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SYMMETRIZED KRONECKER PRODUCTS OF GROUP REPRESENTATIONS

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1. Introduction. Certain phases are associated with the Kronecker squares and cubes of representations of the finite and of the compact semi-simple groups. These phases are important in giving the symmetry properties of the $1 - jm$ and $3 - jm$ symbols of the groups [**4**; **9**]. It is our primary purpose to evaluate these phases.

The Frobenius-Schur invariant [**12**, p. 142] for an irreducible representation $\{\lambda\}$ of group G

$$(1.1) \quad c_\lambda = \frac{1}{g} \sum_{R \in G} \chi^\lambda(R^2)$$

takes on three values:

(i) $c_\lambda = 1$ if the representation is equivalent to a real representation (orthogonal);

(ii) $c_\lambda = -1$ if the character is real, but the representation is not equivalent to a real representation (symplectic);

(iii) $c_\lambda = 0$ if the character is complex (complex). The representation with the complex conjugate character is written as $\{\lambda\}^*$.

The representation is sometimes said to be of the first, second or third kind, respectively.

These statements are equivalent to locating the identity representation, $\{0\}$ in the symmetrized Kronecker square

$$c_\lambda = 1 \quad if\ \{\lambda\} \otimes \{2\} \supset \{0\}$$
$$c_\lambda = -1 \quad if\ \{\lambda\} \otimes \{1^2\} \supset \{0\}$$
$$c_\lambda = 0 \quad if\ \{\lambda\} \times \{\lambda\} \not\supset \{0\}.$$

When $c_\lambda \neq 0$ this invariant c_λ gives the symmetry property of the $1 - jm$ symbol for $\{\lambda\}$. When $c_\lambda = 0$ we may choose the $1 - jm$ symbol to have either symmetry. Denoting the symmetry phase (the $1 - j$ phase) by ϕ_λ we have

$$\phi_\lambda = c_\lambda \quad if\ c_\lambda = \pm 1$$

and ϕ_λ is arbitrary if $c_\lambda = 0$. It is demonstrated in section 4 that the arbitrary

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phases ϕ_λ may be chosen for most groups in such a way that

$$\phi_\lambda = \phi_{\lambda*} = \pm 1$$

and

$$\phi_\lambda \phi_\mu \phi_\nu = 1 \quad \textit{if } \{\lambda\} \times \{\mu\} \times \{\nu\} \supset \{0\}.$$

There are no exceptions to these rules amongst the compact semi-simple Lie groups although we produce two counterexamples from the finite groups. Apart from such rare exceptions the first part of this result implies that the complex representations of a group may be said to be either quasi-orthogonal or quasi-symplectic, whilst the second part implies that the irreducible representations in the Kronecker product of any two irreducible representations are either all orthogonal or quasi-orthogonal or all symplectic or quasi-symplectic.

The existence of a $3 - jm$ symbol for the three irreducible representations $\{\lambda\}$, $\{\mu\}$, $\{\nu\}$ of a group is associated with the occurrence of $\{0\}$ in the Kronecker product $\{\lambda\} \times \{\mu\} \times \{\nu\}$. The symmetry of such a symbol is

(i) arbitrary if $\{\lambda\}$, $\{\mu\}$ and $\{\nu\}$ are inequivalent,

(ii) partially determined by the occurrence of $\{\nu\}^*$ in $\{\lambda\} \otimes \{2\}$ and $\{\lambda\} \otimes \{1^2\}$ if $\{\lambda\} \cong \{\mu\}$, and

(iii) completely determined by the occurrence of $\{0\}$ in $\{\lambda\} \otimes \{3\}$, $\{\lambda\} \otimes \{21\}$ and $\{\lambda\} \otimes \{1^3\}$ if $\{\lambda\} \cong \{\mu\} \cong \{\nu\}$. The symmetry relations involve only phase factors if and only if $\{\lambda\} \otimes \{21\}$ does not contain $\{0\}$ for every irreducible representation $\{\lambda\}$. For such a case the group is said to be a simple phase group [**4**, **23**]. It is shown in section 5 that in addition to the well-known example, S_6, of a non-simple phase finite group almost all compact semi-simple Lie groups are non-simple phase. In view of this it is preferable to associate the terms simple phase and non-simple phase with representations rather than groups. In this context the Frobenius-Schur invariant for an irreducible representation $\{\lambda\}$ can be generalised. The invariant [**6**]

$$(1.2) \quad m_\lambda = \frac{1}{g} \sum_R \frac{1}{3} [\chi^\lambda(R)^3 - \chi^\lambda(R^3)]$$

is just the coefficient of $\{0\}$ in $\{\lambda\} \otimes \{21\}$. The possible values of m_λ are the non-negative integers which may be divided into two classes:

(i) $m_\lambda = 0$ if $\{\lambda\}$ is simple phase,

(ii) $m_\lambda > 0$ if $\{\lambda\}$ is non-simple phase.

In section 5 we give a number of examples illustrating this classification scheme.

From this preamble it should be clear that it is important to know the resolution of symmetrized Kronecker squares and cubes. Littlewood [**17**] has given some closed formulae in terms of the algebra of S-functions, for both products and symmetrized products of representations of the orthogonal, symplectic and symmetric groups. These formulae, together with some similar formulae for the unitary groups, are given in a modern notation in section 2. They form the basis of a rapid derivation of c_λ. The results for ϕ_λ

are tabulated and discussed in section 4. We use the work of Mal'cev [**19**] and Dynkin [**10**] to derive ϕ_λ for the exceptional Lie groups.

In section 5, we use the formulae of section 2 to evaluate m_λ. Modification rules [**14**] are particularly important. Use is also made of Littlewood's [**16**] modular technique for evaluating plethysms, giving that among the groups SU_n, only SU_2 and SU_3 are simple phase.

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2. Products of representations. Various methods are available for analysing the Kronecker products of irreducible representations of the classical groups. The algebra of S-functions plays a prominent role in the most powerful methods. In preparing tables with the aid of a computer, one of the authors [**22**] used formulae given by Littlewood [**18**; **22**, pp. 39-42] relating characters of the orthogonal and symplectic groups O_n and Sp_n to S-functions. These formulae had the advantage, for specific values of n, of requiring only the use of a standardization technique [**22**, p. 44] appropriate to SU_n rather than the use of the more complicated modification [**22**, p. 43] rules appropriate to O_n and Sp_n. However, with the aid of the simplified statement of these modification rules given by the other author [**14**] it makes sense to exploit some later results of Littlewood [**17**]. These will be quoted here.

Notation. Previous authors [**18**; **22**] have not always clearly distinguished between (the class of equivalent) irreducible representations $\{\!\!\{\lambda\}\!\!\}$, their characters χ^λ and the label itself λ. Robinson [**21**] makes the distinction by the use of different brackets (e.g., $\langle\lambda\rangle$ and $\{\lambda\}$ for a representation and its character for GL_d). Different brackets are used here to partially distinguish between different groups. If an absolute distinction is required subscripts may be added. We shall denote the representations of all U_n, O_n and Sp_n as $\{\lambda\}$, $[\lambda]$, $\langle\lambda\rangle$ respectively, and the representation of S_l as $[\lambda]$. (λ) is a partition of l, but as is well known we also require the concept of a partition generalized to half-integer or negative parts [**22**, p. 133], both for some of the above groups, as well as other groups. For U_n, SU_n and S_n we also use alternate labelling schemes: the composite notation [**1**; **13**; **14**] $\{\lambda;\mu\}$ for U_n and SU_n and the reduced notation [**5**] $\langle\lambda\rangle$ for S_n. The reduced notation is especially powerful.

We use $\times$ and $\otimes$ for Kronecker and symmetrized Kronecker products of representations of all groups. This suggests using $\cdot$ and $\odot$ for direct and symmetrized direct (wreath) products for all groups.

An S-function λ is specified by a partition (λ) and is a polynomial symmetric function of degree l on any given variables. If the partition is known or specified, e.g. (31) or $(\lambda_1, \lambda_2 \ldots)$ we denote the S-function by enclosing the partition in braces, e.g. $\{31\}$ or $\{\lambda_1\lambda_2 \ldots\}$. We have the ordinary algebraic operations of addition, subtraction and multiplication $(\lambda + \mu, \lambda - \mu$ and $\lambda\mu)$ on two S-functions λ and μ. The last operation gives rise to a sum of S-functions ν given by the Littlewood-Richardson rule

$$\lambda\mu = \sum_{\nu} \Gamma_{\lambda\mu\nu}\nu.$$

The coefficient $\Gamma_{\lambda\mu\nu}$ is used to define division [**18**, p. 110; **22** p. 143]

$$\lambda/\nu = \sum_{\mu} \Gamma_{\lambda\mu\nu}\mu.$$

A product of two S-functions of the same degree but on different variables gives rise to inner multiplication $\lambda \circ \mu$. Outer plethysm $\lambda \otimes \mu$ and inner plethysm $\lambda \odot \mu$ are defined by substitutive or inductive processes.

Kronecker and direct products of representations (or equivalently characters) of many groups, finite or continuous, can be related to operations on S-functions. For example, for the orthogonal groups we write

$$[\lambda] \times [\mu] \cong \sum_{\alpha} [(\lambda/\alpha)(\mu/\alpha)]$$

which we read as, "the Kronecker product representation obtained from the two irreducible representations $[\lambda]$ and $[\mu]$ of O_n, is equivalent to the direct sum of irreducible representations, labelled by the partitions which occur in the S-function expression $(\lambda/\alpha)(\mu/\alpha)$". Notice that the brackets ( ) are used here, and elsewhere in similar expressions, for the purpose of punctuation.

The usual notation $(\tilde{\lambda})$ for transposed partitions (interchanging rows of columns of a Young diagram) is generalized to k successive transpositions, thus for S-functions we define

$$\tilde{\lambda}^k = \begin{cases} \lambda \text{ if } k \text{ is even} \\ \tilde{\lambda} \text{ if } k \text{ is odd.} \end{cases}$$

Throughout the paper we use a Greek letter to designate an arbitrary partition of a number designated by the corresponding Latin letter. Sums are over all partitions consistent with this notation (including the S function $\{0\}$).

In view of the fact that we are concerned with symmetrised squares and cubes of representations we consistently use σ and τ to denote members of the sets of S-functions $\{2\}$, $\{1^2\}$ and $\{3\}$, $\{21\}$, $\{1^3\}$ respectively. f^{τ} is the degree of the S-function τ, which is also the dimension of the representation $[\tau]$ of S_3, taking on the values of 1, 2, 1 for the appropriate members of the set.

(i) *The Orthogonal Groups*, O_n. Littlewood's theorems [**17**] are derived from the knowledge that the metric tensor for O_n is quadratic and symmetric.

$$(2.1a)\qquad [\lambda] \times [\mu] \cong \sum_{\alpha} [(\lambda/\alpha)(\mu/\alpha)]$$

$$(2.1b)\qquad [\lambda] \otimes \sigma \cong \sum_{\alpha} [(\lambda/\alpha) \otimes \sigma]$$

$$(2.1c)\qquad [\lambda] \times [\mu] \times [\nu] \cong \sum_{\alpha,\beta,\gamma} [(\lambda/\alpha\beta)(\mu/\beta\gamma)(\nu/\gamma\alpha)]$$

$$\begin{aligned}(2.1d)\qquad [\lambda] \otimes \tau \cong f^{\tau} & \sum_{\alpha<\beta<\gamma} [(\lambda/\alpha\beta)(\lambda/\beta\gamma)(\lambda/\gamma\alpha)] \\ & + \sum_{\alpha\neq\beta} [(\lambda/\alpha \otimes \{2\})((\lambda/\alpha\beta) \otimes (\tau/\{1\})) \\ & \qquad + (\lambda/\alpha \otimes \{1^2\})((\lambda/\alpha\beta) \otimes (\tilde{\tau}/\{1\}))] \\ & + \sum_{\alpha,\epsilon} [((\lambda/\alpha \otimes \{2\}) \otimes (\tilde{\tau}^{e}/\epsilon))((\lambda/\alpha \otimes \{1^2\}) \otimes \epsilon)].\end{aligned}$$

(ii) *The Symplectic Groups*, Sp_n. In this case there again exists a second order invariant or metric tensor to be divided or contracted out. The anti-symmetry of this form ($\{1^2\}$ rather than $\{2\}$) leads to

$$(2.2a)\qquad \langle\lambda\rangle \times \langle\mu\rangle \cong \sum_{\alpha} \langle(\lambda/\alpha)(\mu/\alpha)\rangle$$

$$(2.2b)\qquad \langle\lambda\rangle \otimes \sigma \cong \sum_{\alpha} \langle(\lambda/\alpha) \otimes \tilde{\sigma}^{a}\rangle$$

$$(2.2c)\qquad \langle\lambda\rangle \times \langle\mu\rangle \times \langle\nu\rangle \cong \sum_{\alpha} \langle(\lambda/\alpha\beta)(\mu/\beta\gamma)(\nu/\gamma\alpha)\rangle$$

$$\begin{aligned}(2.2d)\qquad \langle\lambda\rangle \otimes \tau \cong f^{\tau} & \sum_{\alpha<\beta<\gamma} \langle(\lambda/\alpha\beta)(\lambda/\beta\gamma)(\lambda/\gamma\alpha)\rangle \\ & + \sum_{\alpha\neq\beta} \langle(\lambda/\alpha \otimes \{2\})((\lambda/\alpha\beta) \otimes (\tilde{\tau}^{b}/\{1\})) \\ & \qquad + (\lambda/\alpha \otimes \{1^2\})((\lambda/\alpha\beta) \otimes (\tilde{\tau}^{b+1}/\{1\}))\rangle \\ & + \sum_{\alpha,\epsilon} \langle((\lambda/\alpha \otimes \{2\}) \otimes (\tilde{\tau}^{e+a}/\epsilon))((\lambda/\alpha \otimes \{1^2\}) \otimes \epsilon)\rangle.\end{aligned}$$

(iii) *The Unitary Groups*, U_n. The S-functions on the appropriate n variables are simple characters of U_n (but not a complete set) and thus the (symmetrised) Kronecker products of U_n are given directly by outer products (plethysms) of S-functions.

$$(2.3a)\qquad \{\lambda\} \times \{\mu\} \cong \{\lambda\mu\}$$
$$(2.3b)\qquad \{\lambda\} \otimes \mu \cong \{\lambda \otimes \mu\}$$

Certain results may often be derived more simply by using the composite tableaux notation. In the composite tableaux notation the irreducible representations of U_n are specified by a pair of partitions and denoted by $\{\lambda; \mu\}$. Forming Kronecker products of such representations again involves contractions with a second order invariant, which in this case is neither symmetric nor antisymmetric in that it involves both upper and lower tensor indices. It is not difficult to obtain [**1**; **13**; **14**].

$$\{\lambda;\mu\}\times\{\nu;\rho\}\cong\sum_{\alpha,\beta}\{(\lambda/\alpha)(\nu/\beta);\ (\mu/\beta)(\rho/\alpha)\} \tag{2.3c}$$

$$\begin{aligned}\{\lambda;\mu\}\otimes\sigma\cong&\sum_{\alpha<\beta}\{(\lambda/\alpha)(\lambda/\beta);\ (\mu/\alpha)(\mu/\beta)\}\\ &+\sum_{\alpha}\{(\lambda/\alpha)\otimes\sigma;\ (\mu/\alpha)\otimes\{2\}\}\\ &+\sum_{\alpha}\{(\lambda/\alpha)\otimes\tilde{\sigma};\ (\mu/\alpha)\otimes\{1^2\}\}\end{aligned} \tag{2.3d}$$

$$\{\lambda;\mu\}\times\{\nu;\rho\}\times\{\theta;\phi\}\cong\sum_{\alpha\ldots\eta}\{(\lambda/\alpha\beta)(\nu/\gamma\delta)(\theta/\epsilon\eta);\ (\mu/\gamma\epsilon)(\rho/\alpha\eta)(\phi/\beta\delta)\} \tag{2.3e}$$

with a more complicated expression for the symmetrised cubes.

(iv) *The Symmetric Groups*, S_n. The internal structure of the S-function is linked to the characters of the symmetric groups [**22**, p. 21]. It is thus very well-known that

(2.4a) $[\lambda]\times[\mu]\cong[\lambda\circ\mu]$ (note $l = m = n$)

(2.4b) $[\lambda]\otimes\mu\cong[\lambda\odot\mu]$ (note $l = n$).

Littlewood however followed up ideas of Murnaghan on the imbedding of S_n in O_n, to produce an n-independent notation [**5**; **17**]. This notation allows us to evaluate the Kronecker product in terms of outer products of S-functions, and a very much simpler inner product. The consequences have been fully discussed in [**5**].

$$\langle\lambda\rangle\times\langle\mu\rangle\cong\sum_{\alpha\beta\gamma}\langle(\lambda/\alpha\beta)(\mu/\alpha\gamma)(\beta\circ\gamma)\rangle \tag{2.4c}$$

$$\begin{aligned}\langle\lambda\rangle\otimes\sigma\cong&\sum_{\alpha}\sum_{\beta<\gamma}\langle(\lambda/\alpha\beta)(\lambda/\alpha\gamma)(\beta\circ\gamma)\rangle\\ &+\sum_{\alpha\beta\epsilon}\langle((\lambda/\alpha\beta)\otimes\epsilon)(\beta\odot(\sigma\circ\epsilon))\rangle.\end{aligned} \tag{2.4d}$$

The additional complexity of these expressions arises from the fact that the symmetric group possesses a large number of invariants. The occurrence of the inner product and inner plethysm of S-functions foils any simple generalization of these results for cubes and symmetrised cubes. They must contain many terms.

(v) *The Unimodular Groups*, SU_n, R_n *and* A_n. The elements of the unitary group U_n are the direct product of a phase factor (an element of U_1) and an element of the unimodular group SU_n. The above product formulae for U_n thus hold for SU_n, with the proviso that all representations with the nth part of the partition other than zero, are equal to one with zero

$$SU_n \qquad \{\lambda_1, \lambda_2, \ldots, \lambda_n\} = \{\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, \ldots, \lambda_{n-1} - \lambda_n, 0\}.$$

The rotation group, R_n or SO_n, is a subgroup of O_n. Under restriction the representation $[\lambda]$ remains irreducible unless $n = 2k$ and $\lambda_k \neq 0$, in which case under the restriction

$$O_n \to R_n, \quad [\lambda] \to [\lambda]_+ + [\lambda]_-.$$

Similarly the alternating group A_l is a subgroup of S_l and under the restriction $S_l \to A_l$

$$[\lambda] \to \begin{cases} [\lambda] & \text{if } (\lambda) < (\tilde{\lambda}) \\ [\lambda]_+ + [\lambda]_- & \text{if } (\lambda) = (\tilde{\lambda}) \\ [\tilde{\lambda}] & \text{if } (\lambda) > (\tilde{\lambda}). \end{cases}$$

The product rules given above for O_n and S_l then apply directly to all representations of R_n and A_l except $[\lambda]_\pm$ for which it is necessary to use techniques such as those associated with Littlewood's difference characters [**22**, p.133] to determine the appropriate formulae.

For the spin representations of O_n and R_n, S-functional techniques are also available [**22**, p. 133].

3. Values of c_λ. Most c_λ follow immediately from the above formulae for symmetrized squares. For R_n and A_n the reality properties of the difference characters are also required. For spin representations the previous references are also required.

For example, looking for the S-function $\{0\}$ in the formulae for O_n(2.1b); Sp_n(2.2b); U_n(2.3d) and S_n(2.4d), we find we require $\alpha = \lambda$; $\alpha = \lambda$; $\alpha = \beta = \lambda = \mu$; $\alpha = \lambda$, $\beta = \gamma = \{0\}$ respectively. These then lead to

$$\begin{aligned} &O_n \quad c_{[\lambda]} = 1 \text{ (true representations only)} \\ &Sp_n \quad c_{\langle\lambda\rangle} = (-)^l \\ &U_n \quad c_{\{\lambda;\mu\}} = \delta_{(\lambda),(\mu)} \text{ (composite notation)} \\ &S_n \quad c_{[\lambda]} = 1. \end{aligned}$$

The Kronecker delta $\delta_{(\lambda),(\mu)}$ tests equality of the partitions (λ) and (μ). It is not difficult to show that the respective modification rules have no effect on these results.

However the modification rules are important for SU_n. All (and only) representations labelled as $\{a^n\}$ are equivalent to $\{0\}$. If one draws the Young diagram for $\{\pi\}_{SU_n}$ and notes the correspondence to the composite notation $\{\lambda; \mu\}_{SU_n}$, it is easy to verify that if $\{\pi\} \cong \{\pi\}^*$ then necessarily

$$\begin{aligned} &(\lambda) = (\mu) \text{ for } n = 2r + 1 \\ &(\lambda) = (\mu \oplus a^r) \text{ for } n = 2r \text{ (for some } a). \end{aligned}$$

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The partition $(\mu \oplus a^r)$ is the partition obtained by adding a to each of the r parts of μ

$$(\mu \oplus a^r) \equiv (\mu_1 + a, \mu_2 + a, \ldots, \mu_r + a)$$

(it is often called the principle part of (μ) and (a^r)). For n even, the partitions for the composite notation are not unique and one simplifies matters if one notes that a can be chosen to take on the two values 0 and 1 only. The use of these correspondences in (2.3d) gives that $c_{\{\pi\}}$ for SU_{2r+1} equals $c_{\{\pi\}}$ for U_{2r+1}. The plethysm for $n = 2r$ ($a = 0$ or 1 only) gives

$$c_{\{\mu;\mu\}} = 1, \; c_{\{\mu\oplus 1^r;\mu\}} = (-)^r.$$

Generalizing to the other separations of (λ) and (μ) gives

$$c_{\{\mu\oplus a^r;\mu\}} = (-)^{ar}.$$

This result may be expressed in several different forms. Still considering the n even case, we have

$$c_{\{\pi\}} = (-)^{pr}\delta_{\{\pi\},\{\pi\}^*}$$

with the delta function zero unless the character is real, or

$$= (-)^{\pi_1}\delta_{\{\pi\},\{\pi\}^*}$$

this being Mal'cev's form of the result [**19**] (π_1 being the first part of the partition (π)). Alternatively

$$c_{\{\lambda;\mu\}} = \sum_{all\ a} (-)^{ar}\delta_{(\lambda),(\mu\oplus a^r)}$$
$$= (-)^{(l-m)r} \sum_{a} \delta_{(\lambda),(\mu\oplus a^r)}.$$

Weight space techniques have been used to give c_λ for all semi-simple Lie groups. We use Mal'cev's [**19**] results for the exceptional groups. His results have been rederived by Dynkin [**10**] and others [**3**].

4. Values of ϕ_λ. It is clear from the introduction that we may write c_λ as a product of two factors, a phase ϕ_λ giving the symmetry properties of the $1 - jm$ symbol and a Kronecker delta between a representation and its complex conjugate $\delta_{\lambda\lambda^*}$,

(4.1) $\quad c_\lambda = \phi_\lambda \delta_{\lambda\lambda^*}.$

When ϕ_λ is undefined (when $c_\lambda = 0$) requirements of simplicity for the Racah algebra ask us to choose [**4**]

(4.2) $\quad \phi_\lambda = \phi_{\lambda^*} = \pm 1$

and also to satisfy

(4.3) $\quad \phi_\lambda\phi_\mu\phi_\nu = 1 \quad$ if $\{\lambda\} \times \{\mu\} \times \{\nu\} \supset \{0\}.$

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Knowing c_λ and the multiplication rules for representations of a group, it is trivial to find the ϕ_λ which best satisfy (4.1) to (4.3).

For example, for SU_6, the S-functional methods of the previous section give

$$SU_6 \quad c_{\{\lambda\}} = (-)^l \delta_{\{\lambda\},\{\lambda\}^*},$$

a form which satisfies all requirements as it stands. This is because the Kronecker product rule gives

$$SU_6 \quad l + m + n \quad \text{is even if } \{\lambda\} \times \{\mu\} \times \{\nu\} \supset \{0\}.$$

It is easy to verify that no choice for $\phi_{\{\lambda\}}$ other than $(-)^l$ exists satisfying the restrictions (4.2), (4.3), However for SU_4, although there are no symplectic representations, the condition on $l + m + n$ is unaltered. If $\{\lambda\} = \{\lambda\}^*$ then l is even and thus we still have a choice for $\phi_{\{\lambda\}}$:

$$SU_4 \quad \phi_{\{\lambda\}} = 1 \text{ or } (-)^l.$$

Once the choice is made for one representation, then no further choices remain.

By these arguments it is possible to derive ϕ_λ for all Lie groups. For certain groups, the modification rules affect the product rules and one must check this does not cause a failure in (4.3). For the exceptional groups E_6 and E_7, Mal'cev's results are insufficient to test (4.3). However the form of the α-series of weights for these groups [**10**] ensures that either all weights are "integer" or "half-integer", and thus the product of two half-integer representations contains only integer representations.

The results are summarized in the table. Results for the finite groups S_n and A_n are also given.

Although (4.2) and (4.3) can be chosen to hold for all these groups, we have two examples demonstrating the failure of the appropriate symmetry property of the *jm* symbols. Frame [**11**] has obtained a direct counterexample to the general validity of (4.3). For the finite group ${}^2F_4(2)$ of order $2^{12}\, 3^3\, 5^2\, 13$ the product of the representations of dimension 325 and 2600 (both orthogonal) contains the symplectic representation of dimension 52, twice:

$${}^2F_4(2) \quad c_{325}c_{2600}c_{52} = -1.$$

From the tables of Biedenharn, Brouwer and Sharp [**2**] it is easy to verify that only one finite group of order up to 32 fails to satisfy both requirements. This is the group $\langle -2, 2, 3\rangle$ of Coxeter, of order 24. For this group one may satisfy (4.3) if certain ϕ_λ are chosen complex.

5. Non-simple phase representations. The formulae of section 2 may be used to determine the coefficient m_λ. Once again it is only necessary to look for the occurrence of $\{0\}$, this time in the symmetrised products $\otimes \{21\}$. Clearly if there exists any S-function λ and three distinct S-functions α, β and γ such that $\lambda/\alpha\beta$, $\lambda/\beta\gamma$ and $\lambda/\gamma\alpha$ all contain $\{0\}$ then $[\lambda]_{O_n}$ and $\langle\lambda\rangle_{Sp_n}$ will not be simple phase. The same is true of the representation $\langle\lambda\rangle_{S_n}$. An example is

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TABLE

Values of the $1 - j$ phase, ϕ_λ, and the reality condition, $\delta_{\lambda\lambda*}$, for representations of various groups.

Group	Representation	Reality Condition	1 − j Phase
U_n	$\{\lambda; \mu\}$	$\delta_{(\lambda),(\mu)}$	$(\pm)^{l-m}$
O_n	tensor	real	1
	spinor	real	$\begin{cases} 1 & n \equiv 3, 4, 5, 6 \pmod 8 \\ -1 & n \equiv 0, 1, 2, 7 \pmod 8 \end{cases}$
Sp_n	$\langle\lambda\rangle$	real	$(-)^l$
SU_n	$\{\pi\} \cong \{\lambda; \mu\}$	$\delta_{(\lambda),(\mu)}$	$1 \quad n \equiv 1, 3 \pmod 4$
		$\sum_a \delta_{(\lambda),(\mu \oplus a^r)}$	$(-)^{l-m} = (-)^p \; n = 2r \equiv 2 \pmod 4$
			$(\pm)^{l-m} = (\pm)^p \; n = 2r \equiv 0 \pmod 4$
SO_n	tensor	real	$1 \quad n \equiv 0, 1, 3, \pmod 4$
		$\delta_{\lambda_k, 0}$	$1 \quad n = 2k \equiv 2 \pmod 4$
	spin	real	$1 \quad n \equiv 0, 1, 7 \pmod 8$
		real	$-1 \quad n \equiv 3, 4, 5 \pmod 8$
		complex	$\pm 1 \quad n \equiv 2, 6 \pmod 8$
G_2	all	real	1
F_4	all	real	1
E_6	integral	real	1
	half-integral	complex	± 1
E_7	integral	real	1
	half-integral	real	-1
E_8	all	real	1
S_n	all	real	1
A_n	$[\nu] \neq [\tilde{\nu}]$	real	1
	$[\nu]_\pm$	$\sum_a \delta_{\frac{1}{2}(n-r), 2a}$	1

afforded by the case $\lambda = \{521\}$, $\alpha = \{4\}$, $\beta = \{31\}$ and $\gamma = \{2^2\}$. This alone is sufficient to prove that, for large n, O_n, R_n, Sp_n, S_n and A_n are not simple phase since the representations $[521]$, $\langle 521\rangle$ and $[n - 8, 521] \cong \langle 521\rangle$ are not simple phase.

A simpler example of a non-simple phase representation of O_n is provided by $[31]$. In this case taking $\alpha = \{2\}$, $\beta = \{1^2\}$ in the expression

$$[(\lambda/\alpha \otimes \{1^2\})((\lambda/\alpha\beta) \otimes (\{21\}/\{1\}))]$$

gives $[0]$. For low values of n it is necessary to make use of modification rules in applying the formulae of section 2. In the particular case of the representation $[31]$ these modification rules are not needed for $n \geqq 12$ since only the cube of $[31]$ is involved. For $5 \leqq n < 12$ moreover the appropriate equivalence relations are such that no terms give rise to cancellation of the $[0]$ occurring in $[31] \otimes \{21\}$. Hence $[31]$ is a non-simple phase representation of O_n for $n \geqq 5$. The same is true of R_n for $n \geqq 5$.

The importance of the modification rules is brought out by the consideration of $[21] \otimes \{21\}$ in the group R_n. The formulae of section 2 imply that since (21) is a partition of an odd number the product $[21] \times [21] \times [21]$ does not contain $[0]$. However for R_n the representations $[1^n]$, $-[2, 1^n]$, $-[2^{n+1}]$, $[3, 1^{n+1}]$, are all equivalent to $[0]$. It follows that for the group R_5, $[21] \otimes \{21\} \supset [0]$. Through the isomorphism between R_5 and Sp_4 this corresponds to the

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statement that the representation $\langle 31 \rangle$ of Sp_4 is not simple phase. In fact $\langle 31 \rangle$ is not simple phase for any of the groups Sp_n with $n \geqq 4$.

Of course the group $R_3 \cong SU_2 \cong Sp_2$ is simple phase. The only other simple Lie group which is simple phase is SU_3 as has been established from (1.3) by Derome [**7**]. An alternative derivation of this result may be obtained through the use of a modular technique [**20**] of evaluating plethysms due to Littlewood [**16**]. The appropriate plethysm is given by

$$(5.1) \qquad \lambda \otimes \{21\} = \tfrac{1}{3}(\lambda \otimes S_1{}^3 - \lambda \otimes S_3)$$

where $\lambda \otimes S_1{}^3 = \lambda\lambda\lambda$ and $\lambda \otimes S_3$ may be evaluated by determining the 3-quotient of partitions of $3l$ having null residue. The relevant partition associated with the representation $\{0\}$ of SU_3 is (l^3). It is a straightforward task to show that $\lambda\lambda\lambda = a\{l^3\} + \ldots$ where $a = 1 + \min\ (p, q)$ if $\lambda = \{p + q, q\}$. Similarly if $\lambda \otimes S_3 = b\{l^3\} + \ldots$ then b is just the coefficient of λ in the product of the form $\{s\}\ \{s\}\ \{t\}$ with $t = s$ or $s \pm 1$ and $l = 2s + t$. This coefficient is again $1 + \min\ (p, q)$. Hence $a = b$ and for the group SU_3 $\{p + q, q\} \otimes \{21\}$ does not contain $\{0\}$.

(5.1) may be usefully applied to other groups SU_n. It is found that the simplest representation which is non-simple phase is $\{21; 21\}$. This result is valid for $n \geqq 4$ even when account is taken of the modification rules. In fact the coefficient m_λ takes on the values 1, 4, 4 ... for $n = 4, 5, 6, \ldots$. This result in the case $n = 4$ has been noted elsewhere [**8**] similarly for $\{31; 2^2\}$, m_λ takes on the value 0 for $n = 4$ but the value 1 for $n \geqq 5$.

From this discussion it is clear that simple phase groups are the exception and not the rule, although finite groups of small order are often simple phase [**23**].

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