J. Austral. Math. Soc. **19** (Series B), (1975), 129–139.

SPECTRAL RESOLUTION OF THE IDENTITY FOR MATRICES OF ELEMENTS OF A LIE ALGEBRA

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(Received 21 April 1975)

Abstract

This is an application of the characteristic identity satisfied by matrices whose elements are also elements of a semi-simple Lie algebra. Generalized eigenvectors are determined for matrices consisting of generators of $GL(n)$, $O(n)$ and $Sp(n)$, and it is shown how to resolve the identity into idempotents constructed from such eigenvectors. By this means rather general functions of the matrices may be defined. It is also shown how to determine traces of such functions, in terms of the invariants of the Lie algebra.

1. Introduction

The generators of various classical groups may be assembled to form a square matrix a, of which the elements a_{pq} $(p, q = 1, 2, \cdots n)$ are also elements of a semi-simple Lie algebra [1]. Although the elements of such a matrix do not commute with one another in general, it has many properties analogous to those of numerical matrices. Powers a^j of the matrix can be defined in the usual way by the formulae

$$\begin{aligned}(a^0)_{pq} &= \delta_{pq} \\ (a^j)_{pq} &= a_{pr}(a^{j-1})_{rq}\end{aligned} \tag{1}$$

(where repeated affixes like p, q, r are considered to be summed from 1 to n). From these powers of a, polynomials in a may also be constructed. If the generators are suitably chosen, the traces of arbitrary polynomials in a, in particular the

$$\operatorname{tr}(a^j) = (a^j)_{pp}, \tag{2}$$

are invariants, i.e., commute with all elements of the Lie algebra.

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It is known from the work of Ilamed [2] that any matrix a whose elements are also elements of an associative algebra must satisfy a characteristic identity, analogous to the Cayley–Hamilton identity satisfied by numerical matrices. Explicit identities of the type

$$\prod_{k=1}^{n} (a - \alpha_k) = 0 \tag{3}$$

have been derived by Bracken and Green [3] and Green [4] for matrices whose elements are generators of $O(n)$, $Sp(n)$, $GL(n)$ and related groups; the 'eigenvalues' α_k were determined as invariant multiples of the identity matrix a^0, and it was shown how to relate the α_k to the invariants $\mathrm{tr}(a^i)$. The transpose b of a, defined as usual by

$$b_{pq} = a_{qp} \tag{4}$$

naturally satisfies a similar identity

$$\prod_{k=1}^{n} (b - \beta_k) = 0, \tag{5}$$

where the β_k are also invariant multiples of a^0.

The matrix a may be applied from the left to a vector ψ, with components ψ_p ($p = 1, 2, \cdots n$) which are, in general, linear operators and do not necessarily commute with one another, with the a_{pq}, or indeed with the α_k. The transpose b may be applied to such a vector from the right. The vector ψ may be resolved into right 'eigenvectors' of a, or left 'eigenvectors' of b, thus:

$$\begin{gathered} \psi = \sum_{j=1}^{n} f_j\psi = \sum_{j=1}^{n} \psi g_j \\ f_j = \prod_{k \neq j} \left(\frac{a - \alpha_k}{\alpha_j - \alpha_k} \right), \qquad g_j = \prod_{k \neq j} \left(\frac{b - \beta_k}{\beta_j - \beta_k} \right) \end{gathered} \tag{6}$$

where, because of (3) and (5),

$$(a - \alpha_j)(f_j\psi) = (\psi g_j)(b - \beta_j) = 0. \tag{7}$$

With a non-degenerate numerical matrix a, it is possible to achieve spectral resolutions of the identity of the types

$$\begin{aligned} a^0 &= \sum_{j=1}^{n} u_j c_j v_j \\ &= \sum_{j=1}^{n} v_j d_j u_j \end{aligned} \tag{8}$$

where

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$$u_j = f_j\psi, \qquad v_j = g_j\phi \tag{9}$$

are eigenvectors of a and its transpose b, respectively, and the c_j and d_j are numerical constants. Such resolutions are useful, because they allow the definition of rather general functions of a and b, by means of the formulae

$$\begin{aligned} \theta(a) &= \sum_{j=1}^{n} \theta(\alpha_j) u_j c_j v_j \\ \theta(b) &= \sum_{j=1}^{n} \theta(\beta_j) v_j d_j u_j. \end{aligned} \tag{10}$$

In this paper, we shall investigate the generalization of results such as (8) and (10) when the elements a_{pq} of a are also elements of semi-simple Lie algebras. Of special interest is a decomposition in which the vectors u_j and v_j are in the enveloping algebra of the Lie algebra, and therefore commute with α_k and β_k.

2. Resolution for $GL(n)$

If the a_{pq} are generators of $GL(n)$, (see [5])

$$[a_{pq}, a_{rs}] = \delta_{rq} a_{ps} - \delta_{ps} a_{rq}, \tag{11}$$

where, as usual, $[A, B]$ means $AB - BA$. The generators of $U(n)$ satisfy similar commutation relations, the only restriction arising from the condition that the elements b_{pq} of the transpose of a must also be the hermitean conjugate of a_{pq}. In any event, the generators satisfy characteristic identities of the types (3) and (5), where, according to [4],

$$\begin{aligned} \alpha_k &= \lambda_k + n - k \\ \beta_k &= \lambda_k + 1 - k \end{aligned} \tag{12}$$

and the eigenvalues l_k of the λ_k serve to identify irreducible representations of $GL(n)$; in the tensor representations, these eigenvalues are integers such that $l_1 \geqq l_2 \geqq \cdots \geqq l_n \geqq 0$. Also, the traces $\operatorname{tr}(a^r)$ are given by relations

$$\begin{aligned} \operatorname{tr}(a) &= \Sigma_k \lambda_k, \\ \operatorname{tr}(a^2) &= \Sigma_k \lambda_k (\lambda_k + n + 1 - 2k), \end{aligned} \tag{13}$$

etc., which can be used if desired to define the λ_k independently, in an algebraic extension of the enveloping algebra of the Lie algebra. An independent method of obtaining these relations will be outlined in the last section of this paper.

Let us introduce two vectors ψ and χ, whose components satisfy

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$$[a_{pq}, \psi_r] = \delta_{qr}\psi_p$$
$$[a_{pq}, \chi_r] = \delta_{qr}\chi_p. \tag{14}$$

These vectors are not defined within a particular irreducible representation of $GL(n)$, since, as we shall soon have occasion to notice, they have components which change one irreducible representation to another. Vectors with the required properties can, however, be found among the generators of groups, such as $GL(n+1)$ and $GL(n+2)$, of which $GL(n)$ is a subgroup; for instance, we could take $\psi_r = a_{r\,n+1}$ and $\chi_r = a_{r\,n+2}$, and the commutation relations (11), extended for values $n+1$ and $n+2$ of p, q, r and s, would then include (14). The vectors ψ and χ may be resolved, as shown in (6), into right and left eigenvectors of a and b respectively; it is known from earlier work of the author [4] that

$$\lambda_k(f_j\psi) = (f_j\psi)(\lambda_k + \delta_{jk}), \tag{15}$$

and it follows from (13) that

$$a(f_j\psi) = (f_j\psi)(b+n), \tag{16}$$

so that $f_j\psi$ is also a left eigenvector of b:

$$f_j\psi = \psi g_j. \tag{17}$$

Now, it can be seen from (15) that the tensor

$$T_{pq} = (\psi g_j)_p\,(f_j\chi)_q\,. \tag{18}$$

increases the eigenvalue l_j of λ_j in an irreducible representation of $GL(n)$ by two units, leaving the other eigenvalues l_k unchanged. This (see [6]) is well known to be a property of symmetric tensors only, and it follows that

$$\psi_r\,[(g_j)_{rp}\,(f_j)_{qs} - (g_j)_{rq}\,(f_j)_{ps}\,]\chi_s = 0. \tag{19}$$

We now notice that, because the vectors ψ and χ are arbitrary, they may be dropped from the equation, leaving the identity

$$(g_j)_{rp}(f_j)_{qs} = (g_j)_{rq}(f_j)_{ps}. \tag{20}$$

[For instance, to eliminate the factor ψ_r, we may set $\psi_r = a_{r\,n+1}$, and take the commutator of (18) with $a_{n+1\,r}$.] If, instead of ψ and χ, we had introduced two adjoint vectors, satisfying

$$[a_{pq}, \phi_r] = -\delta_{pr}\phi_q, \tag{21}$$

we should have obtained the similar result

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$$(f_j)_{rp}(g_j)_{qs} = (f_j)_{rq}(g_j)_{ps} \tag{22}$$

with f_j and g_j interchanged.

The results (20) and (22) have important consequences. We consider, if necessary, a subspace in which

$$c_j = (f_j)_{nn}, \qquad d_j = (g_j)_{nn} \tag{23}$$

are non-singular, and define a set of vectors u_j and v_j with components

$$\begin{aligned} u_{jp} &= (f_j)_{pn}\, c_j^{-1} \\ v_{jp} &= (g_j)_{pn}\, d_j^{-1}. \end{aligned} \tag{24}$$

By setting $q = r = s = n$ in (20) and (22) and dividing by c_j and d_j from right and left, we obtain

$$\begin{aligned} u_{jp} &= d_j^{-1}(g_j)_{np}, \\ v_{jp} &= c_j^{-1}(f_j)_{np}. \end{aligned} \tag{25}$$

By setting $r = s = n$ in (20) and (22), and again dividing by c_j and d_j from right and left, we have also

$$[u_{jp}, u_{jq}] = [v_{jp}, v_{jq}] = 0. \tag{26}$$

Thus, the components of either of these vectors commute among themselves; as $u_{jn} = v_{jn} = 1$, this is a non-trivial result only for $n \geqq 3$. Finally, by setting $r = p = n$ in (20) and (22), we have

$$\begin{aligned} (f_j)_{qs} &= u_{jq} c_j v_{js} \\ (g_j)_{qs} &= v_{jq} d_j u_{js}. \end{aligned} \tag{27}$$

It follows, with the help of (6), that

$$\begin{aligned} \delta_{pq} &= \Sigma_j (f_j)_{pq} = \Sigma_j (u_{jp} c_j v_{jq}), \\ \delta_{pq} &= \Sigma_j (g_j)_{pq} = \Sigma_j (v_{jp} d_j u_{jq}). \end{aligned} \tag{28}$$

This is the required generalization of (8); clearly, c_j and d_j are no longer numerical constants, but linear operators which do not commute with the vectors u_j and v_j; but u_j and v_j are eigenvectors of a and b, so that the analogy is a good one. In particular, it is easy to deduce results of the form (10) when $\theta(a)$ is any polynomial function of a; and the results may be extended to transcendental functions $\theta(a)$, provided $\theta(\alpha_j)$ and $\theta(\beta_j)$ are defined for all admissible eigenvalues of α_j and β_j.

The argument leading to (28) involved vectors $\psi, \chi, \phi, \cdots$ which could not be constructed from generators of $GL(n)$, but the final results do not involve

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such vectors and can be verified independently. However, the verification is by no means trivial even in the simplest instances. Thus, for $n = 2$, the formula

$$(f_1)_{11} = u_{11}c_1v_{11}$$

reduces to a generalized determinantal identity

$$(a_{11} - \lambda_2) = a_{12}(a_{22} - \lambda_2)^{-1}a_{21},$$

which is verifiable with the help of (11) and (13), but is hardly obvious.

3. Extension to $O(n)$ and $Sp(n)$

We now extend the results of the previous section to matrices constructed from generators of $O(n)$ and $Sp(n)$, and their pseudo-orthogonal and pseudo-symplectic analogues. As in [4], we introduce a metric tensor g_{pq}, which is either symmetric or antisymmetric:

$$g_{pq} = \eta g_{qp}, \tag{29}$$

where $\eta = +1$ for $O(n)$ or -1 for $Sp(n)$. We require the existence of a corresponding contravariant tensor g^{pq}, satisfying

$$g^{pr}g_{rq} = g_{qr}g^{rp} = \delta^p_q; \tag{30}$$

this imposes the usual limitation to even values of n for $Sp(n)$. The elements of the matrix a in this instance will be denoted by $a^p{}_q$, and those of the 'transpose' by

$$b_q{}^p = a^p{}_q. \tag{31}$$

The commutation relations are

$$[a^p_q, a^r_s] = \delta^r_q a^p_s - \delta^p_s a^r_q - g^{rp}g_{qt}a^t_s + g_{sq}g^{pt}a^r_t. \tag{32}$$

The characteristic identities can still be written in the form (3) and (5), where

$$\begin{aligned} \alpha_k &= \lambda_k + n - \eta - k, \\ \beta_k &= \lambda_k + 1 - k = -\alpha_{n+1-k}. \end{aligned} \tag{33}$$

The relation between the α_k and β_k follows from the formulae

$$\begin{aligned} &\lambda_k + \lambda_{n+1-k} = \eta && k \neq \tfrac{1}{2}(n+1) \\ &\lambda_k = h = \tfrac{1}{2}(n-1) && k = \tfrac{1}{2}(n+1) \end{aligned} \tag{34}$$

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defining the λ_k for $k \geqq \frac{1}{2}(n+1)$ in terms of those for $k < \frac{1}{2}(n+1)$. Here again, the λ_k are invariants whose eigenvalues l_k, for $k < \frac{1}{2}(n+1)$, serve to label irreducible representations, and can be defined implicitly by formulae connecting them with the traces of the *even* powers of a, e.g.,

$$\operatorname{tr}(a^2) = 2 \sum_{k=1}^{h} \lambda_k(\lambda_k + n + 1 - \eta - 2k), \tag{35}$$

where $h = \frac{1}{2}n$ when n is even, but $h = \frac{1}{2}(n-1)$ when n is odd.

We next introduce two vectors ψ and χ, with components which satisfy

$$\begin{aligned} [a^p_q, \psi^r] &= \delta^r_q \psi^p - g^{rp} g_{qt} \psi^t, \\ [a^p_q, \chi^r] &= \delta^r_q \chi^p - g^{rp} g_{qt} \chi^t. \end{aligned} \tag{36}$$

The corresponding adjoint vectors satisfy commutation relations of the type

$$[a^p_q, \phi_s] = -\delta^p_s \phi_q + g_{sq} g^{pt} \phi_t. \tag{37}$$

Again it follows from earlier work of Bracken and Green [3] and Green [4] that

$$\lambda_j(f_k \psi) = (f_k \psi)(\lambda_j + \delta_{jk})$$

and from (20) it follows that

$$a(f_k \psi) = (f_k \psi)(b + n - \eta) \tag{38}$$

so that

$$(a - \alpha_k)(f_k \psi) = (f_k \psi)(b - \beta_k) = 0,$$

and

$$f_k \psi = \psi g_k. \tag{39}$$

From this point, the argument leading to the result (8) parallels that leading from (17) to (28) in the last section. As the tensor T_{pq} defined by (18) increases the eigenvalue l_j of λ_j by two units, when $j \neq \frac{1}{2}(n+1)$ it corresponds to a representation of $O(n)$ or $Sp(n)$ labelled $(2, 0, \cdots)$ and is necessarily symmetric. Thus (20) and (22) are still valid, for $j \neq \frac{1}{2}(n+1)$, and we can proceed to deduce (24), (25), (26) and (27). A special derivation of (27) is evidently needed only for the 'idempotents' f_m and g_m, where $m = h + 1 = \frac{1}{2}(n+1)$, for $O(n)$ when n is odd. For this purpose, let us define

$$\begin{aligned} u^q_m &= \varepsilon^{qrs\cdots xy} a_{rs} \cdots a_{xy} \\ v_{mq} &= \varepsilon_{qrs\cdots xy} a^{rs} \cdots a^{xy}, \end{aligned} \tag{40}$$

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where

$$a_{rs} = g_{rt} a^t_s, \qquad a^{rs} = a^r_t g^{ts}, \tag{41}$$

and ε is the permutation symbol. Then, by a calculation similar to that given by Bracken and Green [3, Appendix A], we verify that

$$\begin{aligned} a^p_q u^q_m &= h\, u^p_m, \\ v_{mq} a^q_p &= h\, v_{mp}. \end{aligned} \tag{42}$$

Thus, u_m and v_m are the right and left eigenvectors of a, corresponding to the eigenvalue $\lambda_m = h$, and we need only define

$$\begin{aligned} c_m &= (v_{mq} u^q_m)^{-1} \\ d_m &= (u^q_m v_{mq})^{-1} \end{aligned} \tag{43}$$

to secure the correct normalization of f_m and g_m, as given by (27). It should be noticed that, unlike the other c_i and d_j, c_m and d_m thus defined are invariants which commute with all elements of the Lie algebra.

As the formulae (27) are now established for all values of j, (28) follows.

4. Evaluation of traces

We shall now apply the foregoing to evaluate the trace of any well defined function $\theta(a)$ of a matrix a of the type considered in the previous two sections. Since

$$\mathrm{tr}[\theta(a)] = \Sigma_j \theta(\alpha_j) \mathrm{tr}(f_j), \tag{44}$$

it is clear that our purpose will be accomplished by the evaluation of

$$t_j = \mathrm{tr}(f_j). \tag{45}$$

We consider first the application to matrices consisting of generators of $GL(n)$. It is evident from $\delta_{pr}\delta_{rq} = \delta_{pq}$ and (28) that

$$\begin{aligned} v_{jp} u_{jp} &= c_j^{-1}, \\ u_{jp} v_{jp} &= d_j^{-1}, \end{aligned} \tag{46}$$

and from (27) that

$$t_j = u_{jp} c_j v_{jp}. \tag{47}$$

But as the c_j, do not, in general, commute with the vectors u_j and v_j, there is some difficulty in the direct application of this formula. From the definition of f_j in (6), however, it follows that

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$$t_j = p_j \Big/ \left[\prod_{k \neq j} (\alpha_k - \alpha_j) \right],$$
$$p_j = \operatorname{tr}\left[\prod_{k \neq j} (a - \alpha_j) \right] \tag{48}$$

where p_j must reduce to a polynomial of the $(n-1)$-th degree in the α_k.

The polynomial can in fact be determined completely by general considerations. First we note that, if a_{pq} is replaced by $a_{pq} + c\delta_{pq}$, where c is any numerical constant or invariant, the commutation relations (11) are unchanged; also, although the 'eigenvalues' α_k of a are replaced by $\alpha_k + c$, the 'eigenvectors' are unchanged; thus, f_j and t_j cannot depend on c. It follows that the polynomials p_j can only depend on the *differences* $\alpha_k - \alpha_l$ of the 'eigenvalues' of a. Moreover, by symmetry, p_j must remain unchanged when α_k and α_l are interchanged, provided $k \neq j$ and $l \neq j$. We now observe that, according to (12),

$$\alpha_k = \lambda_k + n - k;$$

also the eigenvalues l_k of the λ_k must satisfy $l_1 \geqq l_2 \geqq \cdots \geqq l_n$ in tensor representations. It follows that, if ψ is any vector, $f_j\psi$ must vanish when applied to a tensor eigenvector of the λ_k such that $l_j = l_{j+1}$. Hence, the trace of f_j must possess a factor $(\alpha_j - \alpha_{j+1} - 1)$. Taking account of the requirements of symmetry already noticed, we must have

$$p_j = c \prod_{k \neq j} (\alpha_j - \alpha_k - 1) \tag{49}$$

where c is a numerical constant. From the condition

$$\operatorname{tr}(a^0) = \Sigma_j p_j \Big/ \left[\prod_{k \neq j} (\alpha_j - \alpha_k) \right] = n, \tag{50}$$

we infer that $c = 1$, so that

$$t_j = \prod_{k \neq j} \left(\frac{\alpha_j - \alpha_k - 1}{\alpha_j - \alpha_k} \right). \tag{51}$$

From this result, and the formula (44), the relations (13) are easily confirmed, and a general formula for $\operatorname{tr}(a^k)$ can be written down.

Next, we consider the application to matrices consisting of generators of $O(n)$ or $Sp(n)$. Again the invariants t_j are given by (48), and again t_j can only depend on differences of the 'eigenvalues' α_k of a, given this time by

$$\alpha_k = \lambda_k + n - \eta - k.$$

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However, the λ_k cannot be assigned arbitrarily, but must satisfy (34). We must consider separately even and odd values of n.

For even values of n, symmetry required that p_j must remain unchanged when α_k and α_l are interchanged, except when k or l takes one of the values j or $n+1-j$. The eigenvalues l_k of the λ_k must satisfy $l_1 \geqq l_2 \geqq \cdots \geqq |l_h| \geqq 0$, where $h = \frac{1}{2}n$, in tensor or spinor representations, and so, if $j \neq h$, $f_j\psi$ must vanish when applied to a tensor or spinor eigenvector of the λ_k such that $l_j = l_{j+1}$. This and the requirements of symmetry already noted determine $n-2$ factors of p_j, of the type $(\alpha_j - \alpha_k - 1)$ where $k \neq j$ and $k \neq n+1-j$. The additional factor, $(\alpha_j - \alpha_{n+1-j})$, cancels a corresponding factor in the denominator of t_j in (48); it is required so that t_j does not become singular in spinor representations. If we again determine the numerical factor with the help of (50), we obtain

$$t_j = \prod_{k \neq j} \left(\frac{\alpha_j - \alpha_k - 1 + \eta\delta_{j\,n+1-k}}{\alpha_j - \alpha_k} \right) \quad (n \text{ even}). \tag{52}$$

For odd values of n, we are restricted to $O(n)$, and λ_j has the fixed value $h = \frac{1}{2}(n-1)$ when $j = m = h+1$. To determine p_m, we note that the vector f_m does not change the eigenvalue of any of the λ_k, so that the condition $l_1 \geqq l_2 \geqq \cdots \geqq l_h \geqq 0$ does not impose any restriction (on p_m). But t_m must not be singular in any representation, so that p_m must possess a factor $\alpha_m - \alpha_k$ corresponding to every factor in the denominator of t_m, which therefore reduces to a numerical constant. As f_m is an 'idempotent' (satisfying the matrix identity $f_m^2 = f_m$), with a unique 'eigenvector' u_m satisfying $f_m u_m = u_m$,

$$t_m = 1 \qquad [m = \tfrac{1}{2}(n+1)]. \tag{53}$$

When $j \neq m$ and $j \neq h$, $f_j\psi$ must vanish when applied to a tensor or spinor eigenvector of the λ_k such that $l_j = l_{j+1}$, so that again p_j must possess a factor $(\alpha_j - \alpha_{j+1} - 1)$. This and the requirements of symmetry determine $n-3$ factors of p_j. A factor $(\alpha_j - \alpha_m)$ is clearly needed to cancel a corresponding factor in the denominator of t_j. But in this instance t_j cannot become singular on account of the divisor $(\alpha_j - \alpha_{n+1-j})$; the shift to a representation in which this divisor vanishes must be prevented by a factor $(\alpha_j - \alpha_{n+1-j} - 2)$ in the numerator of t_j. Hence

$$t_j = \prod_{k \neq j} \left(\frac{\alpha_j - \alpha_k - 1 + \delta_{km} - \delta_{j\,n+1-k}}{\alpha_j - \alpha_k} \right). \tag{54}$$

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These results may be used, in conjunction with (44), to confirm the formula (35); of course, they may also be used to compute $\mathrm{tr}(a^4)$ and other traces which can only be determined with much greater labour by earlier methods.

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