4.10 TRIGONOMETRY IN TWO SIXTEENTH CENTURY WORKS; THE DE REVOLUTIONIBUS ORBIUM COELESTIUM AND THE SIDRA AL-MUNTAHĀ

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In Greece, Autolycos (4th cent. B.C.), Aristarchos of Samos (3rd cent.B.C.), Hipparchos (2nd cent.B.C.), Menelaos (1st cent. A.D.), and Ptolemaos (2nd cent. A.D.) are the forerunners of trigonometry. The Greeks used chords and prepared a table of chords.

Later, the Hindus produced Siddhāntas (4th cent.A.D.). The most important feature of these works is the use of jyā instead of chords, and utkramajyā (versed sine).

In Islam, al-Battānī al-Ṣābī (858-929) used the sine, cosine, tangent, and cotangent with clear consciousness of their individual characteristics.

As is known, trigonometry developed as a branch of astronomy. Although in the thirteenth century Naṣīr al-Dīn al-Ṭūsī (in the Islamic world) and in the fifteenth century Regiomontanus (in the West) established trigonometry as a science independent of astronomy, the essential situation did not change, and the subject went on developing as before.

As we come to the sixteeenth century, Copernicus complete some of the work left unfinished by Regiomontanus, in his famous book De Revolutionibus Orbium Coelestium. Later, the chapter devoted to trigonometry was published separately by this pupil Rhaeticus.

On the other hand, the first book of the Sidrat al-Muntahā, by Taqī al-Dīn of Istanbul, was devoted to trigonometry.

The purpose of my work is to make a comparison between these two chapters, and to show what Taqī al-Dīn accomplished with trigonometry in the sixteenth century.

Copernicus in Book 1, Section 12 of Revolutionibus divided the circle into 360^0 and the diameter into 200000 parts.

Taqī al-Dīn in Book 1, Section 1 divided the circle into 360^0 and the diameter into 120 or 2. The first geometrician to divide the diameter into 2 was Abū'l-Wafaᵓ (940-998). In the West, the first mathematician to adopt the simpler form

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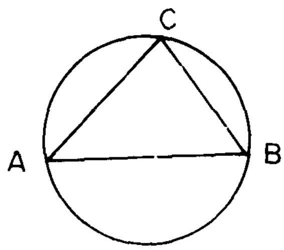
$$X = 4/3\,X^3 - 1/3\,AD \quad ,$$

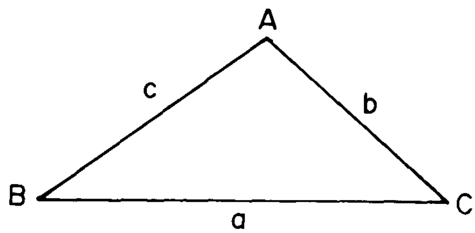
where $X = \sin 1^0, \quad AD = \sin 3^0$.

As is seen, Taqī al-Dīn puts everything in terms of the sine or cosine. By contrast, Copernicus mentions only the halves of chords subtending twice the arc. Of course, half the chord of twice the arc is the sine.

On Plane Triangles

In Book 1, Section 13, Copernicus says: Let there be the triangle ABC about which a circle is circumscribed. Therefore arcs AB, BC and CA will be given.





He proves that
(1) if the sides of a triangle, or
(2) two sides and an angle, or
(3) a side and two angles, are given, the triangle is known.

Taqī al-Dīn says in Book 1, Section 4 that the sides of a triangle (it may be acute, obtuse, or right) are proportional to the sines of the angles subtending the sides.

$$\frac{AB}{AC} = \frac{\sin C}{\sin A} \qquad \text{Sine Theorem}$$

As is seen, Copernicus, in solving plane triangles, repeats very nearly what Euclid said in his Elements.

On the other hand, Taqī al-Dīn mentions the important relation known as the sine theorem.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

While the theorem was recognised by Abū'l-Wafā᾿ (940-998) and al-Bīrūnī (973-1048), Naṣīr al-Dīn al-Ṭūsī (1201-1274) was the first to set it forth with clarity.

On Spherical Triangles

Copernicus defines and gives in Book 1, Section 14 the principal properties of a spherical triangle, that is to say, sides neither equal to or greater than the halves of great circles.

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$$\sin 90^{o} = 1$$

that is to say

$$\text{diameter} = 2$$

was Just Bürgi in the seventeenth century. This is an important point in Taqī al-Dīn's favour.

In the same section Copernicus gives the following formulas for finding crd 2A, crd A/2, crd (A-B), crd (A-B). He also shows that

$$\text{arc } BC/ \text{ arc } AB > \text{crd } BC/ \text{ crd } AB$$

Copernicus calculated crd 1^{o} or crd 2^{o} by proving that the arc is always greater than the straight line subtending it, but in going from greater to lesser sections of the circle, the inequality approaches equality so that finally the circular lines go out of existence simultaneously at the point of tangency on the circle. Therefore it is necessary that just before that moment they differ from one another by no discernible difference. He established crd 1^{o} as subtended by 1745 approximately.

Taqī al-Dīn also gives the same formulas for crd 2A, crd A/2, crd (A-B), crd(A-B), and he proves that

$$\text{arc } BC/ \text{ arc } AB > \text{crd } BC/ \text{ crd } AB$$

As we come to Taqī al-Dīn, he says the following, "The Ancients could not find a correct way to obtain crd 1^{o} or crd 2^{o}, in consequence of this, they depended upon approximate methods which do not merit description. " The late Ulug Beg said that we had an inspiration about extracting crd 1^{o} and sin 1^{o}. This method involves the solution of a cubic equation of the form

$$X = \frac{X - AD}{3} \quad ,$$

where X = crd 2^{o} and AD = crd 6^{o}. The leading position of Taqī al-Dīn is obvious.

On Sines

At the end of the 12th section, Copernicus, without using the concept of sine, says, "Nevertheless I think it will be enough if in the table we give only the halves of the chords subtending twice the arc, whereby we may concisely comprehend in the quadrant what used necessarily to be spread out over a semicircle, and especially because the halves come more frequently into use in demonstration and calculation than the whole chords do ".

In Book 1, Section 2 Taqī al-Dīn defines the sine, cosine, and versed sine, and gives the formulas for sin 2A, sin A/2, sin(A-B), sin(A+B).

He says that the radius is greater than the sine, and in the limit the sine of 90^{o} is equal to the radius, that is to say

$$\sin 90^{o} = 1$$

Later he calculates sin 1^{o} by using a third equation, In a spherical triangle having a right angle,

$$\frac{1/2 \text{ crd } 2 \text{ AB}}{1/2 \text{ crd } 2 \text{ BC}} = \frac{\text{radius}}{1/2 \text{ crd } 2\text{C}}$$

That is to say,

$$\frac{\sin \text{AB}}{\sin \text{BC}} = \frac{\text{radius}}{\sin \text{C}}$$

Beside this, in right spherical triangles

(1) if a side and an acute angle, or
(2) three sides, or
(3) two acute angles,

are given, the triangles are known.

Later he mentions the equality of spherical triangles.

In Book 1, Section 5 and a part of Section 6 Taqi al-Dīn also defines and gives the principal properties of the spherical triangle, the sides being neither equal to nor greater than half a great circle.

Spherical triangle ABC having a right angle B.

$$\frac{\sin \text{A}}{\sin \text{BC}} = \frac{\sin 90^0}{\sin \text{CA}} \qquad \text{Muġnī theorem}$$

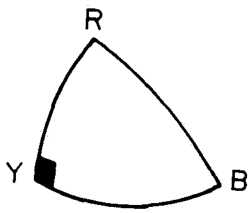
Taqī al-Dīn attributes the Muġnī and its conclusion to Abū Naṣr ibn ᶜIrāq (10th cent.).

1. The first conclusion of the Muġnī:

$$\frac{\cos \text{YR}}{\cos \text{BR}} = \frac{\sin 90}{\cos \text{BY}} \qquad \text{triangle RYB}$$

2. The second conclusion of the Muġnī:

$$\frac{\cos \text{B}}{\cos \text{YR}} = \frac{\sin \text{R}}{\sin 90^0}$$



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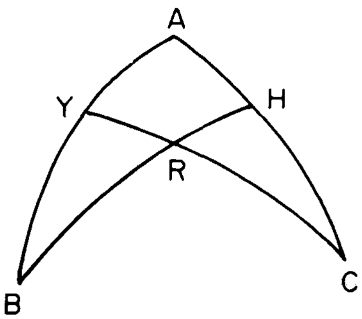
3. The third conclusion of the Muġnī:

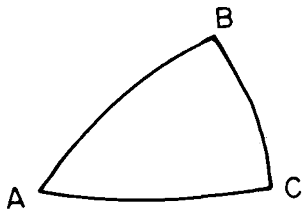
$$\frac{\sin RY}{\sin BY} = \frac{\sin CR}{\sin CH}$$

The sides corresponding to equal angles are proportional.



In Section 6, in any triangle having obtuse or acute angles, he gives

$$\frac{\sin AB}{\sin BC} \qquad \frac{\sin C}{\sin A}$$ sine theorem



As is seen, there is not only a quantitative, but also a qualitative difference betweeen Copernicus and Taqī al-Dīn who used the Muġnī, the conclusions of the Muġnī and the sine theorem.

On the Tangent and Cotangent

Copernicus says nothing on this subject.

In Book 1, Section 6 Taqī al-Dīn defines the Umbra versa (tangent) and the Umbra recta (cotangent), and gives the following formulas:

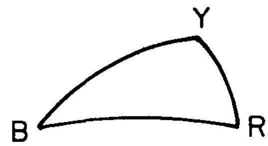
$$\frac{tg\ A}{ctg\ A} = \frac{radius}{\cos A} \qquad tg\ A\ ctg\ A = radius^2 \text{ if radius } = 1$$
$$tg\ A\ ctg\ A = 1$$

He gives the tangent theorem and its conclusions.

$$\frac{tg\ A}{tg\ BC} = \frac{\sin A}{\sin AB}$$ tangent theorem

1. The first conclusion of the tangent theorem:

$$\frac{\cos B}{\sin 90^\circ} = \frac{\operatorname{ctg} BR}{\operatorname{ctg} BY}$$

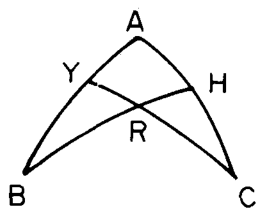


2. The second conclusion of the tangent theorem: (in the same triangle)

$$\frac{\cos BR}{\sin 90^\circ} = \frac{\operatorname{ctg} B}{\operatorname{tg} R}$$

3. The third conclusion of the tangent theorem:

$$\frac{\sin RY}{\operatorname{tg} YB} \qquad \frac{\sin RH}{\operatorname{tg} HC}$$



While Taqī al-Dīn presents almost all the arguments for the tangents and cotangents, Copernicus says nothing on this subject.

Conclusion: As is seen, in the sixteenth century, the Islamic World is ahead in the field of trigonometry. Detailed knowledge in trigonometry implies precision in practical astronomy.

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