colon \mathcal{X} \to \mathcal{M}$ with $\xi^{-1}(0) \cong W$ for some $0 \in \mathcal{M}$ and $\dim \mathcal{M} = \dim H^1(\Theta_W)$. We consider three cases of the isomorphism classes of W. The families are given by parametrizing the transition function of W as a $\mathbb{P}^2$-bundle. Furthermore, we prove that these families are effectively parametrized and complete.

2 Preliminaries

Let $\mathcal{E}_B(r, d)$ be a set of the isomorphism classes of indecomposable vector bundles of rank r and of degree d over an elliptic curve B.

Theorem 2.1 (*cf.* [1]) *If B is an elliptic curve, then we have $\mathcal{E}_B(r, d) \neq \varnothing$ for any $r \in \mathbb{Z}_{>0}$ and $d \in \mathbb{Z}$. For any $E_1, E_2 \in \mathcal{E}_B(r, d)$, $E_1 \cong E_2 \otimes L$ holds for some $L \in \mathcal{E}_B(1, 0)$. For any $r \in \mathbb{N}$, there is a unique element $F_r \in \mathcal{E}_B(r, 0)$ with $H^0(B, F_r) \cong \mathbb{C}$. For any $E \in \mathcal{E}_B(r, 0) \setminus \{F_r\}$, we have $H^0(B, E) = 0$.*

Remark 2.2 We use the notation $\mathcal{E}_B(r, d)$ and F_r freely in the rest of the paper.

Lemma 2.3 (*cf. e.g.,* [4]) *If B is an elliptic curve, then the following hold:*

(i) *There exists an automorphism $\varphi \in \operatorname{Aut}(B)$ satisfying $L_1 \cong \varphi^* L_2$ for any two line bundles L_1 and L_2 of degree d, where d is any positive integer.*

(ii) *Let $p \in B$ be any fixed point. For any $L \in \operatorname{Pic}^0(B)$, there exists a point $p' \in B$ satisfying $L \cong \mathcal{O}_B(p - p')$, where $-$ is an operator of divisors.*

(iii) *Let U_1 be the local coordinate neighbourhood near some point $p \in B$, and let u_1 be the coordinate on U_1. Put $U_0 := B \setminus \{p\}$, and let v_0 be a complex number. Then the line bundle obtained by patching $(u_1, z_0) \in U_0 \times \mathbb{C}$ and $(u_1, z_1) \in U_1 \times \mathbb{C}$ when they satisfy $z_0 = e^{v_0/u_1} z_1$ for $u_1 \in U_0 \cap U_1$ is contained in $\mathcal{E}_B(1, 0)$. Furthermore, this line bundle is trivial if and only if v_0/u_1 is in the image of $H^1(B, \mathbb{Z}) \to H^1(B, \mathcal{O}_B)$.*

3 Classification of the Isomorphism Classes

Lemma 3.1 *Let d_1, d_2, d_1', d_2' be integers with $0 < d_1 < d_2$ and $0 < d_1' < d_2'$. Put $E := \mathcal{O}_B \oplus L_1 \oplus L_2$ for $L_i \in \mathcal{E}_B(1, d_i)$ $(i = 1, 2)$ and $E' := \mathcal{O}_B \oplus L_1' \oplus L_2'$ for $L_j' \in \mathcal{E}_B(1, d_j')$ $(j = 1, 2)$. Then $\mathbb{P}(E) \cong \mathbb{P}(E')$ holds if and only if there exists an automorphism $\varphi \in \operatorname{Aut}(B)$ with $\varphi^* L_1 \cong L_1'$ and $\varphi^* L_2 \cong L_2'$. Necessarily, we have $d_1 = d_1'$ and $d_2 = d_2'$.*

Proof Denote by $\pi\colon W := \mathbb{P}(E) \to B$ and $\pi'\colon W' := \mathbb{P}(E') \to B$ the $\mathbb{P}^2$-bundles. Let $\Psi\colon W \to W'$ be an isomorphism. Since there exists no surjective morphism from $\mathbb{P}^2$ to B, Ψ maps any fiber of π to the fiber of π'. Then we get the following commutative diagram:

$$\begin{array}{ccc} W & \xrightarrow{\Psi} & W' \\ {\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi'} \\ B & \xrightarrow[\varphi]{} & B, \end{array}$$

where φ is an automorphism of B induced by Ψ. If T' is a tautological divisor of W', then we have $\pi_* \Psi^* \mathcal{O}_{W'}(T') \cong \varphi^* E'$ by the base change theorem. Hence, we obtain the line bundle $\mathcal{L}$ with $E \cong (\varphi^* E') \otimes \mathcal{L}$. We obtain $\mathcal{L} \cong \mathcal{O}_B$, $L_1' \cong \varphi^* L_1$ and $L_2' \cong \varphi^* L_2$.

Next, we prove the converse. Let W'' be the fiber product $W' \times_B B$ for $\pi'\colon W' := \mathbb{P}(E') \to B$ and $\varphi\colon B \to B$. Then we obtain the morphisms $\Psi'\colon W'' \to W'$ and $\pi''\colon W'' \to B$. Clearly, Ψ' is an isomorphism. On the other hand, we obtain

$$\pi''_* \Psi'^* \mathcal{O}_{W'}(T') \cong \varphi^* E' \cong E,$$

by the base change theorem, where T' is the tautological divisor of W'. Hence we obtain $W'' \cong W := \mathbb{P}(E)$. ■

Remark 3.2 For $d_1 \in \mathbb{Z}_{>0}$, there exists $\varphi \in \operatorname{Aut}(C)$ with $\varphi^* L_1 \cong L_1'$ for any $L_1, L_1' \in \mathcal{E}_C(1, d_1)$. Hence, when we consider the isomorphism class of $\mathbb{P}^2$-bundles as in Lemma 3.1, we can fix the second component of the direct sum and have only to consider the case where the isomorphism class of the third component varies.

Lemma 3.3 *Put $E := \mathcal{O}_B \oplus E_0$ and $E' := \mathcal{O}_B \oplus E_0'$ for $E_0 \in \mathcal{E}_B(2, e)$ and $E_0' \in \mathcal{E}_B(2, e')$, where e and e' are both nonzero integers. Then $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}(E')$ if and only if $e = e'$ holds.*

Proof The proof of the only if part is similar to Lemma 3.1. In order to prove the if part, it is sufficient to prove that there exists an automorphism $\psi \in \operatorname{Aut}(B)$ with $E_0 \cong \psi^* E_0'$. If e is odd, there exists ψ with $\det E_0 \cong \det(\psi^* E_0')$ by Lemma 2.3(i). Since the map $\det\colon \mathcal{E}_B(2, e) \to \mathcal{E}_B(1, e)$ is bijective (*cf.* [1, Corollary to Theorem 7]), we obtain $E_0 \cong \psi^* E_0'$. If e is even, there exist line bundles L_0 and L_0' with $E_0 \cong L_0 \otimes F_2$, and $E_0' \cong L_0' \otimes F_2$. Hence, the statement is obtained. ■

Remark 3.4 When $e = e' = 0$, the "if" part of Lemma 3.3 does not hold. Put $E := \mathcal{O}_B \oplus F_2$, and $E' := \mathcal{O}_B \oplus E_0$ for $E_0 \in \mathcal{E}_B(2, 0) \setminus \{F_2\}$. Then the cohomology groups of the holomorphic tangent bundles for $\mathbb{P}(E)$ and $\mathbb{P}(E')$ have different dimension each other. See Theorem 4.1(ix) and (xii).

The following is trivial by the results of [1].

Lemma 3.5 *The isomorphism classes of the projective plane bundles defined by indecomposable vector bundles over an elliptic curve of rank three are the following:*

(i) $\mathbb{P}(F_3)$ $(F_3 \in \mathcal{E}_B(3, 0))$,

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(ii) $\mathbb{P}(F_{3,1})$ $(F_{3,1} \in \mathcal{E}_B(3,1))$,
(iii) $\mathbb{P}(F_{3,2})$ $(F_{3,2} \in \mathcal{E}_B(3,2))$.

4 The Cohomology Groups of the Holomorphic Tangent Sheaf

In this section, we consider the dimension of the cohomology groups of the holomorphic tangent sheaf for each isomorphism class of $\mathbb{P}^2$-bundle. Let E be a vector bundle of rank 3 over an elliptic curve B, and $W := \mathbb{P}(E)$ the $\mathbb{P}^2$-bundle. Then from the results of the previous section, we have to consider the following cases:

(i) $E \cong \mathcal{O}_B^{\oplus 3}$.
(ii) $E \cong \mathcal{O}_B \oplus \mathcal{O}_B \oplus L_2$, where $L_2 \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$.
(iii) $E \cong \mathcal{O}_B \oplus L_1 \oplus L_2$, where $L_1, L_2 \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $L_1 \not\cong L_2$.
(iv) $E \cong \mathcal{O}_B \oplus \mathcal{O}_B \oplus L_2$, where $L_2 \in \mathcal{E}_B(1, d_2)$, $d_2 > 0$.
(v) $E \cong \mathcal{O}_B \oplus L_1 \oplus L_1$, where $L_1 \in \mathcal{E}_B(1, d_1)$, $d_1 > 0$.
(vi) $E \cong \mathcal{O}_B \oplus L_1 \oplus L_1'$, where $L_1, L_1' \in \mathcal{E}_B(1, d_1)$, $L_1 \not\cong L_1'$, $d_1 > 0$.
(vii) $E \cong \mathcal{O}_B \oplus L_1 \oplus L_2$, where $L_1 \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$ and $L_2 \in \mathcal{E}_B(1, d_2)$, $d_2 > 0$.
(viii) $E \cong \mathcal{O}_B \oplus L_1 \oplus L_2$, where $L_i \in \mathcal{E}_B(1, d_i)(i = 1, 2)$ and $0 < d_1 < d_2$.
(ix) $E \cong \mathcal{O}_B \oplus F_2$.
(x) $E \cong \mathcal{O}_B \oplus E_0$, where $E_0 \in \mathcal{E}_B(2, e)$, $e > 0$ and $e \equiv 0(2)$.
(xi) $E \cong \mathcal{O}_B \oplus E_0$, where $E_0 \in \mathcal{E}_B(2, e)$, $e > 0$ and $e \equiv 1(2)$.
(xii) $E \cong L \oplus F_2$, where $L \in \mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$.
(xiii) $E \cong F_2 \oplus L$, where $L \in \mathcal{E}_B(1, d)$ and $d > 0$.
(xiv) $E \cong F_{2,1} \oplus L$, where $L \in \mathcal{E}_B(1, d)$ and $d > 0$.
(xv) $E \cong F_3$.
(xvi) $E \cong F_{3,1}$.
(xvii) $E \cong F_{3,2}$.

Let B be defined as $B = \mathbb{C}/\langle 1, \omega\rangle$, where ω is an element of the upper half plane of $\mathbb{C}$. For any $u_0 \in \mathbb{C}$, let $[u_0] \in B$ be a class whose representative element is u_0.

For each case, W is obtained as following:

In cases (i), (iv), (v), (ix), (x), (xi), (xiii), (xiv), (xv), (xvi), and (xvii), let U_0 and U_1 be as in Lemma 2.3(iii). Then W is obtained by patching $U_0 \times \mathbb{P}^2$ and $U_1 \times \mathbb{P}^2$. Let u_1 be a local coordinate near p. Denote by $(u_i, (X_i : Y_i : Z_i))$ $(i = 0, 1)$ the element of $U_i \times \mathbb{P}^2$, where $(X_i : Y_i : Z_i)$ is a homogeneous coordinate of $\mathbb{P}^2$.

In case (i), we obtain W by patching

$$(u_0, (X_0 : Y_0 : Z_0)) \in U_0 \times \mathbb{P}^2 \quad \text{and} \quad (u_1, (X_1 : Y_1 : Z_1)) \in U_1 \times \mathbb{P}^2$$

when they satisfy $X_0 = X_1$, $Y_0 = Y_1$, $Z_0 = Z_1$ and $[u_0] = p + u_1$, where $p + u_1$ means $[u' + u_1]$ for $u' \in \mathbb{C}$ such that $p = [u']$.

Similarly, in other cases, W is obtained by the following patching data with the equality $[u_0] = p + u_1$:

In case (iv), $X_0 = X_1$, $Y_0 = Y_1$, $Z_0 = \frac{1}{u_1^{d_2}} Z_1$.

In case (v), $X_0 = X_1$, $Y_0 = \frac{1}{u_1^{d_1}} Y_1$, $Z_0 = \frac{1}{u_1^{d_1}} Z_1$.

In case (ix), $X_0 = X_1$, $Y_0 = Y_1 + \frac{1}{u_1} Z_1$, $Z_0 = Z_1$.

In case (x), $X_0 = X_1$, $Y_0 = \frac{1}{u_1^{e_0}} Y_1 + \frac{1}{u_1^{e_0+1}} Z_1$, $Z_0 = \frac{1}{u_1^{e_0}} Z_1$, $\left(e_0 = \frac{e}{2}\right)$.

In case (xi), $X_0 = X_1, Y_0 = \frac{1}{u_1^{e_0}} Y_1 + \frac{1}{u_1^{e_0+2}} Z_1, Z_0 = \frac{1}{u_1^{e_0+1}} Z_1, \left(e_0 = \frac{e-1}{2} \right)$.
In case (xiii), $X_0 = X_1 + \frac{1}{u_1} Y_1, Y_0 = Y_1, Z_0 = \frac{1}{u_1^d} Z_1$.
In case (xiv), $X_0 = X_1 + \frac{1}{u_1^2} Y_1, Y_0 = \frac{1}{u_1} Y_1, Z_0 = \frac{1}{u_1^d} Z_1$.
In case (xv), $X_0 = X_1 + \frac{1}{u_1} Y_1, Y_0 = Y_1 + \frac{1}{u_1} Z_1, Z_0 = Z_1$.
In case (xvi), $X_0 = X_1 + \frac{1}{u_1} Y_1, Y_0 = Y_1 + \frac{1}{u_1^2} Z_1, Z_0 = \frac{1}{u_1} Z_1$.
In case (xvii), $X_0 = X_1 + \frac{1}{u_1} Z_1, Y_0 = Y_1 + \frac{1}{u_1^3} Z_1, Z_0 = \frac{1}{u_1^2} Z_1$.

Next, we consider cases (ii), (vi), (vii), (viii), and (xii). Let $p \in B$ and $p' \in B$ $(p \neq p')$ be points with $L_2 \cong \mathcal{O}_B(p - p')$ in case (ii), $L_1 \cong \mathcal{O}_B(d_1 p)$ and $L_1' \cong \mathcal{O}_B(d_1 p')$ in case (vi), $L_1 \cong \mathcal{O}_B(p - p')$ and $L_2 \cong \mathcal{O}_B(d_2 p)$ in case (vii), $L_1 \cong \mathcal{O}_B(d_1 p)$, and $L_2 \cong \mathcal{O}_B(d_2 p')$ in case (viii) and $L \cong \mathcal{O}_B(p - p')$ in case (xii). Put $U_0 := B \smallsetminus \{p, p'\}$, and let (U_1, u_1) and (U_2, u_2) be local coordinate system near p and p', respectively. We can assume that $U_1 \cap U_2 = \varnothing$. Denote the element of $U_i \times \mathbb{P}^2$ by $(u_i, (X_i : Y_i : Z_i))$ $(i = 0, 1, 2)$.

In case (ii), we obtain W by patching $(u_0, (X_0 : Y_0 : Z_0))$ and $(u_1, (X_1 : Y_1 : Z_1))$ when they satisfy

$$X_0 = X_1, Y_0 = Y_1, Z_0 = \frac{1}{u_1} Z_1 \quad \text{and} \quad [u_0] = p + u_1;$$

$(u_0, (X_0 : Y_0 : Z_0))$ and $(u_2, (X_2 : Y_2 : Z_2))$ when they satisfy

$$X_0 = X_2, Y_0 = Y_2, Z_0 = u_2 Z_2 \quad \text{and} \quad [u_0] = p' + u_2.$$

Similarly, in other cases, W is obtained by the following patching data with the equalities $[u_0] = p + u_1$ and $[u_0] = p' + u_2$:
In case (vi),

$$X_0 = X_1, Y_0 = \frac{1}{u_1^{d_1}} Y_1, Z_0 = Z_1 \quad \text{and} \quad X_0 = X_2, Y_0 = Y_2, Z_0 = \frac{1}{u_2^{d_1}} Z_2.$$

In case (vii),

$$X_0 = X_1, Y_0 = \frac{1}{u_1} Y_1, Z_0 = \frac{1}{u_1^{d_2}} Z_1 \quad \text{and} \quad X_0 = X_2, Y_0 = u_2 Y_2, Z_0 = Z_2.$$

In case (viii),

$$X_0 = X_1, Y_0 = \frac{1}{u_1^{d_1}} Y_1, Z_0 = Z_1 \quad \text{and} \quad X_0 = X_2, Y_0 = Y_2, Z_0 = \frac{1}{u_2^{d_2}} Z_2.$$

In case (xii),

$$X_0 = \frac{1}{u_1} X_1, Y_0 = Y_1 + \frac{1}{u_1} Z_1, Z_0 = Z_1 \quad \text{and} \quad X_0 = u_2 X_2, Y_0 = Y_2, Z_0 = Z_2.$$

Finally, we consider case (iii). Let $p, p', p'' \in B$ be the points with $L_1 \cong \mathcal{O}_B(p - p')$, $L_2 \cong \mathcal{O}_B(p - p'')$. Put $U_0 := B \smallsetminus \{p, p', p''\}$, and let (U_1, u_1), (U_2, u_2), and (U_3, u_3) be the local coordinate systems near p, p', and p'', respectively. We can assume that $U_i \cap U_j = \varnothing$ $(i, j = 1, 2, 3, i \neq j)$. Denote by $(u_i, (X_i : Y_i : Z_i))$ $(i = 0, 1, 2, 3)$ the element of $U_i \times \mathbb{P}^2$.

We obtain W by patching $(u_0,(X_0:Y_0:Z_0))$ and $(u_1,(X_1:Y_1:Z_1))$ with

$$X_0 = X_1, Y_0 = \frac{1}{u_1}Y_1, Z_0 = \frac{1}{u_1}Z_1 \quad \text{and} \quad [u_0] = p + u_1;$$

$(u_0,(X_0:Y_0:Z_0))$ and $(u_2,(X_2:Y_2:Z_2))$ with

$$X_0 = X_2, Y_0 = u_2Y_2, Z_0 = Z_2 \quad \text{and} \quad [u_0] = p' + u_2;$$

$(u_0,(X_0:Y_0:Z_0))$ and $(u_3,(X_3:Y_3:Z_3))$ with

$$X_0 = X_3, Y_0 = Y_3, Z_0 = u_3Z_3 \quad \text{and} \quad [u_0] = p'' + u_3.$$

Theorem 4.1 *Let $\pi\colon W := \mathbb{P}(E) \to B$ be a $\mathbb{P}^2$-bundle over an elliptic curve B defined by a vector bundle E of rank 3, and let Θ_W be the holomorphic tangent bundle.*

(i) *We have* $\dim H^2(W,\Theta_W) = \dim H^3(W,\Theta_W) = 0$.

(ii) *We have* $\dim H^0(W,\Theta_W) = \dim H^1(W,\Theta_W)$. *If we let N be this dimension, then it has the following values:*

Case	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)
N	9	5	3	$2d_2+4$	$2d_1+4$	$2d_1+2$	$2d_2+2$	$2d_2+2$

Case	(ix)	(x)	(xi)	(xii)	(xiii)	(xiv)	(xv)	(xvi)	(xvii)
N	5	$e+2$	$e+1$	3	$2d+2$	$2d$	3	1	1

Proof Consider the following two exact sequences:

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_W(T)\otimes\pi^*E^\vee \longrightarrow \Theta_{W/B} \longrightarrow 0,$$
$$0 \longrightarrow \Theta_{W/B} \longrightarrow \Theta_W \longrightarrow \mathcal{O}_W \longrightarrow 0.$$

Since $H^j(W,\mathcal{O}_W) = 0$ and $H^j(W,\mathcal{O}_W(T)\otimes\pi^*E^\vee) = 0$ hold for $j \geq 2$, we obtain $H^j(W,\Theta_{W/B}) = 0$ and $H^j(W,\Theta_W) = 0$. Since

$$\begin{aligned}\dim H^0(W,\mathcal{O}_W(T)\otimes\pi^*E^\vee) &= \dim H^0(B,E\otimes E^\vee)\\ &= \dim H^1(B,E\otimes E^\vee) = \dim H^1(W,\mathcal{O}_W(T)\otimes\pi^*E^\vee)\end{aligned}$$

holds by Serre duality theorem, we obtain

$$\dim H^0(W,\Theta_W) = \dim H^1(W,\Theta_W).$$

Hence, it is sufficient to prove the statement for $\dim H^0(W,\Theta_W)$.

Put $x_i = X_i/Z_i$ and $y_i = Y_i/Z_i$ for $i = 0,1,2$. On U_0, a global holomorphic vector field $\theta \in H^0(W,\Theta_W)$ is written as

$$\begin{aligned}\theta|_{U_0} = (a_0 + a_1x_0 + a_2y_0 + a_3x_0^2 + a_4x_0y_0)\frac{\partial}{\partial x_0}\\ + (b_0 + b_1x_0 + b_2y_0 + a_3x_0y_0 + a_4y_0^2)\frac{\partial}{\partial y_0} + c\frac{\partial}{\partial u_0},\end{aligned}$$

where a_i $(i = 0,1,2,3,4,5)$, b_j $(j = 0,1,2)$, and c are holomorphic functions on U_0.

Since each case can be proved similarly, we prove only cases (ii) and (xiii), and leave the details of the other cases to readers.

Consider case (ii). On $U_0 \cap U_1$ and on $U_0 \cap U_2$, θ can be written as

$$\begin{aligned}\theta|_{U_0\cap U_1} = a_0\frac{\partial}{\partial x_1} + a_1x_1\frac{\partial}{\partial x_1} + \frac{a_2}{u_1}y_1\frac{\partial}{\partial x_1} + a_3x_1^2\frac{\partial}{\partial x_1} + \frac{a_4}{u_1}x_1y_1\frac{\partial}{\partial x_1}\\ + b_0u_1\frac{\partial}{\partial y_1} + b_1u_1x_1\frac{\partial}{\partial y_1} + \Big(b_2 + \frac{c}{u_1}\Big)y_1\frac{\partial}{\partial y_1}\\ + a_3x_1y_1\frac{\partial}{\partial y_1} + \frac{a_4}{u_1}y_1^2\frac{\partial}{\partial y_1} + c\frac{\partial}{\partial u_1},\\ \theta|_{U_0\cap U_2} = a_0\frac{\partial}{\partial x_2} + a_1x_2\frac{\partial}{\partial x_2} + a_2u_2y_2\frac{\partial}{\partial x_2} + a_3x_2^2\frac{\partial}{\partial x_2} + a_4u_2x_2y_2\frac{\partial}{\partial x_2}\\ + \frac{b_0}{u_1}\frac{\partial}{\partial y_2} + \frac{b_1}{u_1}x_2\frac{\partial}{\partial y_2} + \Big(b_2 - \frac{c}{u_2}\Big)y_2\frac{\partial}{\partial y_2}\\ + a_3x_2y_2\frac{\partial}{\partial y_2} + a_4u_2y_2^2\frac{\partial}{\partial y_2} + c\frac{\partial}{\partial u_2}.\end{aligned}$$

We obtain that a_0, a_1, a_3, and c are arbitrary constants. Furthermore, we obtain $a_2 = 0$, $a_4 = 0$, $b_0 = 0$, and $b_1 = 0$. We have that b_2 is a meromorphic function with two poles p and p' of order one. If ζ is the zeta function with period $(1,\omega)$, there exists a constant c' with $b_2 = c' - c\zeta(u_0 - p) + c\zeta(u_0 - p')$. Consequently, we obtain five arbitrary constants a_0, a_1, a_3, c and c', which leads us to $\dim H^0(W,\Theta_W) = 5$.

Let W be case (xiii). On $U_0 \cap U_1$, θ is written as

$$\begin{aligned}\theta|_{U_0\cap U_1} = \Big(\frac{a_0}{u_1^d} - \frac{b_0}{u_1^{d+1}}\Big)\frac{\partial}{\partial x_1} + \Big(a_1 - \frac{b_1 + dc}{u_1}\Big)x_1\frac{\partial}{\partial x_1}\\ + \Big(a_2 + \frac{a_1 - b_2}{u_1} + \frac{c - b_1}{u_1^2}\Big)y_1\frac{\partial}{\partial x_1} + a_3u_1^dx_1^2\frac{\partial}{\partial x_1} + (a_3u_1^{d-1} + a_4u_1^d)x_1y_1\frac{\partial}{\partial x_1}\\ + \frac{b_0}{u_1^d}\frac{\partial}{\partial y_1} + b_1x_1\frac{\partial}{\partial y_1} + \Big(b_2 + \frac{b_1 - c}{u_1}\Big)y_1\frac{\partial}{\partial y_1}\\ + a_3u_1^dx_1y_1\frac{\partial}{\partial y_1} + (a_3u_1^{d-1} + a_4u_1^d)y_1^2\frac{\partial}{\partial y_1} + c\frac{\partial}{\partial u_1}.\end{aligned}$$

We obtain that a_0 and b_0 are identically zero, and b_1 and c are constants. Since there is no meromorphic function that has p as a pole of order 1 and no other pole, a_1 and b_2 are constants, and $b_1 = c = 0$ holds. For the same reason as above, we obtain $a_1 = b_2$ and a_2 is a constant. Then a_3 and a_4 determine the meromorphic functions which have p as a unique pole of order at most d and $d+1$, respectively. The coefficient of the term for $u_1^{-(d+1)}$ of a_4 depends on a_3. Hence, we obtain $\dim H^0(W,\Theta_W) = 2d+2$. ∎

5 Construction of the Families

In this section, we construct a family of the projective plane bundles over an elliptic curve that is complete and effectively parametrized. Let the notation be as before. Recall that B is defined as $B = \mathbb{C}/\langle 1,\omega\rangle$.

(i) Let $\mathcal{M} \subset \mathbb{C}^8$ be a domain containing the origin, and let $(t_1,\dots,t_8,\omega)$ be the parameter on $\mathcal{M}\times\mathcal{H}$. Let $W_{t_1,t_2,t_3,t_4,t_5,t_6,t_7,t_8,\omega}$ be a $\mathbb{P}^2$-bundle obtained by patching

$(u_0, (X_0 : Y_0 : Z_0)) \in U_0 \times \mathbb{P}^2$ and $(u_1, (X_1 : Y_1 : Z_1)) \in U_1 \times \mathbb{P}^2$ with

$$X_0 = e^{\frac{t_1}{u_1}} X_1 + \frac{t_2}{u_1} Y_1 + \frac{t_3}{u_1} Z_1, \qquad Y_0 = \frac{t_4}{u_1} X_1 + e^{\frac{t_5}{u_1}} Y_1 + \frac{t_6}{u_1} Z_1,$$
$$Z_0 = \frac{t_7}{u_1} X_1 + \frac{t_8}{u_1} Y_1 + Z_1, \qquad [u_0] = p + u_1.$$

Then we have $W_{0,0,0,0,0,0,0,0,\omega} \cong \mathbb{P}(\mathcal{O}_B^{\oplus 3})$. For $i = 1,2,3,4,5,6,7,8$, denote by W_{t_i} the $\mathbb{P}^2$-bundle $W_{t_1,\dots t_8,\omega}$ when $t_i \neq 0$ and $t_j = 0$ ($j \in \{1,\dots,\widehat{i},\dots,8\}$), where $\widehat{i}$ is a sign for elimination. By applying the result of [4, §3], we have $W_{t_1} \cong W_{t_5} \cong \mathbb{P}(\mathcal{O}_B \oplus \mathcal{O}_B \oplus L)$ for some $L \in \mathrm{Pic}^0(B)$, and $W_{t_2} \cong W_{t_3} \cong W_{t_4} \cong W_{t_6} \cong W_{t_7} \cong W_{t_8} \cong \mathbb{P}(\mathcal{O}_B \oplus F_2)$. For $i, i' = 1,2,3,4,5,6,7,8$ ($i < i'$), denote by $W_{t_i,t_{i'}}$ the $\mathbb{P}^2$-bundle $W_{t_1,\dots t_8,\omega}$ when $t_i, t_{i'} \neq 0$ and $t_j = 0$ ($j \in \{1,\dots,\widehat{i},\dots,\widehat{i'},\dots,8\}$). Then we have $W_{t_2,t_6} \cong W_{t_4,t_8} \cong \mathbb{P}(F_3)$.

If we put $x_i = X_i/Z_i$ and $y_i = Y_i/Z_i$ ($i = 1,2$), then we have

$$x_0 = \frac{e^{\frac{t_1}{u_1}} x_1 + \frac{t_2}{u_1} y_1 + \frac{t_3}{u_1}}{\frac{t_7}{u_1} x_1 + \frac{t_8}{u_1} y_1 + 1}, \quad y_0 = \frac{\frac{t_4}{u_1} x_1 + e^{\frac{t_5}{u_1}} y_1 + \frac{t_6}{u_1}}{\frac{t_7}{u_1} x_1 + \frac{t_8}{u_1} y_1 + 1}. \tag{5.1}$$

Theorem 5.1 *The above family is complete and effectively parametrized at the point* $(0,0,0,0,0,0,0,0,\omega)$.

Proof Put $O = (0,0,0,0,0,0,0,0,\omega)$ and let $T_{M\times\mathcal{H},O}$ be the tangent space of $M\times\mathcal{H}$ at O. Put $W := W_{0,0,0,0,0,0,0,0,\omega}$ and let $\sigma\colon T_{M\times\mathcal{H},O} \to H^1(W, \Theta_W)$ be Kodaira–Spencer map. It is sufficient to prove that σ is an isomorphism. Let $\partial W/\partial t_i$ and $\partial W/\partial\omega$ be the image of $\partial/\partial t_i$ ($i = 1,\dots 8$) and $\partial/\partial\omega$ by σ, respectively. It is clear that $\partial W/\partial\omega$ can not be written as a linear combination of $\partial W/\partial t_i$ ($i = 0,\dots,8$). Hence, it is sufficient to prove that $\partial W/\partial t_i$ ($i = 1,\dots,8$) are linearly independent. Consider that these are the elements of the cohomology group $H^1(\mathcal{U}, \Theta_W)$ corresponding to the open covering $\mathcal{U} := \{U_0, U_1\}$. Let $\partial W/\partial t_i$ be expressed as a 1-cocycle $\theta^{(t_i)} = \{\theta_{01}^{(t_i)}\}$. Then by (5.1), we obtain the following:

$$\theta_{01}^{(t_1)} = \frac{x_1}{u_1}\frac{\partial}{\partial x_0}, \quad \theta_{01}^{(t_2)} = \frac{y_1}{u_1}\frac{\partial}{\partial x_0}, \quad \theta_{01}^{(t_3)} = \frac{1}{u_1}\frac{\partial}{\partial x_0}, \quad \theta_{01}^{(t_4)} = \frac{x_1}{u_1}\frac{\partial}{\partial y_0}, \quad \theta_{01}^{(t_5)} = \frac{y_1}{u_1}\frac{\partial}{\partial y_0},$$
$$\theta_{01}^{(t_6)} = \frac{1}{u_1}\frac{\partial}{\partial y_0}, \quad \theta_{01}^{(t_7)} = -\frac{x_1^2}{u_1}\frac{\partial}{\partial x_0} - \frac{x_1 y_1}{u_1}\frac{\partial}{\partial y_0}, \quad \theta_{01}^{(t_8)} = -\frac{x_1 y_1}{u_1}\frac{\partial}{\partial x_0} - \frac{y_1^2}{u_1}\frac{\partial}{\partial y_0}.$$

Assume that $\sum_{i=1}^{8} \alpha_i \theta^{(t_i)} \sim 0$ holds for constants $\alpha_1, \dots, \alpha_8$. Then there exist $\theta_0 \in \Gamma(U_0, \Theta_W)$ and $\theta_1 \in \Gamma(U_1, \Theta_W)$ such that

$$\sum_{i=1}^{8} \alpha_i \theta_{01}^{(t_i)} = \theta_0 - \theta_1. \tag{5.2}$$

If θ_0 and θ_1 are written as

$$\theta_0 = (a_0 + a_1x_0 + a_2y_0 + a_3x_0^2 + a_4x_0y_0)\frac{\partial}{\partial x_0} + (b_0 + b_1x_0 + b_2y_0 + a_3x_0y_0 + a_4y_0^2)\frac{\partial}{\partial y_0} + c_0\frac{\partial}{\partial u_0},$$

$$\theta_1 = (a_{10} + a_{11}x_1 + a_{12}y_1 + a_{13}x_1^2 + a_{14}x_1y_1)\frac{\partial}{\partial x_1} + (b_{10} + b_{11}x_1 + b_{12}y_1 + a_{13}x_1y_1 + a_{14}y_1^2)\frac{\partial}{\partial y_1} + c_1\frac{\partial}{\partial u_1},$$

then by (5.2), we have

$$\begin{aligned}
&\frac{\alpha_1x_1}{u_1}\frac{\partial}{\partial x_0} + \frac{\alpha_2y_1}{u_1}\frac{\partial}{\partial x_0} + \frac{\alpha_3}{u_1}\frac{\partial}{\partial x_0} + \frac{\alpha_4x_1}{u_1}\frac{\partial}{\partial y_0} + \frac{\alpha_5y_1}{u_1}\frac{\partial}{\partial y_0}\\
&\quad + \frac{\alpha_6}{u_1}\frac{\partial}{\partial y_0} - \frac{\alpha_7x_1^2}{u_1}\frac{\partial}{\partial x_0} - \frac{\alpha_7x_1y_1}{u_1}\frac{\partial}{\partial y_0} - \frac{\alpha_8x_1y_1}{u_1}\frac{\partial}{\partial x_0} - \frac{\alpha_8y_1^2}{u_1}\frac{\partial}{\partial y_0}\\
&= (a_0 - a_{10})\frac{\partial}{\partial x_0} + (a_1 - a_{11})x_1\frac{\partial}{\partial x_0} + (a_2 - a_{12})y_1\frac{\partial}{\partial x_0} + (a_3 - a_{13})x_1^2\frac{\partial}{\partial x_0}\\
&\quad + (a_4 - a_{14})x_1y_1\frac{\partial}{\partial x_0} + (b_0 - b_{10})\frac{\partial}{\partial y_0} + (b_1 - b_{11})x_1\frac{\partial}{\partial y_0} + (b_2 - b_{12})y_1\frac{\partial}{\partial y_0}\\
&\quad + (a_3 - a_{13})x_1y_1\frac{\partial}{\partial y_0} + (a_4 - a_{14})y_1^2\frac{\partial}{\partial y_0} + (c_0 - c_1)\frac{\partial}{\partial u_0},
\end{aligned}$$

and hence, we have

$$\begin{aligned}
&a_0 - a_{10} = \frac{\alpha_3}{u_1}, &&a_1 - a_{11} = \frac{\alpha_1}{u_1}, &&a_2 - a_{12} = \frac{\alpha_2}{u_1}, &&a_3 - a_{13} = -\frac{\alpha_7}{u_1},\\
&a_4 - a_{14} = -\frac{\alpha_8}{u_1}, &&b_0 - b_{10} = \frac{\alpha_6}{u_1}, &&b_1 - b_{11} = \frac{\alpha_4}{u_1}, &&b_2 - b_{12} = \frac{\alpha_5}{u_1}.
\end{aligned}$$

We can write $a_0 = \alpha_3/u_1 + a_{10}$. Since p cannot be a pole of order 1 for a_0, we have $\alpha_3 = 0$ ($a_0 = a_{10}$ =constant). Similarly, we obtain $\alpha_i = 0$ ($i = 1, 2, 4, 5, 6, 7, 8$). Therefore, $\partial W/\partial t_i$ ($i = 1, 2, 3, 4, 5, 6, 7, 8$) are linearly independent. ■

(ii) Let $\mathcal{M} \subset \mathbb{C}^4$ be a domain containing the origin, and let $(t_1, t_2, t_3, t_4, \omega)$ be the parameter on $\mathcal{M} \times \mathcal{H}$. Further, let $t \in \mathbb{C}$ be a complex number such that the line bundle whose transition function on $U_0 \cap U_1$ is represented as e^{t/u_1} is contained in $\mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$. Let $W_{t_1,t_2,t_3,t_4,\omega}$ be the $\mathbb{P}^2$-bundle over B obtained by patching $(u_0, (X_0 : Y_0 : Z_0)) \in U_0 \times \mathbb{P}^2$ and $(u_1, (X_1 : Y_1 : Z_1)) \in U_1 \times \mathbb{P}^2$ with

$$X_0 = X_1 + \frac{t_1}{u_1}Y_1, \quad Y_0 = \frac{t_2}{u_1}X_1 + e^{\frac{t_3}{u_1}}Y_1, \quad Z_0 = e^{\frac{t+t_4}{u_1}}Z_1, \quad [u_0] = p + u_1.$$

Then we have $W_{0,0,0,0,\omega} \cong \mathbb{P}(\mathcal{O}_B^{\oplus 2} \oplus L_2)$ for some $L_2 \in \mathrm{Pic}^0(B)$. Furthermore,

$$W_{t_1,0,0,0,\omega} \cong W_{0,t_2,0,0,\omega} \cong \mathbb{P}(F_2 \oplus L_2)$$

when $t_1 \neq 0$ and $t_2 \neq 0$, $W_{0,0,t_3,0,\omega} \cong \mathbb{P}(\mathcal{O}_B \oplus L_1 \oplus L_2)$ for some $L_1 \in \mathrm{Pic}^0(B)$ when $t_3 \neq 0$ and $W_{0,0,0,t_4,\omega} \cong \mathbb{P}(\mathcal{O}_B^{\oplus 2} \oplus L_2')$ for some $L_1' \in \mathrm{Pic}^0(B)$ when $t_4 \neq 0$.

We can prove the following theorem by the same argument as Theorem 5.1.

Theorem 5.2 *The above family is complete and effectively parametrized at* $(0, 0, 0, 0, \omega)$.

(iii) Let $\mathcal{M} \subset \mathbb{C}^2$ be a domain containing the origin, and let (t_1, t_2, ω) be the parameter on $\mathcal{M} \times \mathcal{H}$. Further, let t and t' be complex numbers such that the line bundles whose transition functions on $U_0 \cap U_1$ are represented as e^{t/u_1} and e^{t'/u_1} are contained in $\mathrm{Pic}^0(B) \setminus \{\mathcal{O}_B\}$. Let $W_{t_1,t_2,\omega}$ be the $\mathbb{P}^2$-bundle over B obtained by patching $(u_0, (X_0 : Y_0 : Z_0)) \in U_0 \times \mathbb{P}^2$ and $(u_1, (X_1 : Y_1 : Z_1)) \in U_1 \times \mathbb{P}^2$ with

$$X_0 = X_1, \quad Y_0 = e^{\frac{t+t_1}{u_1}} Y_1, \quad Z_0 = e^{\frac{t'+t_2}{u_1}} Z_1, \quad [u_0] = p + u_1.$$

It is clear that $W_{t_1,t_2,\omega}$ is defined by $\mathcal{O}_B \oplus L_1' \oplus L_2'$ for some $L_i' \in \mathrm{Pic}^0(B)$ $(i = 1, 2)$.

We can prove the following theorem by the same argument as Theorem 5.1.

Theorem 5.3 *The above family is complete and effectively parametrized at* $(0, 0, \omega)$.

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