Canad. Math. Bull. Vol. **64** (4), 2023, pp. 1231–1236
http://dx.doi.org/10.4153/S0008439523000309

Toric degenerations of low-degree hypersurfaces

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Abstract. We show that a sufficiently general hypersurface of degree d in $\mathbb{P}^n$ admits a toric Gröbner degeneration after linear change of coordinates if and only if $d \leq 2n-1$.

1 Introduction

When does a projective variety X admit a flat degeneration to a toric variety? Among other applications, such degenerations are used in the mirror-theoretic approach to the classification of Fano varieties [CCG+13], the construction of integrable systems [HK15], and in bounding Seshadri constants [Ito14]. The many applications of toric degenerations notwithstanding, there is as of yet no general method for determining if a given variety admits a toric degeneration.

In this note, we will consider the special case of toric degenerations of some $X \subset \mathbb{P}^n$ obtained as the flat limit of X under a $\mathbb{G}_m$-action on $\mathbb{P}^n$. In the case that the $\mathbb{G}_m$-action arises as a one-parameter subgroup of the standard torus on $\mathbb{P}^n$, the situation may be well understood by studying the Gröbner fan and tropicalization of X [MS15]. However, if we consider arbitrary $\mathbb{G}_m$-actions on $\mathbb{P}^n$, the situation becomes more complicated. As a test case, we investigate the existence of such toric degenerations when X is a hypersurface.

In order to state our result, we introduce some notation. Throughout the paper, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. Let $\omega \in \mathbb{R}^{n+1}$. Consider any polynomial $f \in \mathbb{K}[x_0, \ldots, x_n]$, where we write

$$f = \sum_{u \in \mathbb{Z}_{\geq 0}^{n+1}} c_u x^u \tag{1.1}$$

using multi-index notation. The *initial term* of f with respect to the weight vector ω is

$$\mathrm{In}_\omega(f) = \sum_{u:\langle u,\omega\rangle=\lambda} c_u x^u,$$

Received by the editors September 1, 2022; revised April 11, 2023; accepted April 12, 2023.
Published online on Cambridge Core April 20, 2023.
N.I. was supported by an NSERC Discovery Grant. O.L. was supported by an NSERC undergraduate student research award.
AMS subject classification: **13P10**, **14J70**, **14M25**.
Keywords: Toric degeneration, hypersurfaces, Khovanskii basis.

where λ is the maximum of $\langle u, \omega \rangle$ as u ranges over all $u \in \mathbb{Z}_{\geq 0}$ with $c_u \neq 0$. For an ideal $J \subset \mathbb{K}[x_0, \ldots, x_n]$, its initial ideal with respect to the weight vector ω is

$$\mathrm{In}_\omega(J) = \langle \mathrm{In}_\omega(f) \mid f \in J \rangle.$$

The *weight* of a monomial x^u with respect to ω is the scalar product $\langle u, \omega \rangle \in \mathbb{R}$.

Definition 1.1 Let $X \subset \mathbb{P}^n$ be a projective variety over $\mathbb{K}$. We say that X *admits a toric Gröbner degeneration up to change of coordinates* if there exist a $\mathrm{PGL}(n+1)$ translate X' of X and a weight vector $\omega \in \mathbb{R}^{n+1}$ such that the initial ideal

$$\mathrm{In}_\omega(I(X'))$$

of the ideal $I(X') \subseteq \mathbb{K}[x_0, \ldots, x_n]$ of X' is a prime binomial ideal.

We can now state our result.

Theorem 1.2 *Let $d, n \in \mathbb{N}$. There is a non-empty Zariski open subset U of the linear system of degree d hypersurfaces in $\mathbb{P}^n$ with the property that every hypersurface in U admits a toric Gröbner degeneration up to change of coordinates if and only if $d \leq 2n - 1$.*

Before proving this theorem in the following section, we discuss connections to the existing literature.

A common source of toric degenerations of a projective variety $X \subset \mathbb{P}^n$ arises by considering the Rees algebra associated with a full-rank homogeneous valuation $\mathfrak{v}$ on the homogeneous coordinate ring of X [And13]. As long as the homogeneous coordinate ring of X contains a finite set $\mathcal{S}$ whose valuations generate the value semigroup of $\mathfrak{v}$, one obtains a toric degeneration. Such a set $\mathcal{S}$ is called a *finite Khovanskii basis* for the coordinate ring of X. This construction is in fact quite general: essentially any $\mathbb{G}_m$-equivariant degeneration of X over $\mathbb{A}^1$ arises by this construction (see [KMM23, Theorem 1.11] for a precise statement). There has been some work on algorithmically constructing valuations with finite Khovanskii bases (see, e.g., [BLMM17] for applications to degenerations of certain flag varieties), but as of yet, there is no general effective criterion for deciding when such a valuation exists.

Drawing on [KM19] which connects Khovanskii bases and tropical geometry, we may rephrase our results in the language of Khovanskii bases. It is straightforward to show that X admits a toric Gröbner degeneration up to change of coordinates if and only if there is some full-rank homogeneous valuation $\mathfrak{v}$ for which the homogeneous coordinate ring has a finite Khovanskii basis consisting of degree one elements. Thus, our theorem shows the existence of finite Khovanskii bases for general hypersurfaces of degree at most $2n - 1$, and shows that any finite Khovanskii basis for a general hypersurface of larger degree necessarily contains elements of degrees larger than one. In fact, we suspect that a general hypersurface of sufficiently large degree does not admit any finite Khovanskii basis at all.

We note in passing that a general hypersurface of arbitrary degree will admit a toric degeneration in a weaker sense. Indeed, the universal hypersurface over the linear system of degree d hypersurfaces is a flat family, and for any degree d, there is a toric hypersurface of degree d. However, such a degeneration is not $\mathbb{G}_m$-equivariant.

An interesting comparison of our result can be made with [KMM21], which states that after a *generic* change of coordinates, any arithmetically Cohen Macaulay variety $X \subset \mathbb{P}^n$ has a Gröbner degeneration to a (potentially non-normal) variety equipped with an effective action of a codimension-one torus. Such varieties, called complexity-one T-varieties, are in a sense one step away from being toric. The hypersurfaces we consider in our main result (Theorem 1.2) are of course arithmetically Cohen Macaulay, so they admit Gröbner degenerations to complexity-one T-varieties. Our result characterizes when we can go one step further and Gröbner degenerate to something toric. When $d \leq 2n-1$ and we are in the range for which this is possible for a generic hypersurface, the change of coordinates required is a special one as opposed to the generic change of coordinates of [KMM21].

2 Proof of the theorem

2.1 Setup

Throughout, we will assume that $d, n > 1$ since the theorem is clearly true if $d = 1$ or $n = 1$. We will view the coefficients c_u of f in (1.1) as coordinates on affine space $\mathbb{A}^{\binom{d+n}{n}}$. To indicate the dependence of f on the choice of coefficients c, we will often write $f = f_c$. Let K be the subset of all $u \in \mathbb{Z}_{\geq 0}^{n+1}$ such that $u_0 + u_1 = d$, $u_i = 0$ for $i > 1$, and $u_1 < d$. We then set

$$W = V(\langle c_u \rangle_{u \in K}) \subset \mathbb{A}^{\binom{d+n}{n}}.$$

The family of polynomials parameterized by W consists of all degree d forms such that the only monomial involving only x_0 and x_1 is x_1^d.

We will be considering the map

$$\begin{aligned} \phi : \mathrm{GL}(n+1) \times W &\to \mathbb{K}[x_0, \ldots, x_n]_d \\ (A, c) &\mapsto A.f_c, \end{aligned}$$

where $A.f_c$ denotes the action of $A \in \mathrm{GL}(n+1)$ on a polynomial $f_c = \sum c_u x^u$ via linear change of coordinates. We will be especially interested in the differential of ϕ at (e, c), where $e \in \mathrm{GL}(n+1)$ is the identity. A straightforward computation shows that the image of the differential at (e, c) is generated by

$$x^u \qquad u \notin K, \tag{2.1}$$

$$\frac{\partial f_c}{\partial x_i} \cdot x_j \qquad 0 \leq i, j \leq n. \tag{2.2}$$

The following lemma is the key to our proof.

Lemma 2.1 *The differential ϕ is surjective at (e, c) for general $c \in W$ if and only if $d \leq 2n-1$.*

Proof Consider the image of the differential of ϕ at (e, c). From (2.1), we obtain the span of all monomials of $\mathbb{K}[x_0, \ldots, x_n]_d$ with the exceptions of the d monomials $x_0^d, x_0^{d-1}x_1, \ldots, x_0 x_1^{d-1}$. From (2.2) with $i = 1$ and $j = 0$, modulo (2.1), we additionally

https://doi.org/10.4153/S0008439523000309 Published online by Cambridge University Press

obtain the monomial $x_0 x_1^{d-1}$. We do not obtain anything new from (2.2) when $i = 1$ and $j = 1$, when $i = 0$, or when $j > 1$.

It remains to consider the contributions to the image from (2.2) with $i > 1$ and $j = 0, 1$. For $2 \leq i \leq n$ and $1 \leq m \leq d-1$, let $u(i,m) \in \mathbb{Z}^{n+1}$ be the exponent vector with $u_i = 1$, $u_0 = m$, $u_1 = d - m - 1$. Modulo the span of (2.1) and $x_0 x_1^{d-1}$, from (2.2), we obtain

$$
\begin{aligned}
\frac{\partial f_c}{\partial x_i} \cdot x_0 &\equiv c_{u(i,d-1)} x_0^d + c_{u(i,d-2)} x_0^{d-1} x_1 + \cdots + c_{u(i,1)} x_0^2 x_1^{d-2},\\
\frac{\partial f_c}{\partial x_i} \cdot x_1 &\equiv \qquad\qquad\qquad c_{u(i,d-1)} x_0^{d-1} x_1 + \cdots + c_{u(i,2)} x_0^2 x_1^{d-2}.
\end{aligned}
$$

Varying i from 2 to n, we obtain $2n-2$ polynomials of degree d. The $(2n-2) \times (d-1)$ matrix of their coefficients has the form

$$
\begin{pmatrix}
c_{u(2,d-1)} & c_{u(2,d-2)} & \cdots & c_{u(2,1)}\\
0 & c_{u(2,d-1)} & \cdots & c_{u(2,2)}\\
c_{u(3,d-1)} & c_{u(3,d-2)} & \cdots & c_{u(3,1)}\\
0 & c_{u(3,d-1)} & \cdots & c_{u(3,2)}\\
\vdots & \vdots & & \vdots\\
c_{u(n,d-1)} & c_{u(n,d-2)} & \cdots & c_{u(n,1)}\\
0 & c_{u(n,d-1)} & \cdots & c_{u(n,2)}
\end{pmatrix}.
$$

Since $c \in W$ is general, this matrix has full rank, that is, its rank is $\min\{d-1, 2n-2\}$. Hence, the image of the differential of ϕ has codimension

$$d - 1 - \min\{d-1, 2n-2\},$$

so the differential is surjective if and only if $d \leq 2n-1$. ∎

We now move on to prove the theorem.

2.2 Existence

We will first show that if $d \leq 2n-1$, a general degree d hypersurface admits a toric Gröbner degeneration up to change of coordinates. As noted above, the family of polynomials parameterized by W consists of all degree d forms such that the only monomial involving only x_0 and x_1 is x_1^d. Consider any $\omega \in \mathbb{R}^{n+1}$ such that

$$\omega_0 > \omega_1 > \omega_2 > \cdots > \omega_n \qquad (d-1)\omega_0 + \omega_2 = d\omega_1.$$

For general $c \in W$, the initial term of f_c is

$$a x_1^d + b x_0^{d-1} x_2$$

for some $a, b \neq 0$; this is a prime binomial. Thus, we will be done with our first claim if we can show that the image of ϕ contains a non-empty open subset of $\mathbb{K}[x_0, \ldots, x_n]_d$.

To this end, we consider the image of the differential at (e, c) for general $c \in W$. By Lemma 2.1, we conclude that ϕ has surjective differential at (e, c) for general $c \in W$; it follows that ϕ has surjective differential at a general point of $\mathrm{GL}(n+1) \times W$.

Thus, the dimension of the image of ϕ is the dimension of $\mathbb{K}[x_0, \dots, x_n]_d$, and the image of ϕ contains a non-empty open subset of $\mathbb{K}[x_0, \dots, x_n]_d$.

2.3 Nonexistence

Assume now that $d > 2n - 1$. We first give an overview of the proof strategy. There are only finitely many prime binomials g of degree d. Likewise, there are only finitely many linear orderings $\prec$ of the variable indices $0, \dots, n$. We say that a weight vector ω is *compatible* with $\prec$ and g if whenever $i \prec j$ in the linear ordering, then $\omega_i \geq \omega_j$, and the two monomials of g have the same weight with respect to ω.

For fixed g and linear ordering on the variables, we may consider the set S of all polynomials f in $\mathbb{K}[x_0, \dots, x_n]_d$ for which there exists a compatible weight vector $\omega \in \mathbb{R}^{n+1}$ such that initial term of f with respect to ω is g. We will show that up to permutation of the coordinates, this set S can be identified as a subfamily of W. By Lemma 2.1, the map ϕ has nowhere surjective differential. Thus, by generic smoothness, the dimension of the image of ϕ must be strictly less than the dimension of $\mathbb{K}[x_0, \dots, x_n]_d$. It follows that there cannot be a Zariski-open subset of $\mathbb{K}[x_0, \dots, x_n]_d$ such that every hypersurface in this subset admits a toric Gröbner degeneration up to change of coordinates.

To complete the proof, we will fix a prime binomial $g = g' + g''$ of degree d and a linear ordering of the variables. Here, g' and g'' are the two terms of g. After permuting the variables and appropriately adapting g, we may assume without loss of generality that the indices are ordered as $0 \prec 1 \prec 2 \prec \cdots \prec n$. The irreducibility of g implies that g involves at least three distinct variables, and no variable appears in both g' and g''. Let p be the smallest index such that x_p appears in g; we denote the corresponding term by g'. Let q be the smallest index such that x_q appears in the term g''.

If g' only involves variables x_i with indices $i < q$, then any compatible term order ω must satisfy $\omega_p = \omega_q = \omega_j$ for all $p \leq j \leq q$. Indeed, if not, the term g'' would necessarily have smaller weight. Without loss of generality, we may thus permute indices without changing the set of compatible weight vectors to also assume that g' involves some x_i with $i > q$. For this, we are using that the irreducibility of g guarantees that at least one of g' and g'' is not a dth power.

Consider the set S of polynomials f_c such that there is a compatible weight ω for which f_c has g as its initial term. We claim that S is a subset of the family parameterized by W. Indeed, since $q > 0$, g'' has weight at most equal to the weight of x_1^d. The monomials $x_0^d, x_0^{d-1}x_1, \dots, x_0 x_1^{d-1}$ all have weight at least as big as the weight of x_1^d, and are not scalar multiples of g' or g''. Hence, none of these monomials can appear in any element of S, and the claim follows.

The proof of the theorem now follows from the argument given above.

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