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SEMI-BAER MODULES OVER DOMAINS

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For a commutative domain R with 1, an R-module M is called a semi-Baer module if $\operatorname{Ext}_R^1(M, D) = 0$ for all divisible R-modules D. We show that finitely generated modules of projective dimension at most 1 are semi-Baer modules and if R is Prüfer or Matlis, then all modules of projective dimension at most 1 are semi-Baer modules.

In his seminal paper [**1**] on mixed Abelian groups, Baer proved that a countable Abelian group B had to be free if $\operatorname{Ext}_Z^1(B, T) = 0$ for all torsion Abelian groups T. The problem of determining those uncountable groups B with this property turned out to be extremely difficult. Only 30 years later was it settled by Griffith [**7**] who showed that B had to be free, no matter what its cardinality was.

The problem of characterising Baer modules B over arbitrary domains R (that is, R-modules B with $\operatorname{Ext}_R^1(B, T) = 0$ for all torsion R-modules T) was raised by Kaplansky [**8**]. He established two lemmas which led Eklof and Fuchs [**2**] to show that Baer modules over valuation domains likewise had to be free. With a lemma in that paper which dealt with regular cardinals as well as a version of Shelah's compactness theorem, Eklof, Fuchs and Shelah [**3**] proved a reduction theorem which reduces the problem of Baer modules to countably generated modules. As an application of the theorem, they proved that a module over an h-local Prüfer domain is a Baer module exactly if it is projective. Very recently, Fuchs and Salce [**6**, p.568] eliminated h-localness from the hypothesis and proved that a module over a Prüfer domain is a Baer module if and only if it is projective.

In this note, we study a weaker form of Baer modules. An R-module M is called a *semi-Baer module* if $\operatorname{Ext}_R^1(M, D) = 0$ for all divisible R-modules D (D is divisible if $rD = D$ for all $0 \neq r \in R$). It turns out that these modules can be characterised in the same way as Baer modules (see Theorem 1). We also obtain a characterisation of divisible modules which might lead to a new study of divisible modules (see Corollary 4).

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Throughout this note, R is a commutative domain with 1 which is not a field. For unexplained terminology, we refer to Fuchs and Salce [**5**].

In view of Kaplansky [**8**], we have

LEMMA 1. *Semi-Baer modules have projective dimension at most* 1.

This is one of the two key properties of Baer modules which played a major role in their characterisations. The other one, flatness, is no longer guaranteed for semi-Baer modules as is illustrated by the module ∂ (see Fuchs and Salce [**5**, p.124]). Note also that we can restrict ourselves to torsion modules of projective dimension 1 regarding their semi-Baer property; to see this consider an exact sequence $0 \to F \to M \to T \to 0$ where F is a free submodule of M of projective dimension 1. Then T is a torsion module of projective dimension 1 and the induced exact sequence $\mathrm{Ext}^1_R(T, D) \to \mathrm{Ext}^1_R(M, D) \to \mathrm{Ext}^1_R(F, D) = 0$ for a divisible module D establishes the claim.

Recall that a submodule N of M is called *tight* if the projective dimension of N and the projective dimension of M/N are both less than or equal to the projective dimension of M. As in the case of Baer modules, we have

LEMMA 2. *Tight submodules of semi-Baer modules are again semi-Baer modules.*

PROOF: Let N be a tight submodule of a semi-Baer module M. Consider the exact sequence $0 \to N \to M \to M/N \to 0$ and the induced exact sequence $0 = \mathrm{Ext}^1_R(M, D) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^2_R(M/N, D) \to \dots$ where D is a divisible module. Since the projective dimension of M/N is at most 1 by Lemma 1, $\mathrm{Ext}^2_R(M/N, D)=0$. Hence, $\mathrm{Ext}^1_R(N, D)=0$, that is, N is a semi-Baer module. ∎

It is shown in [**3**] that countably generated, flat modules have projective dimension at most 1 and are countably presented. More generally, it can be shown that any countably generated module of projective dimension at most 1 is countably presented. Hence we have

COROLLARY 1. *Countably generated semi-Baer modules are countably presented.*

A submodule N of M is said to be a *DEP-submodule* (Divisible Extension Property) if, for each divisible module D, the map $\mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(N, D)$ induced by the inclusion $N \to M$ is surjective.

LEMMA 3. *Let N be a submodule of a semi-Baer module M. Then it is a DEP-submodule of M exactly if M/N is a semi-Baer module.*

PROOF: The induced exact sequence

$$\mathrm{Hom}_R(M, D) \to \mathrm{Hom}_R(N, D) \to \mathrm{Ext}^1_R(M/N, D) \to \mathrm{Ext}^1_R(M, D) = 0$$

induced from the inclusion $N \to M$, where D is a divisible module establishes the result. ∎

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Corollary 2. *DEP-submodules of semi-Baer modules are tight. They are therefore semi-Baer modules.*

The proof of the following lemma which is a verbatim version of Baer modules is the same as that of [**2**, Lemma 9] with the only change of TEP-submodules into DEP-submodules. For a module M, we denote the cardinality of a minimal generating set of M by $\operatorname{gen} M$.

Lemma 4. *Let κ be an uncountable regular cardinal and M a module with $\operatorname{gen} M = \kappa$. Suppose*

$$0 = M_0 < M_1 < \cdots < M_\mu < \dots \quad (\mu < \kappa)$$

is a continuous well-ordered ascending chain of submodules of M such that

(a) $M = \bigcup_{\mu<\kappa} M_\mu$

(b) $\operatorname{gen} M_\mu < \kappa$ *for each* $\mu < \kappa$

(c) M_μ *is a semi-Baer module for each* $\mu < \kappa$.

If the set

$$E = \{\mu < \kappa \mid \text{ there exists } \beta > \mu \text{ such that } M_\beta / M_\mu \text{ is not semi-Baer}\}$$

is stationary in κ, then M is not a semi-Baer module.

With the aid of Lemma 4, we can prove:

Theorem 1. *A module M is a semi-Baer module if and only if there exists a well-ordered continuous ascending chain of submodules*

$$0 = M_0 < M_1 < \dots < M_\mu < \dots < M_\kappa = M \quad (\mu < \kappa)$$

for some ordinal κ such that, for each $\mu < \kappa$, $M_{\mu+1}/M_\mu$ is a countably generated semi-Baer module.

Proof: The proof is basically the same as that of [**3**, Theorem 10 and Theorem A] except that we start from the set N of all countably generated semi-Baer modules instead of countably generated Baer modules. They are still countably presented by Corollary 1 and thus fit the proof. ▯

Now we consider the converse of Lemma 1. Recall that a domain R with its field of quotients Q is called a *Matlis domain* if the projective dimension of Q equals 1. For a characterisation of Matlis domains, see Lee [**9**].

Theorem 2. *Let R be a Prüfer or Matlis domain and M an R-module. Then M is a semi-Baer module exactly if the projective dimension of M is at most 1.*

Proof: If M is an R-module of projective dimension 1 over a Prüfer domain R, then M is the union of a well-ordered continuous ascending chain of submodules

$0 = M_0 < M_1 < \cdots < M_\mu < \cdots < M_\kappa = M$ $(\mu < \kappa)$ such that $M_{\mu+1}/M_\mu$ is finitely presented and cyclic for each $\mu < \kappa$. Hence each $M_{\mu+1}/M_\mu$ is cyclic of projective dimension at most 1 and thus is a semi-Baer module by [**5**, p.36]. Then again by [**5**, p.74], M is also a semi-Baer module. Now suppose R is a Matlis domain. Then every divisible module D is h-divisible. The exact sequence $0 \to H \to E \to D \to 0$ where E is an injective module induces an exact sequence $0 = \mathrm{Ext}^1_R(M,E) \to \mathrm{Ext}^1_R(M,D) \to \mathrm{Ext}^2_R(M,H)$. The last Ext is 0 since the projective dimension of M is at most 1, proving the result. ∎

Semi-Baer modules are abundant. Baer modules are trivial examples. The divisible module ∂ and its tight submodules are semi-Baer modules. By the same argument as in the case of ∂, all simply presented modules are also semi-Baer modules (see Fuchs [**4**]). It is shown in [**5**, p.41] that all RD-projective modules are semi-Baer modules. Now we try to identify a new class of semi-Baer modules.

Lemma 5. *If M is a module of projective dimension 1, then $\mathrm{Tor}^R_1(M,A) = 0$ for all torsion-free modules A.*

Proof: Consider a projective resolution $0 \to H \to F \to M \to 0$ of the module M where F is free and H is projective. From the induced exact sequence $0 \to \mathrm{Hom}_R(M,E) \to \mathrm{Hom}_R(F,E) \to \mathrm{Hom}_R(H,E) \to 0$ where E is an injective module, we derive that the injective dimension of $\mathrm{Hom}_R(M,E)$ is at most 1 since both $\mathrm{Hom}_R(F,E)$ and $\mathrm{Hom}_R(H,E)$ are injective. The isomorphism

$$\mathrm{Ext}^1_R(Q,\mathrm{Hom}_R(M,E)) \cong \mathrm{Hom}_R\left(\mathrm{Tor}^R_1(Q,M),E\right) = 0$$

implies that $\mathrm{Hom}_R(M,E)$ is a cotorsion module by [**5**, p.243]. We obtain $0 = \mathrm{Ext}^1_R(A,\mathrm{Hom}_R(M,E)) \cong \mathrm{Hom}_R\left(\mathrm{Tor}^R_1(A,M),E\right)$. The choice of E implies that $\mathrm{Tor}^R_1(M,A) = 0$. ∎

Let K denote the R-module Q/R. We call a module D *K-injective* if $\mathrm{Ext}^1_R(K,D) = 0$. It is easy to show that K-injective modules are h-divisible. Note also that over Matlis domains, K-injectivity, h-divisibility and divisibility are equivalent.

The next lemma is an analog of the well-known theorem that a module F is flat if and only if its character module $F^\flat = \mathrm{Hom}_Z(F,Q/Z)$ is injective if and only if $F^\flat$ is absolutely pure.

Lemma 6. *For a module A, the following are equivalent:*

(a) *A is torsion-free;*
(b) *$\mathrm{Hom}_R(A,E)$ is K-injective for all injective modules E;*
(c) *$\mathrm{Hom}_R(A,E)$ is h-divisible for all injective modules E;*
(d) *$\mathrm{Hom}_R(A,E)$ is divisible for all injective modules E.*

PROOF: (a) $\Rightarrow$ (b) Since $\mathrm{Tor}_1^R(K,A) = 0$ for all torsion-free modules A, $0 = \mathrm{Hom}_R(\mathrm{Tor}_1^R(K,A),E) \cong \mathrm{Ext}_R^1(K,\mathrm{Hom}_R(A,E))$ for all injective modules E. This implies that $\mathrm{Hom}_R(A,E)$ is K-injective.

(b) $\Rightarrow$ (c) $\Rightarrow$ (d) are trivial.

(d) $\Rightarrow$ (a) Let tA be the torsion submodule of A. The inclusion $tA \to A$ induces an RD-exact sequence $0 \to \mathrm{Hom}_R(A/tA,E) \to \mathrm{Hom}_R(A,E) \to \mathrm{Hom}_R(tA,E) \to 0$. Here $\mathrm{Hom}_R(tA,E)$ is reduced and is at the same time divisible as an epic image of the divisible module $\mathrm{Hom}_R(A,E)$. Thus it is 0. The choice of E implies that tA is 0, and A is torsion-free. $\square$

Recall that a finitely generated module has a *finite projective resolution* if it has a long projective resolution with finitely generated projective modules.

LEMMA 7. *For a module M of projective dimension 1, the following are equivalent*

(a) *M is finitely generated;*
(b) *M is finitely presented;*
(c) *M has a finite projective resolution.*

PROOF: We have only to show that (a) $\Rightarrow$ (b). Consider a projective resolution $0 \to H \to F \to M \to 0$ of M where F is finitely generated, free and H is projective. Since H is of finite rank, H is a summand of a finitely generated, free module F_1. Hence H is trivially finitely generated, and M is finitely presented. $\square$

Note that a finitely presented module M of projective dimension 2 has always a finite projective resolution. Now we prove our main result.

THEOREM 3. *All finitely generated modules of projective dimension at most* 1 *are semi-Baer modules.*

PROOF: Let M be a finitely generated module of projective dimension at most 1 and D a divisible module. By Lemma 7, M has a finite projective resolution and thus we obtain a natural isomorphism (see Rotman [**10**, p.257])

$$\mathrm{Tor}_1^R(M,\mathrm{Hom}_R(D,E)) \cong \mathrm{Hom}_R(\mathrm{Ext}_R^1(M,D),E)$$

where E is an injective module. Since $\mathrm{Hom}_R(D,E)$ is torsion-free, the Tor is 0 by Lemma 5 and thus the Hom on the right hand side is 0. The choice of E implies that $\mathrm{Ext}_R^1(M,D) = 0$, and M is a semi-Baer module. $\square$

COROLLARY 3. *Let M be a module of projective dimension at most* 1. *If there is a well-ordered continuous ascending chain*

$$0 = M_0 < M_1 < \cdots < M_\mu < \cdots < M_\kappa = M \quad (\mu < \kappa)$$

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of submodules of M such that for each $\mu < \kappa$, $M_{\mu+1}/M_\mu$ is finitely generated and of projective dimension at most 1, then M is a semi-Baer module.

Note that a module P is absolutely pure if and only if $\mathrm{Ext}^1_R(N,P) = 0$ for all finitely presented modules N. All h-divisible modules H satisfy $\mathrm{Ext}^1_R(M,H) = 0$ for all modules M of projective dimension at most 1. The next corollary is an analog of these results.

Corollary 4. *The following conditions on a module D are equivalent:*

(a) *D is divisible;*
(b) *$\mathrm{Ext}^1_R(R/Rr, D) = 0$ for every $r \in R$;*
(c) *$\mathrm{Ext}^1_R(R/L, D) = 0$ for every projective ideal L of R;*
(d) *$\mathrm{Ext}^1_R(M, D) = 0$ for every finitely generated module M of projective dimension at most 1.*

Proof: See [5, p.36] and Theorem 3. ∎

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