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LOCAL EXISTENCE AND BLOW-UP CRITERION OF HÖLDER CONTINUOUS SOLUTIONS OF THE BOUSSINESQ EQUATIONS

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Abstract. In this paper we prove the local existence and uniqueness of $C^{1+\gamma}$ solutions of the Boussinesq equations with initial data $v_0, \theta_0 \in C^{1+\gamma}$, $\omega_0, \nabla\theta_0 \in L^q$ for $0 < \gamma < 1$ and $1 < q < 2$. We also obtain a blow-up criterion for this local solutions. More precisely we show that the gradient of the passive scalar θ controls the breakdown of $C^{1+\gamma}$ solutions of the Boussinesq equations.

§1. Introduction

The interactive motion of a passive scalar (e.g. temperature) and atmosphere with an external potential force is modeled by the following Boussinesq equations:

$$
\begin{aligned}
&(1) \qquad & v_t + (v\cdot\nabla)v &= -\nabla p + \theta f, \qquad & \text{curl } f = 0\\
&(2) \qquad & \theta_t + v\cdot\nabla\theta &= 0 & (x,t) \in R^2 \times R_+\\
& & \text{div } v &= 0 & \\
& & v|_{t=0} = v_0, \quad \theta|_{t=0} &= \theta_0, & x \in R^2.
\end{aligned}
$$

In (1) p denotes the scalar pressure of the fluid flow.

It is suggested that these equations have strong resemblance with the 3-D Euler equations in many aspects (see e.g.[7]). In particular the problem of finite time blow-up of smooth solutions of the Boussinesq equations is outstanding as in the case of 3-D Euler equations.

In [4], authors proved local existence of solutions of the Boussinesq equations in the Sobolev spaces $H^m(R^2)$, $m > 2$, and obtained a blow-up criterion of the smooth solutions. The proofs are similar to Kato's [6] and Beale-Kato-Majda's [2] respectively for the 3-D Euler equations.

In this paper we extend the previous results to the case of Hölder continuous initial data. We first prove the unique local existence of the solutions of

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the Boussinesq equations (Theorem 3.1), and then establish a link between the maximum norm of the gradient of the passive scalar θ and the formation of singularities for the Boussinesq equations. That is, if the solutions of the Boussinesq equations with Hölder initial data lose their regularity at some later time, then the maximum norm of the gradient of the passive scalar θ necessarily grows without bound as the critical time approaches (Theorem 4.1).

In the section 2 we transform the Boussinesq equations to the equivalent integro-differential equation for particle trajectories of flow.

In the section 3 we apply the Picard type iteration to the integro-differential equation to construct a local solution. The key step here is the Lipschitz estimate in a properly chosen Banach space. The uniqueness follows easily.

We found that in [5] the author used other type of iteration than Picards' in proving local existence of Hölder continuous solutions of the particle trajectories equations associated with the 3-D Euler equations. This method may be applied to the Boussinesq equations also.

In the section 4 we prove the theorem on the blow-up criterion in the Hölder space. We used some ideas in [1]. We found that similarly to the Sobolev space case [4] $\int_0^t \|\nabla\theta(\cdot,s)\|_{L^\infty}\, ds$ controls the blow-up of $\|v(\cdot,t)\|_{C^{1+\gamma}}$, $\|\theta(\cdot,t)\|_{C^{1+\gamma}}$, $\|\omega(\cdot,t)\|_{L^q}$ and $\|\nabla\theta(\cdot,t)\|_{L^q}$.

Although we assumed curl $f = 0$ (*potential force*) *throughout this paper, this condition on the external force could be relaxed without any technical difficulties.*

§2. Preliminaries

In this section we firstly formulate the Boussinesq equations as an integro-differential equation for particle trajectories following the ideas used in the case of Euler equations [7]. After that we list some facts and lemmas needed in the sequel.

We can regard R^2 as the subspace of R^3 with the zero third component. Then from equation (1) we take curl operator to get, for $\omega = \frac{\partial}{\partial x_1}v_2 - \frac{\partial}{\partial x_2}v_1$ and $\omega' = (0,0,\omega)^t$,

$$\omega'_t + (v\cdot\nabla)\omega' = \nabla\theta\times f. \tag{3}$$

Integrating from $t = 0$ to time t, we obtain

$$\omega(\Psi_t(\alpha),t) = \omega_0(\alpha) + \int_0^t (\nabla\theta\times f)_3(\Psi_s(\alpha),s)\, ds \tag{4}$$

where $(\cdot)_3$ means the third component, and $\Psi_t(\alpha)$ is defined by the following ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\Psi_t(\alpha) & = & v(\Psi_t(\alpha),t) \\ \Psi_t(\alpha)\Big|_{t=0} & = & \alpha\ . \end{cases} \tag{5}$$

By the Biot-Savart law we have

$$v = K_2 * \omega \tag{6}$$

with the kernel

$$K_2(x) = \frac{1}{2\pi}\Big(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big)^t\ .$$

On the other hand, from the equation (2), we get

$$\theta(\Psi_t(\alpha),t) = \theta_0(\alpha) \quad \forall t > 0. \tag{7}$$

Thus

$$\nabla\theta(\Psi_t(\alpha),t)\cdot\nabla\Psi_t(\alpha) = \nabla\theta_0(\alpha).$$

If $\det\ \nabla\Psi_t(\alpha) \neq 0$, then we have

$$\nabla\theta(\Psi_t(\alpha),t) = \nabla\theta_0(\alpha)\cdot(\nabla\Psi_t(\alpha))^{-1}. \tag{8}$$

Combining (5) with (4) and (8), we obtain the following integro-differential equation formulation of the Boussinesq equations:

$$\begin{cases} \dfrac{d}{dt}\Psi_t(\alpha) & = & \displaystyle\int_{R^2} K_2\Big(\Psi_t(\alpha)-\Psi_t(\alpha')\Big)\omega(\Psi_t(\alpha'),t)\ d\alpha' \\ \Psi_t(\alpha)\Big|_{t=0} & = & \alpha\ , \end{cases} \tag{9}$$

with

$$\omega(\Psi_t(\alpha'),t) = \omega_0(\alpha') + \int_0^t \Big(\nabla\theta_0(\alpha')(\nabla\Psi_s(\alpha'))^{-1}\times f(\Psi_s(\alpha'),s)\Big)_3\ ds. \tag{10}$$

Now, we show that the above formulation is equivalent to original Boussinesq equations for sufficiently smooth solutions with vorticity rapidly decreasing near infinity.

The integro-differential equation formulation (9)–(10) from the Boussinesq equations is derived above. Below we verify the derivation in the converse direction.

We firstly observe that the Boussinesq equations is equivalent to the following vorticity-stream formulation:

$$\omega'_t + (v'\cdot\nabla)\omega' = \nabla\theta\times f$$
$$\omega'\Big|_{t=0} = \omega'_0 \;\equiv\; \text{curl}\;\, v'_0,$$

where the velocity v' is determined by $v' = (v,0)^t$ and

(11) $$v(x,t) = \int_{R^2} K_2(x-y)\omega(y,t)\; dy\;.$$

Thus it suffices to show that there exists $\omega'(x,t)$ such that curl $v' = \omega'$, div $v' = 0$ and

$$\omega'_t + (v'\cdot\nabla)\omega' = \nabla\theta\times f.$$

Let $\Psi_t(\cdot)$ be the smooth solution to integro-differential equation (9) for particle trajectories so that $\Psi_t(\cdot)$ is 1-1, onto and has the inverse $\Psi_t^{-1}(\cdot)$.

We define the function $\omega'(x,t)$ for $x = \Psi_t(\alpha)$ by $\omega'(x,t) = (0,0,\omega)^t$ and

$$\begin{aligned}\omega(x,t) &= \omega(\Psi_t(\alpha),t)\\ &= \omega_0(\alpha) + \int_0^t \Big(\nabla\theta_0(\alpha)(\nabla\Psi_s(\alpha))^{-1}\times f(\Psi_s(\alpha),s)\Big)_3\, ds.\end{aligned}$$

Then this function solves the equation

$$\frac{d}{dt}\omega' = \omega'_t + (v'\cdot\nabla)\omega' = \nabla\theta\times f.$$

From (11), direct computation shows that the velocity field $v'(x,t)$ is divergence free.

It remains to show that $\omega' = \text{curl}\;\, v'$. Since

$$\text{div}\;\,\omega' = \frac{\partial}{\partial x_3}\omega = \frac{\partial}{\partial x_3}(\nabla\theta\times f)_3 = 0,$$

using the vector identity

$$-\,\text{curl}\;\,\text{curl}\;\,\psi + \nabla\,\text{div}\;\,\psi = \Delta\psi,$$

v' is determined constructively by $v' = \text{curl } \psi$ where ψ is the vector stream function which solves the Poisson equation $-\Delta\psi = \omega'$. In fact, since $\omega \in L^2$ is smooth and vanishes rapidly as $|x| \nearrow \infty$, we can take L^2-inner product in the above vector identity with $\nabla \text{div } \psi$ and get

$$\|\nabla \text{div } \psi\|_{L^2} = -\int_{R^2} \omega' \cdot \nabla \text{div } \psi \, dx + \int_{R^2} \text{curl}(\text{curl } \psi) \cdot \nabla \text{div } \psi \, dx = 0,$$

where we used the orthogonality $\int_{R^2} u \cdot \nabla q = 0$ for u, q sufficiently rapidly vanishing as $|x| \nearrow \infty$ with div $u = 0$.

It is easy to see that the explicit form of v' is given by computing the curl of the convolution with the Newtonian potential i.e. $v' = (v, 0)$ and $v = K_2 * \omega$. This gives the desired results.

We will use the Banach space $\mathcal{B}$ defined as

$$\mathcal{B} = \{\Psi : R^2 \to R^2, \ \|\!|\Psi|\!\|_\gamma < \infty\},$$

where $\|\!|\cdot|\!\|_\gamma$ is the norm defined by

$$\|\!|\Psi|\!\|_\gamma := |\Psi(0)| + \|\nabla\Psi\|_\gamma \quad , \quad 0 < \gamma < 1$$

and $\|\cdot\|_\gamma = \|\cdot\|_{L^\infty} + |\cdot|_\gamma$. Here $|\cdot|_\gamma$ denotes the Hölder seminorm.

The following lemmas can be established without difficulty (see e.g. [7]).

Lemma 2.1. *Let $\Psi : R^2 \to R^2$ be a smooth, invertible transformation with*

$$|\det \nabla\Psi(\alpha)| \geq c_1 > 0.$$

Then for $0 < \gamma < 1$, there exists $C > 0$ such that

$$\|(\nabla\Psi)^{-1}\|_\gamma \leq C\|\nabla\Psi\|_\gamma$$
$$\|\Psi^{-1}\|_\gamma \leq C\|\!|\Psi|\!\|_\gamma.$$

Lemma 2.2. *Let $\Psi : R^2 \to R^2$ be given as above and let $f : R^2 \to R^n$ be a smooth function. Then for $0 < \gamma < 1$, the superposition $f \circ \Psi$ and $f \circ \Psi^{-1}$ satisfies*

$$|f \circ \Psi|_\gamma \leq |f|_\gamma \|\nabla\Psi\|^\gamma_{L^\infty}$$
$$\|f \circ \Psi\|_\gamma \leq \|f\|_\gamma \Big(1 + \|\!|\Psi|\!\|^\gamma_\gamma\Big)$$
$$\|f \circ \Psi^{-1}\|_\gamma \leq \|f\|_\gamma \Big(1 + C\|\!|\Psi|\!\|^\gamma_\gamma\Big).$$

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We define

$$\mathcal{O}_M = \Big\{ \Psi \in \mathcal{B} : \inf_{\alpha \in R^2} |\det\ \nabla\Psi(\alpha)| > \frac{1}{2},\ \|\Psi\|_\gamma < M \Big\}.$$

Let $\Psi_t(\cdot) \in \mathcal{O}_M$ be a solution to our problem. Since the velocity field v is divergence free, $\det\ \nabla\Psi_t(\alpha) \equiv 1,\ \forall t \geq 0$. If $\Psi \in \mathcal{O}_M$, then by the inverse function theorem, Ψ is locally 1-1 (i.e. local homeomorphism). In fact, $\mathcal{O}_M$ consists of functions $\Psi : R^2 \to R^2$ which are 1-1 and onto.

The following is an immediate corollary of the Hadamard's criterion (Theorem 5.15 p.222 [3]) and Lemma 2.1.

PROPOSITION 2.1. *For any $M > 0$, $0 < \gamma \leq 1$, the set $\mathcal{O}_M$ is nonempty, open and it consists of $1-1$ mappings of R^2 onto R^2.*

We set

$$\|\cdot\|_{1+\gamma} := \|\cdot\|_{L^\infty} + \|D\cdot\|_\gamma, \quad \|\cdot\|_{\gamma\cap L^q} := \|\cdot\|_\gamma + \|\cdot\|_{L^q}.$$

From (6) we have

$$\nabla v = P_2 * \omega + C\omega, \tag{12}$$

where P_2 is a kernel defining singular integral operator with $|P_2(x)| \leq C|x|^{-2}$ (see e.g.[7]).

We will use the following well-known singular integral operator type inequalities.

LEMMA 2.3. *Let $v = K_2 * \omega$ with $\omega \in C^\gamma(R^2) \cap L^q(R)$ for some $q \in (1, \infty)$, then we have*

(i) $\|\nabla v\|_{L^q} \leq C_q \|\omega\|_{L^q}$,

(ii) $\|\nabla v\|_{L^\infty} \leq C_\gamma \Big(\|\omega\|_{L^\infty}(1 + \log^+ \|\omega\|_\gamma) + \|\omega\|_{L^q} \Big)$,

(iii) $\|\nabla v\|_\gamma \leq C_{\gamma,q} \|\omega\|_{\gamma\cap L^q}$.

§3. Local existence in the Hölder space

We now state our main local existence theorem in the form:

THEOREM 3.1. *Let the initial data v_0, $\theta_0 \in C^{1+\gamma}(R^2)$ with $\omega_0, \nabla\theta_0 \in L^q(R^2)$ satisfy $\operatorname{div}\ v_0 = 0$ for some $1 < q < 2$ and $0 < \gamma < 1$. Suppose that*

$f \in L^{\infty}_{loc}([0, \infty); W^{1,\infty}(R^2))$. *Then for any* $M > 1$, *there exists* $T(M) > 0$ *such that there exists a unique solution*

$$\Psi \in C^1([0, T(M)); \mathcal{O}_M) \tag{13}$$

to the integro-differential equation for particle trajectories.

Remark. The above theorem combined with (4), (5), (7) and (8) implies that for solutions (v, θ) we have

$$\begin{aligned} v &\in C([0, T(M)); C^{1+\gamma}(R^2) \cap L^2(R^2)), \\ \theta &\in C([0, T(M)); C^{1+\gamma}(R^2) \cap L^q(R^2)), \\ \omega &\in L^{\infty}([0, T(M)); L^q(R^2)), \\ \nabla\theta &\in L^{\infty}([0, T(M)); L^q(R^2)). \end{aligned}$$

Here, for v, $L^2(R^2)$-norm control is obtained easily by taking L^2-norm of (1) with v and, using the L^∞-bound for $\theta(\cdot, t)$, and $L^\infty(R^2)$-norm control is obtained by the interpolation inequality

$$\|v\|_{L^\infty} \le C\|v\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^\infty}^{\frac{1}{2}}.$$

The control of $\|\theta(\cdot, t)\|_{L^q}$, $\|\nabla\theta(\cdot, t)\|_{L^q}$ and $\|\omega(\cdot, t)\|_{L^q}$ follow immediately by using (7), (8) and (4) respectively.

To prove Theorem 3.1 we rewrite the integro-differential equation (9) in the equivalent integral equation form as follows.

$$\Psi_t(\alpha) = \alpha + \int_0^t F_\tau(\Psi)(\alpha)\, d\tau + \int_0^t G_\tau(\Psi)(\alpha)\, d\tau,$$

where

$$\begin{aligned} F_\tau(\Psi)(\alpha) &:= \int_{R^2} K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha'))\omega_0(\alpha')\, d\alpha', \\ G_\tau(\Psi)(\alpha) &:= \int_{R^2} K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha')) \\ &\qquad \int_0^\tau \Big(\nabla\theta_0(\alpha')(\nabla\Psi_s(\alpha'))^{-1} \times f(\Psi_s(\alpha'), s)\Big)_3\, ds\, d\alpha'. \end{aligned}$$

We first establish:

LEMMA 3.1. *Let v_0, θ_0 and f be given as in Theorem 3.1, and let F_τ and G_τ be defined as above. Suppose that $\Psi_\tau, \Phi_\tau \in \mathcal{O}_M$ for all $\tau \geq 0$. Then we get*

(i) $\|F_\tau(\Psi)\|_\gamma \leq CM^4(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})$

(ii) $\|G_\tau(\Psi)\|_\gamma \leq C \sup_{0\leq s\leq \tau} \|f(\cdot,s)\|_{W^{1,\infty}} M^7 \tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})$

(iii) $\|F'_\tau(\Psi)\Phi_\tau\|_\gamma \leq CM^6(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})\|\Phi_\tau\|_\gamma$

(iv) $\|G'_\tau(\Psi)\Phi_\tau\|_\gamma \leq C \sup_{0\leq s\leq \tau} \|f(\cdot,s)\|_{W^{1,\infty}} M^{10} \tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})\|\Phi_\tau\|_\gamma$.

Proof. Let's estimate $F_\tau(\Psi)$. Setting $x = \Psi_\tau(\alpha)$ and $x' = \Psi_\tau(\alpha')$,

$$\begin{aligned} F_\tau(\Psi)(\alpha) &= \int_{R^2} K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha'))\omega_0(\alpha')\, d\alpha' \\ &= \int_{R^2} K_2(x - x')\, \omega_0(\Psi_\tau^{-1}(x'))\ \det\ \nabla\Psi_\tau^{-1}(x')\, dx' \\ &\equiv K_2 * g_1(x,\tau), \end{aligned}$$

where $g_1(x',\tau) = \omega_0(\Psi_\tau^{-1}(x'))\ \det\ \nabla\Psi_\tau^{-1}(x')$.

Using the potential theory estimates, we get

$$|K_2 * g_1(0,\tau)| \leq \|K_2 * g_1(\cdot,\tau)\|_{L^\infty} \leq C(\|g_1(\cdot,\tau)\|_{L^\infty} + \|g_1(\cdot,\tau)\|_{L^q}),$$

and

$$\|\nabla K_2 * g_1(\cdot,\tau)\|_\gamma \leq C\|g_1(\cdot,\tau)\|_{\gamma\cap L^q}.$$

Applying Lemmas 2.1, 2.2, we can estimate $\|g_1(\cdot,\tau)\|_\gamma$ as

$$\begin{aligned} \|g_1(\cdot,\tau)\|_\gamma &\leq C\|\omega_0 \circ \Psi_\tau^{-1}\|_\gamma\ \|\det\ \nabla\Psi_\tau^{-1}\|_\gamma \\ &\leq C\|\omega_0\|_\gamma\ (1 + C\|\Psi_\tau^{-1}\|_\gamma^\gamma)\|\nabla\Psi_\tau^{-1}\|_\gamma^2 \\ &\leq C\|\omega_0\|_\gamma\ (1 + C\|\Psi_\tau^{-1}\|_\gamma^\gamma)\|\Psi_\tau\|_\gamma^2 \\ &\leq CM^3\|v_0\|_{1+\gamma}. \end{aligned}$$

Moreover, we can control the L^q-norm as follows.

$$\begin{aligned} \|g_1(\cdot,\tau)\|_{L^q} &= \Big(\int_{R^2} |\omega_0(\Psi_\tau^{-1}(x'))\ \det\ \nabla\Psi_\tau^{-1}(x')|^q\, dx'\Big)^{\frac{1}{q}} \\ &\leq \|\det\ \nabla\Psi_\tau^{-1}\|_{L^\infty}\Big(\int_{R^2} |\omega_0(\Psi_\tau^{-1}(x'))|^q\, dx'\Big)^{\frac{1}{q}} \end{aligned}$$

$$
\begin{aligned}
&\le C\|\nabla\Psi_\tau^{-1}\|_\gamma^2\|\det\ \nabla\Psi_\tau\|_{L^\infty}^{\frac{1}{q}}\Big(\int_{R^2}|\omega_0(\alpha')|^q\, d\alpha'\Big)^{\frac{1}{q}}\\
&\le C\|\Psi_\tau\|_\gamma^{2+\frac{2}{q}}\Big(\int_{R^2}|\omega_0(\alpha')|^q\, d\alpha'\Big)^{\frac{1}{q}}\\
&\le CM^4\|\omega_0\|_{L^q}.
\end{aligned}
$$

Hence we obtain

$$
\|F_\tau\Psi\|_\gamma \le CM^4(\|v_0\|_{1+\gamma}+\|\omega_0\|_{L^q})
$$

and this proves (i).

We write $G_\tau(\Psi)$ as

$$
\begin{aligned}
G_\tau(\Psi)(x) = &\int_{R^2}\Big\{K_2(x-x')\ \det\ \nabla\Psi_\tau^{-1}(x')\\
&\int_0^\tau\Big(\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1}\times f(\Psi_s(\Psi_\tau^{-1}(x')),s)\Big)_3\, ds\Big\}\, dx'\\
&\equiv K_2 * g_2(x),
\end{aligned}
$$

where

$$
\begin{aligned}
g_2(x) := \int_0^\tau &\Big(\nabla\theta_0(\Psi_\tau^{-1}(x))(\nabla\Psi_s(\Psi_\tau^{-1}(x)))^{-1}\\
&\times f(\Psi_s(\Psi_\tau^{-1}(x)),s)\Big)_3 \det\ \nabla\Psi_\tau^{-1}(x)\, ds.
\end{aligned}
$$

Similarly, it suffices to estimate $\|g_2\|_{\gamma\cap L^q}$. Using lemmas 2.1 , 2.2 again, we estimate

$$
\begin{aligned}
&\|\Big(\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1}\times f(\Psi_s(\Psi_\tau^{-1}(x')),s)\Big)_3\|_\gamma\\
&\le C\|\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1}\|_\gamma\\
&\qquad\times\Big(\|f(\cdot,s)\|_{L^\infty}+\|Df(\cdot,s)\|_{L^\infty}\|\Psi_\tau\|_\gamma(1+\|\Psi_\tau\|_\gamma^\gamma)\Big)\\
&\le CM^2\|f(\cdot,s)\|_{W^{1,\infty}}\|\nabla\theta_0(\Psi_\tau^{-1}(x'))\|_\gamma\|(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1}\|_\gamma\\
&\le C\|f(\cdot,s)\|_{W^{1,\infty}}M^{2+3\gamma}\|\theta_0\|_{1+\gamma}.
\end{aligned}
\tag{14}
$$

This gives

$$
\begin{aligned}
\|g_2\|_\gamma &\le C\|\det\ \nabla\Psi^{-1}\|_\gamma \sup_{0\le s\le\tau}\|f(\cdot,s)\|_{W^{1,\infty}}M^{2+3\gamma}\tau\|\nabla\theta_0\|_\gamma\\
&\le C\sup_{0\le s\le\tau}\|f(\cdot,s)\|_{W^{1,\infty}}M^{4+3\gamma}\tau\|\theta_0\|_{1+\gamma}.
\end{aligned}
$$

For L^q-norm, using the generalized Minkowski's inequality,

$$
\begin{aligned}
&\|g_2(\cdot,\tau)\|_{L^q} \\
&= \Big[\int_{R^2}\Big\{\int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x))(\nabla\Psi_s(\Psi_\tau^{-1}(x)))^{-1} \\
&\qquad\qquad \times f(\Psi_s(\Psi_\tau^{-1}(x)),s)\Big)_3 \det\ \nabla\Psi_\tau^{-1}(x)\ ds\Big\}^q dx'\Big]^{\frac{1}{q}} \\
&\le C\|\!|\Psi_\tau\|\!|_\gamma^2\int_0^\tau \|\!|(\nabla\Psi_s(\Psi_\tau^{-1}(x)))^{-1}\|\!|_\gamma\|f(\cdot,s)\|_{L^\infty}\Big(\int_{R^2}|\nabla\theta_0(\Psi_\tau^{-1}(x))|^q dx'\Big)^{\frac{1}{q}}\ ds \\
&\le C\sup_{0\le s\le\tau}\|f(\cdot,s)\|_{W^{1,\infty}}M^{3+\gamma+\frac{2}{q}}\tau\|\nabla\theta_0\|_{L^q}.
\end{aligned}
$$

Hence we get

$$
\|\!|G_\tau(\Psi)\|\!|_\gamma \le C\sup_{0\le s\le\tau}\|f(\cdot,s)\|_{W^{1,\infty}}M^7\tau(\|\theta_0\|_{1+\gamma}+\|\nabla\theta_0\|_{L^q})
$$

and this proves (ii).

Setting $\alpha = \Psi_\tau^{-1}(x)$ and $\alpha' = \Psi_\tau^{-1}(x')$, we can write $F'_\tau(\Psi)\Phi_\tau$ as

$$
\begin{aligned}
F'_\tau(\Psi)\Phi_\tau &= \frac{d}{d\epsilon}F_\tau(\Psi+\epsilon\Phi)\Big|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int_{R^2}K_2(\Psi_\tau(\alpha)-\Psi_\tau(\alpha')+\epsilon(\Phi_\tau(\alpha)-\Phi_\tau(\alpha'))\ \omega_0(\alpha')\ d\alpha'\Big|_{\epsilon=0} \\
&= \int_{R^2}\nabla K_2(\Psi_\tau(\alpha)-\Psi_\tau(\alpha'))(\Phi_\tau(\alpha)-\Phi_\tau(\alpha'))\omega_0(\alpha')\ d\alpha' \\
&\equiv \int_{R^2}\nabla K_2(x-x')\Big(\Phi_\tau(\Psi_\tau^{-1}(x))-\Phi_\tau(\Psi_\tau^{-1}(x'))\Big)g_1(x')\ dx'.
\end{aligned}
$$

We observe that $|\nabla K_2(x)| \le C|x|^{-2}$ and

$$
|\Phi_\tau(\Psi_\tau^{-1}(x))-\Phi_\tau(\Psi_\tau^{-1}(x'))| \le \|\nabla\Phi_\tau\circ\Psi_\tau^{-1}\ \nabla\Psi_\tau^{-1}\|_{L^\infty}\ |x-x'|.
$$

Using Hölder inequality, we estimate

$$
\begin{aligned}
|F'_\tau(\Psi)\Phi_\tau| &\le C\|\nabla\Phi_\tau\circ\Psi_\tau^{-1}\ \nabla\Psi_\tau^{-1}\|_{L^\infty} \\
&\qquad\qquad \times\Big\{\|g_1\|_{L^\infty}\int_{|x-x'|\le 1}|x-x'|^{-1}\ dx'\ +\ C\|g_1\|_{L^q}\Big\} \\
&\le C\ \|\nabla\Phi_\tau\circ\Psi_\tau^{-1}\|_\gamma\|\nabla\Psi_\tau^{-1}\|_\gamma\|g_1\|_{\gamma\cap L^q} \\
&\le CM^4\ \|\nabla\Phi_\tau\|_\gamma\ (1+\|\!|\Psi_\tau^{-1}\|\!|_\gamma^\gamma)\ \|\!|\Psi_\tau^{-1}\|\!|_\gamma\ (\|v_0\|_{1+\gamma}+\|\omega_0\|_{L^q}) \\
&\le CM^{5+\gamma}(\|v_0\|_{1+\gamma}+\|\omega_0\|_{L^q})\|\!|\Phi_\tau\|\!|_\gamma
\end{aligned}
$$

It remains to estimate the Hölder norm of $\nabla F'_\tau(\Psi)\Phi_\tau$.
Let's compute the distributional derivative of $\partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)$.

$$\begin{aligned}
&\Big(-\partial_{x_k}\phi,\ \partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\Big)\\
&= -\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\partial_{x_k}\phi\ \partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\,dx\\
&= \lim_{\epsilon\to 0}\Big[\int_{|x|\geq\epsilon}\phi(x)\Big\{\partial_{x_k}\partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\\
&\qquad\qquad + \partial_{x_j}K_2(x)\nabla\Phi_\tau\cdot\partial_{x_k}\Psi_\tau^{-1}(x)\Big\}\,dx\\
&\qquad + \int_{|x|=\epsilon}\phi(x)\partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\frac{x_k}{|x|}\,dx\Big].
\end{aligned}$$

The last term can be estimated as follows. Set $x=\epsilon\omega$ with $|\omega|=1$. Then using the mean value theorem, we get

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{|x|=\epsilon}\phi(x)\partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\frac{x_k}{|x|}\,dx\\
&= \lim_{\epsilon\to 0}\int_{|\omega|=1}\phi(\epsilon\omega)\frac{1}{\epsilon}\partial_{\omega_j}K_2(\epsilon\omega)\nabla\Phi_{\tau,j}(\Psi_\tau^{-1}(s\epsilon\omega))\nabla\Psi_\tau^{-1}(s\epsilon\omega)\epsilon\omega\frac{\epsilon\omega_k}{\epsilon}\epsilon^2\,d\omega\\
&= \lim_{\epsilon\to 0}\int_{|\omega|=1}\phi(\epsilon\omega)\partial_{\omega_j}K_2(\omega)\nabla\Phi_{\tau,j}(\Psi_\tau^{-1}(s\epsilon\omega))\nabla\Psi_\tau^{-1}(s\epsilon\omega)\omega\omega_k\,d\omega\\
&= \phi(0)\nabla\Phi_{\tau,j}(\Psi_\tau^{-1}(0))\nabla\Psi_\tau^{-1}(o)\lim_{\epsilon\to 0}\int_{|\omega|=1}\partial_{\omega_j}K_2(\omega)\omega\omega_k\,d\omega\\
&= \nabla\Phi_{\tau,j}(\Psi_\tau^{-1}(0))\nabla\Psi_\tau^{-1}(0)\ C_{jk}(\phi,\delta).
\end{aligned}$$

Thus we obtain, for all $\phi\in C_0^\infty$,

$$\begin{aligned}
&\Big(-\partial_{x_k}\phi,\ \partial_{x_j}K_2(x)\Big(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0))\Big)\Big)\\
=&\Big(\phi,\ \partial_{x_k}\partial_{x_j}K_2(x)(\Phi_{\tau,j}(\Psi_\tau^{-1}(x)) - \Phi_{\tau,j}(\Psi_\tau^{-1}(0)))\\
&+\partial_{x_j}K_2(x)\nabla\Phi_{\tau,j}\partial_{x_k}\Psi_\tau^{-1}(x)\Big)\\
&+\nabla\Phi_{\tau,j}(\Psi_\tau^{-1}(0))\nabla\Psi_\tau^{-1}(0)C_{jk}(\phi,\delta).
\end{aligned}$$

Using this formula, we know that the distributional derivative of $F'_\tau(\Psi)\Phi_\tau$

is given by

$$\partial_{x_k} F'_\tau(\Psi)\Phi_\tau = \oint_{R^2} \partial_{x_k} \nabla K_2(x-x')\Big(\Phi_\tau(\Psi_\tau^{-1}(x)) - \Phi_\tau(\Psi_\tau^{-1}(x'))\Big) g_1(x')\ dx'$$
$$+ \oint_{R^2} \nabla K_2(x-x')\nabla\Phi_\tau(\Psi_\tau^{-1}(x))\partial_{x_k}\Psi_\tau^{-1}(x) g_1(x')\ dx'$$
$$+\nabla\Phi_\tau(\Psi_\tau^{-1}(x))\nabla\Psi_\tau^{-1}(x)\ C_k g_1(x)$$
$$\equiv I_1 + I_2 + I_3.$$

Applying componentwise the mean value theorem with the parameter $0 \leq s \leq 1$ depending on x, x' and the component, we rewrite the first term I_1 as

$$I_1 = \oint_{R^2} \partial_{x_k}\partial_{x_j} K_2(x-x')\nabla\Phi_{\tau,j}\Big(\Psi_\tau^{-1}(x+s(x'-x)\Big)$$
$$\nabla\Psi_\tau^{-1}(x+s(x'-x))(x-x')g_1(x')\ dx'\ .$$

Recall that for $|\alpha| = |\beta| + 1$, the function $x^\beta \partial_x^\alpha K_2(x)$ is homogeneous of degree -2 and it has the mean value zero on the unit sphere. Thus by the singular integral operator type estimate similar to Lemma 2.3 we have

$$\|I_1\|_\gamma \leq C\|\nabla\Phi_\tau \circ \Psi_\tau^{-1}\ \nabla\Psi_\tau^{-1}\|_\gamma\ \|g_1\|_{\gamma\cap L^q}$$
$$\leq C\|\nabla\Phi_\tau\|_\gamma\ (1 + \|\Psi_\tau^{-1}\|_\gamma^\gamma)\ \|\nabla\Psi_\tau\|_\gamma(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})$$
$$\leq CM^6(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})\|\Phi_\tau\|_\gamma.$$

We estimate I_2 and I_3 as we did in the previous estimates to obtain

$$\|I_2\|_\gamma \leq CM^6(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})\|\Phi_\tau\|_\gamma$$
$$\|I_3\|_\gamma \leq C\|\nabla\Phi_\tau \circ \Psi_\tau^{-1}\|_\gamma\|\nabla\Psi_\tau^{-1}\|_\gamma\|g_1\|_\gamma$$
$$\leq CM^5\|v_0\|_{1+\gamma}\|\Phi_\tau\|_\gamma\ ,$$

and

$$\|F'_\tau(\Psi)\Phi_\tau\|_\gamma \leq CM^6(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q})\|\Phi_\tau\|_\gamma\ .$$

It remains to estimate $G'_\tau(\Psi)\Phi_\tau$.

$$G'_\tau(\Psi)\Phi_\tau(x) = \frac{d}{d\epsilon}\int_{R^2}\Big\{K_2\Big(\Psi_\tau(\alpha) - \Psi_\tau(\alpha') + \epsilon(\Phi_\tau(\alpha) - \Phi_\tau(\alpha'))\Big)$$
$$\int_0^\tau \Big(\nabla\theta_0(\alpha')(\nabla(\Psi_s(\alpha') + \epsilon\Phi_s(\alpha')))^{-1}$$

$$
\begin{aligned}
&\qquad\qquad \times f(\Psi_s(\alpha') + \epsilon\Phi_s(\alpha'), s)\Big)_3 \, ds\Big\} \, d\alpha'\Big|_{\epsilon=0} \\
&= \int_{R^2} \Big\{K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha')) \\
&\qquad\qquad \times \int_0^\tau \Big(\nabla\theta_0(\alpha')(\nabla\Phi_s(\alpha'))^{-1} \times f(\Psi_s(\alpha'), s)\Big)_3 \, ds\Big\} \, d\alpha' \\
&\quad - \int_{R^2} \Big\{K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha')) \\
&\qquad\qquad \times \int_0^\tau \Big(\nabla\theta_0(\alpha')(\nabla\Psi_s(\alpha'))^{-1}\nabla\Phi_s(\alpha')(\nabla\Psi_s(\alpha'))^{-1} \\
&\qquad\qquad \times\nabla f(\Psi_s(\alpha'), s)\Phi_s(\alpha')\Big)_3 \, ds\Big\} \, d\alpha' \\
&\quad + \int_{R^2} \Big\{\nabla K_2(\Psi_\tau(\alpha) - \Psi_\tau(\alpha'))(\Phi_\tau(\alpha) - \Phi_\tau(\alpha')) \\
&\qquad\qquad \times \int_0^\tau \Big(\nabla\theta_0(\alpha')(\nabla\Psi_s(\alpha'))^{-1} \times f(\Psi_s(\alpha'), s)\Big)_3 \, ds\Big\} \, d\alpha' \\
&= \int_{R^2} \Big\{K_2(x - x') \ \det \ \nabla\Psi_\tau^{-1}(x') \\
&\times \int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Phi_s(\Psi_\tau^{-1}(x')))^{-1} \times f(\Psi_s(\Psi_\tau^{-1}(x')), s)\Big)_3 \, ds\Big\} \, dx' \\
&\quad - \int_{R^2} \Big\{K_2(x - x') \ \det \ \nabla\Psi_\tau^{-1}(x') \\
&\qquad\qquad \times \int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x')(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1} \\
&\qquad\qquad \times\nabla\Phi_s(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1} \\
&\qquad\qquad \times\nabla f(\Psi_s(\Psi_\tau^{-1}(x')), s)\Phi_s(\Psi_\tau^{-1}(x'))\Big)_3 \, ds\Big\} \, dx' \\
&\quad + \int_{R^2} \Big\{\nabla K_2(x - x')(\Phi_\tau(\Psi_\tau^{-1}(x)) - \Phi_\tau(\Psi_\tau^{-1}(x'))) \ \det \ \nabla\Psi_\tau^{-1}(x') \\
&\quad\quad \int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1} \\
&\qquad\qquad \times f(\Psi_s(\Psi_\tau^{-1}(x')), s)\Big)_3 \, ds\Big\} \, dx' \\
&= \int_{R^2} K_2(x - x')g_{21}(x') \, dx' + \int_{R^2} K_2(x - x')g_{22}(x') \, dx' \\
&\quad + \int_{R^2} \nabla K_2(x - x')\Big(\Phi_\tau(\Psi_\tau^{-1}(x)) - \Phi_\tau(\Psi_\tau^{-1}(x'))\Big)g_2(x') \, dx' \\
&\equiv I_1 + I_2 + I_3,
\end{aligned}
$$

where we set

$$g_{21}(x') := \int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Phi_s(\Psi_\tau^{-1}(x')))^{-1} \\ \times f(\Psi_s(\Psi_\tau^{-1}(x')), s)\Big)_3 \, ds \ \det\ \nabla\Psi_\tau^{-1}(x'),$$

$$g_{22}(x') := \int_0^\tau \Big(\nabla\theta_0(\Psi_\tau^{-1}(x')(\nabla\Psi_s(\Psi_\tau^{-1}(x')))^{-1} \\ \times\nabla\Phi_s(\Psi_\tau^{-1}(x'))(\nabla\Psi_s(\Psi_\tau^{-1}(x'))^{-1} \\ \times\ \nabla f(\Psi_s(\Psi_\tau^{-1}(x')), s)\Phi_s(\Psi_\tau^{-1}(x'))\Big)_3 \, ds \ \det\ \nabla\Psi_\tau^{-1}(x'),$$

For I_1 and I_2, it suffices to estimate $\|g_{21}\|_{\gamma\cap L^q}$ and $\|g_{22}\|_{\gamma\cap L^q}$. Comparing g_{21} with g_2 we observe that, as we did in (14),

$$\|\nabla\theta_0(\Psi_\tau^{-1}(x'))(\nabla\Phi_s(\Psi_\tau^{-1}(x')))^{-1} \times f(\Psi_s(\Psi_\tau^{-1}(x')), s)\|_\gamma \\ \le C\|\nabla\theta_0(\Psi_\tau^{-1}(x'))\|_\gamma\|\nabla\Phi_s(\Psi_\tau^{-1}(x'))\|_\gamma\|f(\Psi_s(\Psi_\tau^{-1}(x')), s)\|_\gamma \\ \le C\|f(\cdot, s)\|_{W^{1,\infty}} M^4\|\theta_0\|_{1+\gamma}|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

This gives

$$\|g_{21}\|_{\gamma\cap L^q} \le C \sup_{0\le s\le\tau} \|f(\cdot, s)\|_{W^{1,\infty}} M^6\tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

Thus

$$\|I_1\|_\gamma \le C \sup_{0\le s\le\tau} \|f(\cdot, s)\|_{W^{1,\infty}} M^6\tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

Similarly we have

$$\|I_2\|_\gamma \le \|g_{22}\|_{\gamma\cap L^q} \le C \sup_{0\le s\le\tau} \|f(\cdot, s)\|_{W^{1,\infty}} M^{10}\tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

We estimate I_3 similarly to the case of $F'_\tau(\Psi)\Phi_\tau$ to obtain

$$\|I_3\|_\gamma \le C \sup_{0\le s\le\tau} \|f(\cdot, s)\|_{W^{1,\infty}} M^9\tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

Hence we have

$$|\!|\!|G'_\tau(\Psi)\Phi_\tau|\!|\!|_\gamma \le C \sup_{0\le s\le\tau} \|f(\cdot, s)\|_{W^{1,\infty}} M^{10}\tau(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})|\!|\!|\Phi_\tau|\!|\!|_\gamma.$$

This completes the proof.

We are now ready to prove our main theorem in this section.

Proof of Theorem 3.1. Since $v(x,t) = K_2 * \omega$ is divergence free, we observe from Liouville's Theorem that

$$\frac{d}{dt}\det\ \nabla\Psi_t(\alpha) = \mathrm{div}\ v(\Psi_t(\alpha),t)\ \det\ \nabla\Psi_t(\alpha) = 0.$$

Hence $\det\ \nabla\Psi_t(\alpha) \equiv 1$.

We define an operator $\mathcal{F}$ which transforms Banach space-valued function Ψ to Banach space-valued function $\tilde{\Psi}$ by the identity

$$\tilde{\Psi}_t(\alpha)\ =\ \mathcal{F}(\Psi)_t(\alpha)\ =\ \alpha + \int_0^t F_\tau(\Psi)(\alpha)\ d\tau + \int_0^t G_\tau(\Psi)(\alpha)\ d\tau.$$

Now, we can start the successive approximation as follows.

$$\Psi_t^0(\alpha) = \alpha$$

(15)
$$\Psi_t^{n+1}(\alpha) = \alpha + \int_0^t F_\tau(\Psi^n)(\alpha)\ d\tau + \int_0^t G_\tau(\Psi^n)(\alpha)\ d\tau, \quad n = 0,1,2,\ldots.$$

Let's check that the above approximation is well defined on $\mathcal{O}_M$ for sufficiently small $0 \le t \le T$.

Clearly, $\Psi_t^0(\alpha) \in \mathcal{O}_M\ \forall t \ge 0$. Using Lemma 3.1 , we estimate

$$\|\!|\Psi_t^{n+1}|\!\|_\gamma \le 1 + CM^7(1 + \sup_{0\le s\le t}\|f(\cdot,s)\|_{W^{1,\infty}})$$
$$\times(\|v_0\|_{1+\gamma} + \|\theta_0\|_{1+\gamma} + \|\omega_0\|_{L^q} + \|\nabla\theta_0\|_{L^q})t(1+t)$$

such that, for $M > 1$, there exists $T > 0$ and

$$\|\!|\Psi_t^{n+1}|\!\|_\gamma < M\ , \quad 0 \le t \le T.$$

Above T can be evaluated roughly as

$$T < \Big(\frac{M-1}{2CM^7(1 + \sup_{0\le t\le T_0}\|f(\cdot,s)\|_{W^{1,\infty}})}\Big)^{\frac{2}{3}}$$
$$\times\Big(\frac{1}{(\|v_0\|_{1+\gamma} + \|\theta_0\|_{1+\gamma} + \|\omega_0\|_{L^q} + \|\nabla\theta_0\|_{L^q})}\Big)^{\frac{2}{3}}.$$

Moreover, from the above observation,

$$\det\ \nabla\Psi_t^{n+1}(\alpha) \equiv 1 > \frac{1}{2}\ , \quad 0 \le t \le T\ .$$

Hence the above approximation is well defined.

We now set

$$N = \sup_{0\le t\le T} (\|F_t(\Psi^0)\|_\gamma + \|G_t(\Psi^0)\|_\gamma) < \infty$$

and

$$L = C(M^6(\|v_0\|_{1+\gamma} + \|\omega_0\|_{L^q}) \\ + \sup_{0\le t\le T} \|f(\cdot,t)\|_{W^{1,\infty}} M^{10} T(\|\theta_0\|_{1+\gamma} + \|\nabla\theta_0\|_{L^q})).$$

From the successive approximation,

$$\|\Psi_t^1 - \Psi_t^0\|_\gamma \le \int_0^t \|F_\tau(\Psi^0)\|_\gamma + \|G_\tau(\Psi^0)\|_\gamma \, d\tau \le Nt \ .$$

Again, by the approximation and Lemma 3.1,

$$\begin{aligned} \|\Psi_t^2 - \Psi_t^1\|_\gamma &\le \int_0^t \|F_\tau(\Psi^1) + G_\tau(\Psi^1) - F_\tau(\Psi^0) - G_\tau(\Psi^0)\|_\gamma \, d\tau \\ &\le L\int_0^t \|\Psi_\tau^1 - \Psi_\tau^0\|_\gamma \, d\tau \\ &\le LN\frac{t^2}{2} \ . \end{aligned}$$

By induction we obtain

$$\|\Psi_t^{n+1} - \Psi_t^n\|_\gamma \le \frac{N(Lt)^{n+1}}{L(n+1)!} \ , \quad n = 0, 1, \cdots .$$

Note that the infinite series

$$\frac{N}{L}\sum_{n=0}^\infty \frac{(Lt)^{n+1}}{(n+1)!}$$

is convergent to $N/L(e^{Lt}-1)$ and uniformly convergent for $0\le t\le T$. So the sequence $\{\Psi_t^n\}$ is uniformly convergent to Ψ_t in $\mathcal{B}$ for each $t\in[0,T]$. Using Lipschitz continuity of F_τ and G_τ we can pass to the limit in (15) to obtain

$$\Psi_t(\alpha) = \alpha + \int_0^t F_\tau(\Psi)(\alpha)\, d\tau + \int_0^t G_\tau(\Psi)(\alpha)\, d\tau.$$

It is easy to check that $\Psi \in \mathcal{O}_M$, and we can conclude that the solution to (9) exists locally in time.

We now prove the uniqueness.

Let Ψ and Φ be two solutions with the same initial conditions. Then we have

$$\Psi_t(\alpha) - \Phi_t(\alpha) = \int_0^t F_\tau(\Psi)(\alpha) - F_\tau(\Phi)(\alpha)\, d\tau + \int_0^t G_\tau(\Psi)(\alpha) - G_\tau(\Phi)(\alpha)\, d\tau.$$

By Lemma 3.1,

$$\begin{aligned} \|\Psi_t - \Phi_t\|_\gamma &\le \int_0^t \|F_\tau(\Psi)(\alpha) - F_\tau(\Phi)(\alpha)\|_\gamma\, d\tau \\ &\quad + \int_0^t \|G_\tau(\Psi)(\alpha) - G_\tau(\Phi)(\alpha)\|_\gamma\, d\tau \\ &\le L\int_0^t \|\Psi_\tau - \Phi_\tau\|_\gamma\, d\tau. \end{aligned}$$

Then Gronwall's inequality gives

$$\|\Psi_t - \Phi_t\|_\gamma \equiv 0 \quad \forall t \in [0, T].$$

This concludes the proof. □

§4. Blow-up criterion

In this section we will prove the following blow-up criterion of the local $C^{1+\gamma}$ solutions constructed in the previous section:

THEOREM 4.1. *Let $v_0, \theta_0 \in C^{1+\gamma}(R^2)$ with ω_0, $\nabla\theta_0 \in L^q(R^2)$ for $0 < \gamma < 1$, and $1 < q < \infty$. Suppose that*

$$f \in L^\infty([0,T]; W^{1,\infty}(R^2)).$$

Let (v, θ) be the local solution constructed via Theorem 3.1. Then we have

$$\limsup_{t \nearrow T}(\|v(\cdot,t)\|_{1+\gamma} + \|\theta(\cdot,t)\|_{1+\gamma} + \|\omega(\cdot,t)\|_{L^q} + \|\nabla\theta(\cdot,t)\|_{L^q}) = \infty$$

$$\textit{if and only if} \quad \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\, d\tau = \infty.$$

We begin with some lemmas.

LEMMA 4.1. *Let $\Psi_t(x)$ denotes the particle trajectories defined by*

$$\begin{cases} \dfrac{d}{dt}\Psi_t(x) & = & v(x,t) \\ \\ \Psi_0(x) & = & x. \end{cases}$$

Then

$$\|\Psi_s(\Psi_t^{-1}(\cdot))\|_{Lip} \le \exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau, \quad s \le t. \tag{16}$$

In particular,

$$\|\Psi_t^{-1}(\cdot)\|_{Lip} \le \exp\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau. \tag{17}$$

Proof. We integrate the first equation over $[t,s]$ and get

$$\Psi_s(x) = \Psi_t(x) + \int_t^s v(\Psi_\tau(x),\tau)\ d\tau.$$

Setting $y = \Psi_t(x)$ we have

$$\Psi_s(\Psi_t^{-1}(y)) = y + \int_t^s v(\Psi_\tau(\Psi_t^{-1}(y)),\tau)\ d\tau$$

so that

$$\nabla_y\Psi_s(\Psi_t^{-1}(y)) = I + \int_t^s \nabla v(\Psi_\tau(\Psi_t^{-1}(y)),\tau)\cdot\nabla_y\Psi_\tau(\Psi_t^{-1}(y))\ d\tau$$

where I is the 2×2 unit matrix. This integral equation is solved as

$$\nabla_y\Psi_s(\Psi_t^{-1}(y)) = I\ \ exp\ [\int_t^s \nabla v(\Psi_\tau(\Psi_t^{-1}(y)),\tau)\ d\tau].$$

Taking L^∞-norm(in y variable) on both sides we obtain (16). (17) follows immediately from (16) by setting $s = 0$. ∎

LEMMA 4.2. *Let v_0, θ_0 and f be given as in Theorem 4.1. Then*

$$\begin{aligned} \|\omega(\cdot,t)\|_\gamma \le \|v_0\|_{1+\gamma}\exp\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau \\ + C\int_0^t \Big(\|\nabla\theta(\cdot,s)\|_\gamma \exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau\Big)\ ds, \end{aligned} \tag{18}$$

where $C = C(v_0,\theta_0,f)$.

Proof. We rewrite (4) as

$$\omega(x,t) = \omega_0(\Psi_t^{-1}(x)) + \int_0^t (\nabla\theta \times f)_3(\Psi_s(\Psi_t^{-1}(x)), s)\ ds \equiv I_1 + I_2. \tag{19}$$

Simple calculation gives

$$\begin{aligned}\|I_1\|_\gamma &\le \|\omega_0\|_{L^\infty} + \sup_{x\neq y}\frac{|\omega_0(x)-\omega_0(y)|}{|x-y|^\gamma}\|\Psi_t^{-1}(\cdot)\|_{Lip}^\gamma \\ &\le \|v_0\|_{1+\gamma}\exp\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau,\end{aligned}$$

where we used Lemma 4.1 in the last inequality.

On the other hand, from

$$\|(\nabla\theta\times f)_3(\cdot,s)\|_\gamma \le \Big(\|f(\cdot,s)\|_\gamma\|\nabla\theta(\cdot,s)\|_{L^\infty} + \|f(\cdot,s)\|_{L^\infty}\|\nabla\theta(\cdot,s)\|_\gamma\Big),$$

and, observing

$$\begin{aligned}\|u(\Psi_s(\Psi_t^{-1}(\cdot)),s)\|_\gamma &\le C\|u(\cdot,s)\|_\gamma\|\Psi_s(\Psi_t^{-1}(\cdot))\|_{Lip}^\gamma \\ &\le C\|u(\cdot,s)\|_\gamma \exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau\end{aligned}$$

for $u(\cdot,s)\in C^\gamma$, we obtain

$$\begin{aligned}&\|(\nabla\theta\times f)_3(\Psi_s(\Psi_t^{-1}(\cdot)),s)\|_\gamma \\ &\le \Big(\|f(\cdot,s)\|_\gamma\|\nabla\theta(\cdot,s)\|_{L^\infty}+\|f(\cdot,s)\|_{L^\infty}\|\nabla\theta(\cdot,s)\|_\gamma\Big)\exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau \\ &\le \sup_{0\le s\le T}\|f(\cdot,s)\|_\gamma\|\nabla\theta(\cdot,s)\|_\gamma \exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau.\end{aligned}$$

Since $\|f\|_\gamma \le 3\|f\|_{W^{1,\infty}}$, we have for I_2,

$$\|I_2\|_\gamma \le C\int_0^t \left(\|\nabla\theta(\cdot,s)\|_\gamma \exp\int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau\right) ds,$$

and this completes the proof. ∎

LEMMA 4.3. *Let v_0, θ_0, f and q be given as in Theorem 4.1. Suppose that*

$$\int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\ dt < \infty.$$

Then we have

$$\|\omega(\cdot,t)\|_{L^q} + \|\nabla\theta(\cdot,t)\|_{L^q} \le C, \quad 0 \le t \le T$$

where $C = C(\omega_0, \nabla\theta_0, f, T, \int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\ dt)$.

Proof. Take ∇ to (2) so that we have

$$(\nabla\theta)_t + (v\cdot\nabla)\nabla\theta = -\nabla v\nabla\theta. \tag{20}$$

Integrating both sides of (20) over $[0,t]$ we get

$$\nabla\theta(x,t) = \nabla\theta_0(\Psi_t^{-1}(x)) - \int_0^t (\nabla v\nabla\theta)(\Psi_s(\Psi_t^{-1}(x)),s)\ ds. \tag{21}$$

Integrating (21) with respect to x, we obtain

$$\begin{aligned}\|\nabla\theta(\cdot,t)\|_{L^q} &\le \|\nabla\theta_0\|_{L^q} + \int_0^t \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\|\nabla v(\cdot,\tau)\|_{L^q}\ d\tau \\ &\le \|\nabla\theta_0\|_{L^q} + C_q\int_0^t \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\|\omega(\cdot,\tau)\|_{L^q}\ d\tau,\end{aligned} \tag{22}$$

where we used (12) and Lemma 2.3 in the last inequality.

On the other hand, integrating (19) with respect to x, we obtain

$$\|\omega(\cdot,t)\|_{L^q} \le \|\omega_0\|_{L^q} + \sup_{0\le\tau\le T}\|f(\cdot,\tau)\|_{L^\infty}\int_0^t \|\nabla\theta(\cdot,\tau)\|_{L^q}\ d\tau. \tag{23}$$

Combining (23) with (23), we obtain

$$\begin{aligned}&\|\omega(\cdot,t)\|_{L^q} + \|\nabla\theta(\cdot,t)\|_{L^q} \\ &\le C(\omega_0,\nabla\theta_0,f)\Big(1+\int_0^t (1+\|\nabla\theta(\cdot,\tau)\|_{L^\infty})(\|\omega(\cdot,\tau)\|_{L^q}+\|\nabla\theta(\cdot,\tau)\|_{L^q})\ d\tau\Big).\end{aligned}$$

Then Gronwall's inequality gives

$$\|\omega(\cdot,t)\|_{L^q} + \|\nabla\theta(\cdot,t)\|_{L^q} \le C\exp\int_0^t (1+\|\nabla\theta(\cdot,\tau)\|_{L^\infty})\ d\tau \le C,$$

where $C = C(\omega_0, \nabla\theta_0, f, T, \int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\ dt)$ and this completes the proof. ∎

LEMMA 4.4. *Under the same assumptions in Lemma 4.3, we have*

$$
\begin{aligned}
(24)\quad & \|\nabla\theta(\cdot,t)\|_\gamma \\
& \le C \exp \int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau \\
& \quad + C \int_0^t \Big(\|\omega(\cdot,s)\|_\gamma + \|\nabla\theta(\cdot,s)\|_\gamma\Big)\Big(\|\nabla\theta(\cdot,s)\|_{L^\infty} + \|\nabla v(\cdot,s)\|_{L^\infty}\Big) \\
& \qquad \times \exp \int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\, ds,
\end{aligned}
$$

where $C = C(v_0, \nabla\theta_0, f, T, \int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\, dt)$.

Proof. From (21) we obtain

$$
\begin{aligned}
\|\nabla\theta(\cdot,t)\|_\gamma &\le \|\nabla\theta_0(\Psi_t^{-1}(\cdot))\|_\gamma + \int_0^t \|(\nabla v\nabla\theta)(\Psi_s(\Psi_t^{-1}(\cdot)),s)\|_\gamma\, ds \\
&\equiv I_1 + I_2.
\end{aligned}
$$

As in the proof of Lemma 4.2 we have

$$
\|I_1\|_\gamma \le \|\theta_0\|_{1+\gamma} \exp \int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau.
$$

Using similar argument as in the proof of Lemma 4.2, and by Lemma 2.3 we estimate

$$
\begin{aligned}
& \|I_2\|_\gamma \\
& \le \int_0^t \Big(\|\nabla v(\cdot,s)\|_\gamma \|\nabla\theta(\cdot,s)\|_{L^\infty} + \|\nabla v(\cdot,s)\|_{L^\infty} \|\nabla\theta(\cdot,s)\|_\gamma\Big) \\
& \qquad \times \exp \int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\, ds \\
& \le C \int_0^t \Big\{(\|\omega(\cdot,s)\|_{L^q} + \|\omega(\cdot,s)\|_\gamma)\|\nabla\theta(\cdot,s)\|_{L^\infty} + \|\nabla v(\cdot,s)\|_{L^\infty}\|\nabla\theta(\cdot,s)\|_\gamma\Big\} \\
& \qquad \times \exp \int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\, ds \\
& \le C \int_0^t \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\, d\tau \ \exp \int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau \\
& \quad + C \int_0^t \Big(\|\omega(\cdot,s)\|_\gamma \|\nabla\theta(\cdot,s)\|_{L^\infty} \exp \int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big)\, ds \\
& \quad + C \int_0^t \Big(\|\nabla\theta(\cdot,s)\|_\gamma \|\nabla v(\cdot,s)\|_{L^\infty} \exp \int_s^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big)\, ds,
\end{aligned}
$$

where $C = C(v_0, \nabla\theta_0, f, T, \int_0^T \|\nabla\theta(\cdot,t)\|_{L^\infty}\ dt)$. Combining above two estimates on I_1 and I_2, we get the desired result (24).

We now introduce some notations as follows:

$$A(\tau) := \|\nabla v(\cdot,\tau)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty} + 1,$$

$$\tilde{\omega}\ := (\omega, \nabla\theta) \quad \text{with norm} \quad \|\tilde{\omega}\|_\gamma := \|\omega\|_\gamma + \|\nabla\theta\|_\gamma.$$

Combining (18) with (24) we have

$$(25)\quad \|\tilde{\omega}(\cdot,t)\|_\gamma \leq C\exp\int_0^t A(\tau)\ d\tau + C\int_0^t \left(\|\tilde{\omega}(\cdot,s)\|_\gamma A(s)\exp\int_s^t A(\tau)\ d\tau\right)\ ds,$$

where $C = C(v_0, \theta_0, \omega_0, \nabla\theta_0, f, T, \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\ d\tau)$.

We are now ready to prove our main theorem in this section.

Proof of the Theorem 4.1. It is obvious that the condition

$$\limsup_{t\nearrow T}(\|v(\cdot,t)\|_{1+\gamma} + \|\theta(\cdot,t)\|_{1+\gamma} + \|\omega(\cdot,t)\|_{L^q} + \|\nabla\theta(\cdot,t)\|_{L^q}) < \infty$$

implies

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\ d\tau < \infty.$$

Now, we prove the sufficiency part. We assume

$$\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\ d\tau < \infty.$$

From the representation of ω in (19) we observe that

$$(26)\quad \begin{aligned}\|\omega(\cdot,t)\|_{L^\infty} &\leq \|\omega_0\|_{L^\infty} + \int_0^t \|(\nabla\theta\times f)_3(\cdot,s)\|_{L^\infty}\ ds\\ &\leq \|v_0\|_{1+\gamma} + C\sup_{0\leq t\leq T}\|f(\cdot,t)\|_{L^\infty}\int_0^t \|\nabla\theta(\cdot,s)\|_{L^\infty}\ ds\\ &\leq C(v_0, f, \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\ d\tau).\end{aligned}$$

First, we claim that

$$\|\tilde{\omega}(\cdot,t)\|_\gamma \le C = C\Big(v_0, \theta_0, \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\, d\tau\Big). \tag{27}$$

If we set $m(t) = \sup_{0\le s\le t} h(s)$, where

$$h(s) = \|\tilde{\omega}(\cdot,s)\|_\gamma \exp\Big(-\lambda \int_0^s A(\tau)\, d\tau\Big),$$

then the inequality (25) gives for $\lambda > 1$,

$$\begin{aligned} h(t) &\le C + Cm(t)\int_0^t A(s)\exp\Big((-\lambda+1)\int_s^t A(\tau)\, d\tau\Big)\, ds \\ &\le C + Cm(t)\frac{1}{\lambda-1}. \end{aligned}$$

Similarly,

$$h(s) \le C + Cm(s)\frac{1}{\lambda-1}.$$

Thus

$$\sup_{0\le s\le t} h(s) \le C + Cm(t)\frac{1}{\lambda-1}.$$

This implies that

$$m(t) \le C + Cm(t)\frac{1}{\lambda-1}.$$

If we set $\lambda \ge 1+2C$, then $m(t) \le 2C$, $0 \le t \le T$, and

$$\begin{aligned} \|\tilde{\omega}(\cdot,t)\|_\gamma &\le 2C\exp\Big(\lambda\int_0^t (\|\nabla v(\cdot,\tau)\|_{L^\infty} + \|\nabla\theta(\cdot,\tau)\|_{L^\infty} + 1)\, d\tau\Big) \\ &\le C\exp\lambda T\exp\Big(\lambda\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big) \\ &\le C\exp\Big(\lambda\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big). \end{aligned} \tag{28}$$

Applying (28) and Lemma 4.3 to Lemma 2.3, we get

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^\infty} &\le C\Big(\|\omega(\cdot,t)\|_{L^\infty}(1+\log^+\|\omega(\cdot,t)\|_\gamma) + \|\omega(\cdot,t)\|_{L^q}\Big) \\ &\le C\|\omega(\cdot,t)\|_{L^\infty}\Big(1+\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big) + C \\ &\le C(\|\omega(\cdot,t)\|_{L^\infty}+1)\Big(1+\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\, d\tau\Big). \end{aligned}$$

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Then Gronwall's inequality and (26) gives

$$\int_0^t \|\nabla v(\cdot,\tau)\|_{L^\infty}\ d\tau \le \exp\Big(C\int_0^t (\|\omega(\cdot,\tau)\|_{L^\infty}+1)\ d\tau\Big) \le C, \tag{29}$$

Hence we can see that our claim is really true, and obtain

$$\|\theta(\cdot,t)\|_{\gamma+1} = \|\theta(\cdot,t)\|_{L^\infty} + \|\nabla\theta(\cdot,t)\|_\gamma \le C,$$

where

$$C = C\Big(\|v_0\|_{1+\gamma}, \|\theta_0\|_{1+\gamma}, \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty}\ d\tau\Big).$$

It remains to estimate $\|v(\cdot,t)\|_{1+\gamma}$. Applying the divergence operator to (1), we get

$$\Delta p = -tr(\nabla v)^2 + \nabla\theta f + \theta\ \operatorname{div} f,$$

and

$$p = \Delta^{-1}\Big(-tr(\nabla v)^2 + \nabla\theta f + \theta\ \operatorname{div} f\Big).$$

Thus the equation (1) becomes

$$v_t + (v\cdot\nabla)v = \nabla\Delta^{-1}\Big(tr(\nabla v)^2 - \nabla\theta f - \theta\ \operatorname{div} f\Big) + \theta f.$$

Integrating both sides over $[0,t]$, we obtain

$$\begin{aligned}
&v(x,t) - v_0(\Psi_t^{-1}(x))\\
&= \int_0^t \Big(\nabla\Delta^{-1} tr(\nabla v)^2 - \nabla\Delta^{-1}(\nabla\theta f + \theta\ \operatorname{div} f) + \theta f\Big)\Big(\Psi_s(\Psi_t^{-1}(x)), s\Big)\ ds\\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

If we set

$$\mathcal{P} := \nabla\Delta^{-1}(tr(\nabla v)^2),$$

then we have

$$\begin{aligned}
|\mathcal{P}(x,t)| &\le \int_{|x-y|\le 1} \frac{|tr(\nabla v(y,t))^2|}{|x-y|}\ dy + \int_{|x-y|\ge 1} \frac{|tr(\nabla v(y,t))^2|}{|x-y|}\ dy\\
&\le C(\|\nabla v(\cdot,t)\|_{L^\infty}^2 + \|\nabla v(\cdot,t)\|_{L^2}^2)\\
&\le C\Big(\|\omega(\cdot,t)\|_{L^\infty}(1+\log^+\|\omega(\cdot,t)\|_\gamma) + \|\omega(\cdot,t)\|_{L^q}\Big)^2\\
&\qquad + C\|\omega(\cdot,t)\|_{L^2}^2\\
&\le C,
\end{aligned}$$

where we used Lemma 2.3, 4.3, (26), (28), (29) and the interpolation inequality

$$\|\omega\|_{L^2} \le \|\omega\|_{L^\infty}^{1-\frac{q}{2}} \|\omega\|_{L^q}^{\frac{q}{2}}.$$

This gives

$$\|I_1\|_{L^\infty} \le Ct. \tag{30}$$

For I_2, let

$$\mathcal{Q} := \nabla\Delta^{-1}(\nabla\theta f + \theta \ \operatorname{div} f).$$

Using $\int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} \ d\tau < \infty$ and $f \in L^\infty([0,T]; W^{1,\infty}(R^2))$, similar argument gives

$$\begin{aligned}
&|\mathcal{Q}(x,\tau)| \\
&\le \int_{|x-y|\le 1} \frac{|(\nabla\theta f + \theta \ \operatorname{div} \ f)(y,\tau)|}{|x-y|} \, dy \\
&\qquad + \int_{|x-y|\ge 1} \frac{|(\nabla\theta f + \theta \ \operatorname{div} \ f)(y,\tau)|}{|x-y|} \, dy \\
&\le C\Big(\|\nabla\theta(\cdot,\tau)\|_{L^\infty} \|f(\cdot,\tau)\|_{L^\infty} \\
&\qquad + \|\theta_0\|_{L^\infty} \|Df(\cdot,\tau)\|_{L^\infty} + \|(\nabla\theta f + \theta \ \operatorname{div} \ f)(\cdot,\tau)\|_{L^q} \Big) \\
&\le C(1 + \|\nabla\theta(\cdot,\tau)\|_{L^\infty}),
\end{aligned}$$

where $C = C(f, \theta_0, \omega_0, T, \int_0^T \|\nabla\theta(\cdot,\tau)\|_{L^\infty} \ d\tau)$. Thus we obtain

$$\|I_2\|_{L^\infty} \le C(t + \int_0^t \|\nabla\theta(\cdot,\tau)\|_{L^\infty} \ d\tau) \le C. \tag{31}$$

It is easy to see that

$$\|I_3\|_{L^\infty} \le \sup_{0\le t\le T} \|f(\cdot,t)\|_{L^\infty} \|\theta_0\|_{L^\infty} t \le C. \tag{32}$$

Combining (30)–(32), we obtain

$$\sup_{0\le t\le T} \|v(\cdot,t)\|_{L^\infty} \le \|v_0\|_{L^\infty} + C.$$

Moreover, combining (27) with Lemma 4.3, we get

$$\|\nabla v(\cdot,t)\|_\gamma \le C\|\omega(\cdot,t)\|_{\gamma\cap L^q} \le C,$$

and by Lemma 4.3 the proof of the theorem ends. ∎

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