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The Langlands Correspondence on the Generic Irreducible Constituents of Principal Series

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Abstract. Let G be a connected semisimple split group over a p-adic field. We establish the explicit link between principal nilpotent orbits and the irreducible constituents of principal series in terms of L-group objects.

1 Introduction

Let G be a split semisimple p-adic group. That is, $G = \mathbf{G}(F)$, where $\mathbf{G}$ is a connected split semisimple algebraic group defined over a p-adic local field F of characteristic zero. Let $\mathfrak{g}$ be the Lie algebra of G. Then the adjoint action on $\mathfrak{g}$ breaks $\mathfrak{g}$ up into G-orbits. The set of all regular nilpotent elements, being G-stable, breaks up into *principal nilpotent orbits*. Let $\mathcal{P}$ be the set of principal nilpotent orbits.

Let B be a Borel subgroup of G. B can be written as $B = TN$, where T is a maximal split torus and N is a maximal nilpotent subgroup. Let λ be a unitary character of T. We can extend λ to B by letting it act trivially on N, and then view λ as a character of B. The induced representation from B to G by λ is denoted by π_λ. A representation π_λ arising in this way is called a *(unitary) principal series representation*. The character λ is unitary, so π_λ is as well. Since a unitary representation is semisimple, we can decompose π_λ into the sum of its irreducible constituents $\pi_\lambda = \bigoplus_{\xi\in\Pi_\lambda} m_\xi \xi$, where Π_λ is the set of all irreducible constituents of π_λ and m_ξ is the multiplicity of ξ in π_λ.

The links between $\mathcal{P}$ and Π_λ have been established in [10]. Let us state those results. Let Γ be the Galois group $\mathrm{Gal}(\overline{F}/F)$, where $\overline{F}$ is the algebraic closure of F. Let $\overline{G} = \mathbf{G}(\overline{F})$, the $\overline{F}$ points of $\mathbf{G}$, and $Z(\overline{G})$ the center of $\overline{G}$. Then the first Galois cohomology group $H^1(\Gamma, Z(\overline{G}))$ acts simply transitively on the set $\mathcal{P}$ of principal nilpotent orbits. On the representation side, a unitary principal series representation is not irreducible in general, and its decomposition is determined by its commuting algebra $\mathcal{C}(\lambda) := \mathrm{End}(\pi_\lambda)$ of intertwining operators. In particular, there is a bijection $j\colon \xi \mapsto r_\xi$ between Π_λ, the set of irreducible constituents of π_λ, and $R_\lambda^\wedge$, the set of irreducible characters of an explicit finite subgroup R_λ of Weyl group $W(T, G)$.

We introduced two more constructions into the picture [10].

- For each $\mathcal{O} \in \mathcal{P}$, we constructed a generic irreducible constituent $\rho(\mathcal{O})$ of π_λ.
- We constructed a canonical pairing $\langle\ ,\ \rangle_\lambda\colon R_\lambda \times H^1(\Gamma, Z(\overline{G})) \to \mathbb{C}^*$.

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The main result in [10, Main Theorem] is the following.

Theorem (i) *The map $\rho\colon \mathcal{P} \to \Pi_\lambda$ induces a bijection $Q_\lambda \backslash \mathcal{P} \xrightarrow{\sim} \Pi_\lambda^{\mathrm{gen}}$, where Q_λ is the right kernel of $\langle\ ,\ \rangle$, and $\Pi_\lambda^{\mathrm{gen}}$ is the subset of generic representations in Π_λ.*

(ii) *The composition $\mathcal{P} \xrightarrow{\rho} \Pi_\lambda \xrightarrow{j} \mathrm{Hom}(R_\lambda, \mathbb{C}^*)$ is the same as the composition*

$$\varrho\colon \mathcal{P} \simeq H^1(\Gamma, Z(\overline{G})) \xrightarrow{\langle\ ,-\rangle_\lambda^{-1}} \mathrm{Hom}(R_\lambda, \mathbb{C}^*).$$

Understanding the irreducible constituents of unitary principal series representations is a very interesting problem in representation theory for the following reason. The Langlands correspondence predicts that the set of equivalence classes of irreducible admissible representations of G breaks up into finite sets, called L-packets, indexed by the *Langlands parameters*. The Langlands parameters are homomorphisms from the Weil–Deligne group W_F' into the L-group ${}^LG$ of G. By the principle of Langlands functoriality, the irreducible constituents of a single unitary principal representation π_λ should be in a single L-packet. The various irreducible constituents in Π_λ are *L-indistinguishable* in the sense of Langlands [3]. In general, understanding the structure of L-packets is a very delicate question. For real and complex groups, Adams, Barbasch, and Vogan [1] refined Langlands' results, and went further by completely parametrizing each L-packet. For p-adic groups, so far there is no well-formulated conjecture. Gelbart and Knapp [4] first investigated this problem for the case of SL_n. The case of unramified principal series was studied as a conjecture by Deligne and Langlands, and later Lusztig modified the conjecture to cover the larger class of so-called "unipotent representations." The unramified part was established by Kazhdan and Lusztig [5], and later on, Lusztig proved his conjecture completely [13]. However, for the other representations, this question still remains open. The main theorem in [10] parametrizes the generic parts of the L-packet Π_λ by $H^1(\Gamma, Z(\overline{G}))/Q_\lambda$. Hence, it offers a new insight into this general problem.

For the same reason, it will be very helpful if we can provide the results in [10] in terms of L-groups objects. The main idea behind it is the Tate duality. Let us explain our results. The pairing $R_\lambda \times H^1(\Gamma, Z(\overline{G})) \to \mathbb{C}^*$ gives rise to a homomorphism $R_\lambda \to \mathrm{Hom}(H^1(\Gamma, Z(\overline{G})), \mathbb{C}^*)$ by $w \mapsto \tilde{\lambda}_w$. In this paper, we first establish a non-trivial interpretation for $\tilde{\lambda}_w$ on the side of dual objects. By the Langlands correspondence for tori, the character λ corresponds to a Langlands parameter ϕ from the Weil group W_F to the dual group ${}^LT^\circ$ of T. We choose a lifting $\tilde{\phi}$ of ϕ from W_F to the dual group ${}^L\tilde{T}^\circ$ of T_{ad} ($\tilde{\phi}$ may not be a homomorphism), where $T_{\mathrm{ad}} = \mathbf{T}_{\mathrm{ad}}(F)$ is the set of F-points of the maximal split torus $\mathbf{T}_{\mathrm{ad}}$ of the adjoint group $\mathbf{G}_{\mathrm{ad}}$. By construction, R_λ is a subgroup of the Weyl group $W(T, G)$, which can be identified with $W({}^LT^\circ, {}^LG^\circ)$, where ${}^LG^\circ$ is the dual group of G. We now define a map from W_F to ${}^L\tilde{T}^\circ$, $\phi_w(\gamma) = \tilde{\phi}(\gamma)\tilde{w}\tilde{\phi}(\gamma)^{-1}\tilde{w}^{-1}$, for $w \in R_\lambda \subset W({}^LT^\circ, {}^LG^\circ), \gamma \in W_F$, where $\tilde{w}$ is a representative of w in ${}^L\tilde{G}^\circ$, the dual group of G_{ad}. There is a natural isomorphism $T_{\mathrm{ad}}/p(T) \xrightarrow{\sim} H^1(\Gamma, Z(\overline{G}))$, where p is the projection map from G to G_{ad}. We extend $\tilde{\lambda}_w$ to a character λ_w of T_{ad} factoring through $H^1(\Gamma, Z(\overline{G}))$. Then we have the following.

Theorem 4.2 *The character λ_w of T_{ad} corresponds to the Langlands parameter $\phi_w\colon W_F \to {}^L\tilde{T}^\circ$ via the Langlands correspondence.*

For the case when $G = \mathrm{SL}_n$, the results in [4, §4] imply that λ_w and ϕ_w are related via the usual Pontryagin duality. Therefore, our theorem here can be viewed as a generalization of their work with a Langlands type interpretation.

It is more complicated to formulate the L-group side version of the map ϱ and the pairing $\langle\ ,\ \rangle_\lambda$ in the Main Theorem of [10] above. First, we have $R_\lambda \cong A_\phi := Z_\phi/Z_\phi^\circ$, where Z_ϕ is the centralizer of the image of ϕ in ${}^L G^\circ$ and Z_ϕ° is its identity component (see Proposition 3.3). Then we can define a pairing $\langle\ ,\ \rangle_\phi$ on $Z_\phi \times W_F$ as follows:

$$\langle w, \gamma\rangle_\phi = \phi_w(\gamma).$$

We can show that this pairing can be extended to $A_\phi \times \Gamma$ whose image is in $\pi_1({}^L G^\circ)$ (see Section 3). It gives a rise to a map

$$\langle\ , - \rangle_\phi \colon A_\phi \to \mathrm{Hom}(\Gamma, \pi_1({}^L G^\circ)) \cong H^1(\Gamma, \pi_1({}^L G^\circ)).$$

Moreover, via the Tate duality, we have

$$H^1(\Gamma, Z(\overline{G}))^\wedge \cong H^1(\Gamma, \pi_1({}^L G^\circ)),$$

where $\pi_1({}^L G^\circ)$ is the fundamental group of ${}^L G^\circ$ (see Lemma 3.4). The following theorem gives us the full Langlands version of the work of [10].

Theorem 3.4 *The homomorphism from A_ϕ to $\mathrm{Hom}(\Gamma, \pi_1({}^L G^\circ))$ defined by $\langle\ , - \rangle_\phi$ coincides with the $\varrho^\vee$, the Pontryagin dual of ϱ by the identifications A_ϕ with R_λ via Proposition 3.3 and $H^1(\Gamma, \pi_1({}^L G^\circ))$ with $H^1(\Gamma, Z(\overline{G}))^\wedge$ via the Tate duality.*

The proof of Theorem 4.2 above needs to use only the basic properties of the Langlands correspondence. Our main theorem, Theorem 3.4, can be derived from Theorem 4.2 by the commutativity of a certain diagram with the correct sign (see Section 5.3). However, the commutativity is not obvious at all (see Section 5). Furthermore, to prove that the sign is correct, it is necessary to go into the details of constructions of the Langlands correspondence and the Tate duality, which involves the computation of cup products in Galois cohomology, as we do in Section 5.3.

This paper is organized as follows. In Section 2, we recall the results in [10]. We review the basic facts about principal series and principal nilpotent orbits. Based on those facts, we state the main theorem from [10]. In Section 3, using the Tate duality with the knowledge of Galois cohomology, we state our main theorem. In Section 4, we devote ourselves to proving the main theorem. In the final section, we prove a cohomological lemma used in the proof of the main theorem.

Preliminary Notations

Let F be a non-archimedean local field of characteristic 0, *i.e.,* a finite extension of $\mathbb{Q}_p$ for some prime number p. We require $p \neq 2$. Let $\overline{F}$ be the algebraic closure of F and Γ be the Galois group of $\overline{F}$ over F. Let W_F be the Weil group of Γ. Given an algebraic group $\mathbf{H}$ defined over F, we denote the set of F-points of $\mathbf{H}$ by H and the set of $\overline{F}$-points of $\mathbf{H}$ by $\overline{H}$. Let $X(H)$ be the group of rational characters of $\overline{H}$ and

$X_*(H)$ the set of 1-parameter subgroups. Given a finite group K, let $K^\wedge$ be the set of characters of K.

Let $\mathbf{G}$ be a connected semisimple split algebraic group defined over F. Fix a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ and write $\mathbf{B} = \mathbf{TN}$, where $\mathbf{T}$ and $\mathbf{N}$ represent a split maximal torus and the unipotent radical of $\mathbf{B}$, respectively. Let Δ be the set of roots and $\Delta^\vee$ be the set of coroots. The Borel subgroup $\mathbf{B}$ determines the set Δ^+ of positive roots. Let $\mathbf{G}_{\mathrm{ad}}$ be the adjoint group of $\mathbf{G}$, p the projection from $\mathbf{G}$ to $\mathbf{G}_{\mathrm{ad}}$, $\mathbf{T}_{\mathrm{ad}}$ the maximal split torus of $\mathbf{G}_{\mathrm{ad}}$ compatible with $\mathbf{T}$. For all roots $\alpha \in \Delta$, $\alpha = \overline{\alpha} \circ p$, where $\overline{\alpha}$ is the root of G_{ad} associated with the root α of G.

For each root α, there is a homomorphism ζ_α from $\mathrm{SL}_2(F)$ to a subgroup of G. Let $W = W(T, G) = N(T)/T$ be the Weyl group of G, where $N(T)$ is the normalizer of T in G. For each root α, let s_α be the reflection of W associated with α. Let $\Delta \subset X(T)$ be the root system of G and $\Delta^\vee \subset X_*(T)$ be the set of coroots. We let $\alpha^\vee \in \Delta^\vee$ denote the coroot associated with the root $\alpha \in \Delta$. The reflection $s_\alpha \in W$ associated with a root α is defined by $t \mapsto t(\alpha^\vee \circ \alpha(t))^{-1}$.

Let $\mathcal{P}$ be the set of all nilpotent orbits consisting of regular nilpotent elements. The orbits in $\mathcal{P}$ are called *principal nilpotent orbits*. Let λ be a (unitary) character of T. The induced representation $\mathrm{Ind}_B^G \lambda$, denoted by π_λ, is called a *(unitary) principal series representation*. Let Π_λ be the set of all irreducible constituents of π_λ.

Finally we set up the notations for dual groups. Let $(X(T), \Delta, X_*(T), \Delta^\vee)$ be the root datum of G. The dual group ${}^LG^\circ$ of G is a complex Lie group with the root datum $(X_*(T), \Delta^\vee, X(T), \Delta)$. Since we only consider split groups, the L-group ${}^LG$ (resp., ${}^LT$) of G (resp., T) is just ${}^LG^\circ \times \Gamma$ (resp., ${}^LT^\circ \times \Gamma$). Let ${}^L\tilde{G}^\circ$ be the algebraic universal cover of ${}^LG^\circ$, which is also the dual group of G_{ad}, and ${}^L\tilde{T}^\circ$ the dual group of T_{ad}. The natural projection from ${}^L\tilde{G}^\circ$ to ${}^LG^\circ$ is denoted by η.

2 Principal Series Representations and Principal Nilpotent Orbits

In this section, we state the known relations between principal series representations and principal nilpotent orbits. The results here are taken from [10].

2.1 Irreducible Constituents and R-Groups

Given a character λ of T, define $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$. Let $\lambda_\alpha = \lambda \circ \alpha^\vee$, where $\alpha \in \Delta$. Define $\Delta_\lambda = \{\alpha \in \Delta^+ \mid \lambda_\alpha \equiv 1\}$ and $W'_\lambda = \langle s_\beta \rangle_{\beta \in \Delta_\lambda}$. It is easy to see that $W'_\lambda \subset W_\lambda$. Define a subgroup R_λ of W_λ, called the *R-group* of π_λ,

$$R_\lambda := \{w \in W_\lambda \mid w(\Delta_\lambda) = \Delta_\lambda\}.$$

The reducibility of a representation (π, V) is determined by its commuting algebra $\mathcal{C}(\pi) := \mathrm{End}(\pi)$. In general, it is not easy to compute commuting algebras. However, in the case of principal series representations of connected split reductive groups, it can be computed explicitly. The following theorem is due to Keys [6, Chapter 1, §3, Theorem 1].

Theorem 2.1 *Keep the notation above. We can write W_λ as a semi-direct product*

$$W_\lambda = W'_\lambda \rtimes R_\lambda.$$

Given a generic character χ of N, there is a homomorphism $\iota_{\lambda,\chi}$ from W_λ to $\mathcal{C}(\pi_\lambda)$ which is trivial on W'_λ. Furthermore, the map from the group algebra $\mathbb{C}[R_\lambda]$ to the commuting algebra $\mathcal{C}(\pi_\lambda)$ induced by $\iota_{\lambda,\chi}$ is indeed an algebra isomorphism.

Let $R_\lambda^\wedge$ be the set of the characters of irreducible representations of R_λ. As a consequence of the theorem above, we have the following corollary.

Corollary 2.2 *With notation as above, the map $\iota_{\lambda,\chi}$ induces a bijection j_χ between Π_λ and $R_\lambda^\wedge$. This bijection is not canonical, but depends on the choice of the generic character χ of N.*

2.2 Principal Nilpotent Orbits

There is a well-known result for the parametrization of principal nilpotent orbits for split groups.

Lemma 2.3 ([10] Proposition 4.1.1) *There is a one-to-one correspondence between principal nilpotent orbits and $H^1(\Gamma, Z(\overline{G}))$, where $Z(\overline{G})$ is the center of $\overline{G}$.*

We have another understanding for $H^1(\Gamma, Z(\overline{G}))$.

Proposition 2.4 ([10] Proposition 4.1.1 and its proof) *The first Galois cohomology group $H^1(\Gamma, Z(\overline{G}))$ is isomorphic to the abelian group $T_{\mathrm{ad}}/p(T)$. Furthermore, there is a transitive T_{ad}-action on $\mathcal{P}$, whose stabilizer is $p(T)$. As a consequence, $T_{\mathrm{ad}}/p(T)$ acts on $\mathcal{P}$ simply transitively.*

2.3 A Whittaker Version of ρ

We say a smooth admissible representation (π, V) *admits a Whittaker model* or is *generic* if there is a generic character χ of N from N to $\mathbb{C}$ such that $\mathrm{Hom}_G(V, \mathrm{Ind}_N^G \chi)$ is non-trivial. Given a principal nilpotent orbit $\mathcal{O}$, one can construct a class $\chi_{\mathcal{O}}$ of generic characters (see [10, Lemma 4.1.1]).

Let $\Pi_\lambda^{\mathrm{gen}}$ be the set of generic irreducible constituents of π_λ. The first result in [10] is the following.

Proposition 2.5 ([10] Proposition 4.1.6) *There is a canonical map ρ from $\mathcal{P}$ to Π_λ. The image of ρ is onto the set $\Pi_\lambda^{\mathrm{gen}}$.*

Define a pairing $\langle\ ,\ \rangle_\lambda$ from $R_\lambda \times T_{\mathrm{ad}}$ to $\mathbb{C}^*$ as follows

$$\langle w, \overline{t}\rangle_\lambda = \prod_{i=1}^{n} (\lambda_i)_{\alpha_i}(\overline{\alpha}_i(\overline{t})).$$

Here $w \in R_\lambda, \overline{t} \in T_{\mathrm{ad}}$, and we factorize $w = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$ in a reduced expression, where α_i are simple roots. For $i \geq 2$ (resp., $i = 1$), write $w'_i = s_{\alpha_{i-1}} \cdots s_{\alpha_1}$ (resp., w'_1 is the identity). Then set $\lambda_i = w'_i \lambda$ and $(\lambda_i)_{\alpha_i} = \lambda_i \circ \alpha_i^\vee$.

The right kernel of $\langle\ ,\ \rangle_\lambda$ contains the subgroup $p(T)$ of T_{ad} [10, Lemma 5.2.4]. Hence, the pairing $\langle\ ,\ \rangle_\lambda$ factors through $R_\lambda \times (T_{\mathrm{ad}}/p(T)) \simeq R_\lambda \times H^1(\Gamma, Z(\overline{G}))$. By abuse of notation, we still denote the induced pairing on $R_\lambda \times H^1(\Gamma, Z(\overline{G}))$ as $\langle\ ,\ \rangle_\lambda$.

The main result in [10] is the following.

Theorem 2.6 ([10, Main Theorem]) *Fix a principal nilpotent orbit $\mathcal{O}$. Using $\mathcal{O}$ and $\chi_{\mathcal{O}}$, we uniquely determine the identifications*

$$i_{\mathcal{O}} : \mathcal{P} \simeq H^1(\Gamma, Z(\overline{G})) \quad \textit{and} \quad j_{\chi_{\mathcal{O}}} : \Pi_\lambda \xrightarrow{\sim} R_\lambda^\wedge.$$

(i) *the map $\rho\colon \mathcal{P} \to \Pi_\lambda$ induces a bijection $Q_\lambda \backslash \mathcal{P} \xrightarrow{\sim} \Pi_\lambda^{\mathrm{gen}}$, where Q_λ is the right kernel of $\langle\ ,\ \rangle_\lambda$;*

(ii) *the map $\varrho := j_{\chi_{\mathcal{O}}} \circ \rho \circ i_{\mathcal{O}}^{-1}$ is independent of the choice of $\mathcal{O}$ and equal to*

$$H^1(\Gamma, Z(\overline{G})) \xrightarrow{\langle\ ,-\rangle_\lambda^{-1}} R_\lambda^\wedge.$$

3 A Langlands Version of ϱ

3.1 The Tate Duality

Let K be a topological group. We let $\widehat{K}$ denote the completion relative to the topology of open subgroups of finite index. Recall the Tate duality from [14, Corollary 2.4].

Proposition 3.1 *Let $\mathbf{A}$ be a commutative algebraic group over a local field F whose identity component $\mathbf{A}^\circ$ is a torus and Γ the Galois group of F. Assume that the order of $\mathbf{A}/\mathbf{A}^\circ$ is not divisible by the characteristic of F. Then the cup-product defines dualities between each of the following:*

(i) *the compact group $\widehat{H^0(\Gamma, \overline{A})}$ and the discrete group $H^2(\Gamma, X(A))$;*

(ii) *the finite groups $H^1(\Gamma, \overline{A})$ and $H^1(\Gamma, X(A))$;*

(iii) *the discrete group $H^2(\Gamma, \overline{A})$ and the compact group $H^0\widehat{(\Gamma, X(A))}$.*

3.2 The Langlands Correspondence on Split Tori

The Langlands parameters for principal series are certain maps from the Weil group W_F into ${}^LT$. Since T is split, we only need to consider the ${}^LT^\circ$ part. Fix a Langlands parameter $\phi\colon W_F \to {}^LT$. By abuse of notations, we also write its restriction to ${}^LT^\circ$ as ϕ. The corresponding character of T is denoted by λ, and the principal series representation is π_λ.

We let $\mathfrak{L}$ denote the functor giving the Langlands correspondence from the characters of tori to the Langlands parameters of dual tori. For a split torus over F, we can write down $\mathfrak{L}$ explicitly. Let H be a split torus, ${}^LH^\circ$ the dual torus, $X_*(H)$ the set of one-parameter subgroups of H, and $X(H)$ the set of rational characters of H. There is a natural perfect pairing $\langle\ ,\ \rangle\colon X_*(H) \times X(H) \to \mathbb{Z}$. We choose bases of $\{e_i\} \subset X(H)$ and $\{e_i^\vee\} \subset X_*(H)$ such that $\langle e_i, e_j^\vee\rangle = \delta_{ij}$. Then we can identify H (resp. ${}^LH^\circ$) as $\{\sum e_i^\vee \otimes f_i \mid f_i \in F^*\}$ (resp. $\{\sum e_i \otimes c_i \mid c_i \in \mathbb{C}^*\}$). For any character ξ of H, there is an expression $\xi = \prod \xi_i \circ e_i$, where ξ_i is a character of F^* and

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$\xi_i = \xi|_{e_i^\vee \otimes F^*} = \xi \circ e_i^\vee$. Then $\mathfrak{L}(\xi) = \prod e_i \circ (\xi_i^{-1})$. Note that we use local class field theory to identify $W_F^{\mathrm{ab}} = W_F/[W_F, W_F]$ with F^*.

3.3 R-groups on the Dual Groups

Recall Keys' theorem in Section 2.1. Let

$$W_\lambda = \{w \in W \mid w\lambda = \lambda\}, \quad \Delta_\lambda = \{\alpha \in \Delta \mid \lambda \circ \alpha^\vee = 1\}, \quad W'_\lambda = \langle w_\alpha \rangle_{\alpha \in \Delta_\lambda},$$

where w_α is the reflection associated with α. Then the R-group R_λ of π_λ is isomorphic to $\{w \in W_\lambda \mid w(\Delta_\lambda) = \Delta_\lambda\}$. Furthermore, $W_\lambda = W'_\lambda \rtimes R_\lambda$.

Let $Z_\phi = Z_{{}^L G^\circ}(\phi(W_F))$ be the centralizer of $\phi(W_F)$ in ${}^L G^\circ$ and Z_ϕ° the identity component of Z_ϕ. The following proposition is due to Langlands.

Proposition 3.2 (Keys [7] and Langlands [12]) *Keep the notation above. There is an expression of the R-group R_λ of π_λ in terms of the L-group as follows:*

$$R_\lambda \cong A_\phi := Z_\phi / Z_\phi^\circ.$$

Moreover, Z_ϕ/Z_ϕ° is isomorphic to the subgroup of W fixing $\phi(W_F)$ modulo the subgroup of W generated by the simple reflections associated with the coroots trivial on $\phi(W_F)$, where we identify the Weyl group of the dual group ${}^L G^\circ$ with that of G.

Proof The subgroup Z_ϕ contains ${}^L T^\circ$. We claim that any complex reductive subgroup H whose identity component H° contains any maximal torus ${}^L T^\circ$ can be generated by its identity component and a subgroup of the Weyl group with respect to the maximal torus.

It is enough to prove that any element h in H can be expressed as a product of a Weyl group element and an element in H°. We have that $h^{-1L}T^\circ h$ is a connected torus in H containing the identity element. Therefore, it is contained in H°. Since its dimension is equal to ${}^L T^\circ$, it is a maximal torus. Hence $h^{-1L}T^\circ h$ conjugates to ${}^L T^\circ$ by an element g^{-1} in H°, *i.e.*, $gh^{-1L}T^\circ hg^{-1} = {}^L T^\circ$. Therefore hg^{-1} normalizes ${}^L T^\circ$. This implies that $hg^{-1} = w$, where w is an element in the Weyl group. Therefore $h = wg$.

By the previous claim, the subgroup of W that we look for is W_ϕ, the subgroup of W generated by elements fixing $\phi(W_F)$. Consider the intersection W'_ϕ of W_ϕ and Z_ϕ°. Let ${\Delta_\phi^\vee}' = \{\alpha^\vee \in \Delta^\vee \mid \alpha(\phi(W_F)) = 1\}$. We claim that $W'_\phi = \langle s_{\alpha^\vee} \rangle_{\alpha \in {\Delta_\phi^\vee}'}$, where $s_{\alpha^\vee}$ is the simple reflection in W with respect to the root $\alpha^\vee$.

Note that the identity component Z_ϕ° can be determined at the Lie algebra level. Because Z_ϕ° contains the maximal torus ${}^L T^\circ$ and is reductive, we can make the root decomposition of Lie Z_ϕ° and its weights are included in ${}^L G^\circ$'s. In fact, those coroots are in W'_ϕ. On the other hand, if a coroot is in W'_ϕ, then the associated root subgroup centralizes $\phi(W_F)$. Therefore we get the other direction.

It remains to prove that R_λ is isomorphic W_ϕ/W'_ϕ. It is enough to prove $W_\lambda \cong W_\phi$ and $W'_\lambda \cong W'_\phi$. These two isomorphisms come from Langlands functoriality. By the Langlands correspondence for tori, the first isomorphism is straightforward. The second one is also easy. It is enough to prove that $\alpha^\vee \in \Delta_\phi^{\vee\prime}$ if and only if $\alpha \in \Delta_\lambda$. For $\alpha \in \Delta_\lambda$, there is a sequence

$$\lambda_\alpha \colon F^* \xrightarrow{\alpha^\vee} T \xrightarrow{\lambda} \mathbb{C}^*.$$

In the sense of Langlands functoriality, the sequence above can be written in terms of L-group as follows:

$$\mathbb{C}^* \xleftarrow{(\alpha^\vee)^{-1}} {}^L T^\circ \xleftarrow{\phi} W_F.$$

Note that ϕ factors through F^* by local class field theory. It completes the proof. ■

3.4 The Map $\varrho^\vee$

Now we are ready to describe the main work of this paper. Since we consider the picture on the Langlands side, it is better to think about the dual version of ϱ. Instead of looking for a map ϱ from $H^1(\Gamma, Z(\overline{G}))$ to $R_\lambda^\wedge$, we search for a map $\varrho^\vee$ from R_λ to $H^1(\Gamma, Z(\overline{G}))^\wedge$. There is a one-to-one correspondence:

$$R_\lambda^\wedge \leftarrow H^1(\Gamma, Z(\overline{G})) \rightleftharpoons A_\phi \to H^1(\Gamma, Z(\overline{G}))^\wedge.$$

We can write $H^1(\Gamma, Z(\overline{G}))^\wedge$ in terms of L-group objects.

Lemma 3.3 *There is a canonical isomorphism $H^1(\Gamma, Z(\overline{G}))^\wedge \cong H^1(\Gamma, \pi_1({}^L G^\circ))$ via the Tate duality.*

Proof By the Tate duality, we know $H^1(\Gamma, Z(\overline{G}))^\wedge \cong H^1(\Gamma, X(Z(\overline{G})))$. Let $Q^*(T)$ be the subgroup of $X(T)$ generated by roots Δ of G. One can prove that $X(Z(\overline{G}))$ is isomorphic to the quotient $X(T)/Q^*(T)$ (see [15, 2.15]). By [2, Proposition 7.4], the abelian group $X(T)/Q^*(T)$ is isomorphic to the fundamental group $\pi_1({}^L G^\circ)$ of ${}^L G^\circ$. Combining these two facts, we get $H^1(\Gamma, Z(\overline{G}))^\wedge \cong H^1(\Gamma, \pi_1({}^L G^\circ))$. ■

Now we can translate all data in terms of L-groups. To sum up, we have a map

$$\varrho^\vee \colon A_\phi \to H^1(\Gamma, \pi_1({}^L G^\circ)),$$

whose dual is ϱ in the main theorem of [10] (*cf.* Theorem 2.6)

3.5 The Pairing $\langle\ ,\ \rangle_\phi$

Now we have a reasonable object $\varrho^\vee$ which can be viewed as a Langlands version of ϱ. Note that $H^1(\Gamma, \pi_1({}^L G^\circ)) = \mathrm{Hom}(\Gamma, \pi_1({}^L G^\circ))$ since G is split. Finding a map from A_ϕ to $H^1(\Gamma, \pi_1({}^L G^\circ))$ is equivalent to finding a pairing

$$\langle\ ,\ \rangle_\phi \colon A_\phi \times \Gamma \to \pi_1({}^L G^\circ) \subset {}^L\tilde{T}^\circ \subset {}^L\tilde{G}^\circ,$$

such that

$$\langle w, \gamma \rangle_\phi = [\varrho^\vee(w)](\gamma),$$

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where $w \in A_\phi, \gamma \in \Gamma$. This pairing can be viewed as the dual of the pairing $\langle\ ,\ \rangle_\lambda$ defined in Section 2.3.

Indeed, there is a natural way to define such a pairing which involves only ϕ. We start with a homomorphism $\phi\colon W_F \to {}^L T^\circ$. Let ${}^L\tilde{G}^\circ$ be the algebraic universal cover of ${}^L G^\circ$. Note that ${}^L\tilde{G}^\circ$ is the dual group of G_{ad}. Denote the natural projection map from ${}^L\tilde{G}^\circ$ to ${}^L G^\circ$ by η. The kernel of η is contained in the center of ${}^L\tilde{G}^\circ$ and it is equal to $\pi_1({}^L G^\circ)$. Let $\tilde{\phi}$ be an arbitrary lifting of ϕ from W_F to ${}^L\tilde{G}^\circ$ (generally not a homomorphism).

Define a pairing $\langle\ ,\ \rangle_\phi$ on $Z_\phi \times W_F$ with values in ${}^L\tilde{G}^\circ$ as follows:

$$\langle w, \gamma\rangle_\phi = \tilde{\phi}(\gamma)\tilde{w}\tilde{\phi}(\gamma)^{-1}\tilde{w}^{-1},$$

where $\tilde{w} \in {}^L\tilde{G}^\circ$ is a lifting of w, *i.e.*, $\eta(\tilde{w}) = w$. We would like to investigate the properties of $\langle\ ,\ \rangle_\phi$.

(i) The definition of $\langle\ ,\ \rangle_\phi$ is independent of the choices of $\tilde{w}$ and $\tilde{\phi}$. This is because the different choices are only up to elements in $\pi_1({}^L G^\circ)$, which is contained in the center of ${}^L\tilde{G}^\circ$. They commute with all items and cancel out each other.

(ii) The values of the pairing $\langle\ ,\ \rangle_\phi$ are contained in $\pi_1({}^L G^\circ)$. Taking η on both sides of the equality, we have

$$\begin{aligned}\eta(\langle w,\gamma\rangle_\phi) &= \eta(\tilde{\phi}(\gamma))\eta(\tilde{w})\eta(\tilde{\phi}(\gamma))^{-1}\eta(\tilde{w})^{-1}\\ &= \phi(\gamma)w\phi(\gamma)^{-1}w^{-1} = \phi(\gamma)\phi(\gamma)^{-1} = 1.\end{aligned}$$

Therefore, the value of $\langle\ ,\ \rangle_\phi$ is taken in $\operatorname{Ker}\eta = \pi_1({}^L G^\circ)$.

(iii) The map $\langle\ ,\ \rangle_\phi$ is a homomorphism on the variables w and γ. For $\gamma_1, \gamma_2 \in \Gamma$, we can choose $\tilde{\phi}(\gamma_1\gamma_2) = \tilde{\phi}(\gamma_1)\tilde{\phi}(\gamma_2)$. Then

$$\begin{aligned}\langle w,\gamma_1\gamma_2\rangle_\phi &= \tilde{\phi}(\gamma_1\gamma_2)\tilde{w}\tilde{\phi}(\gamma_1\gamma_2)^{-1}\tilde{w}^{-1}\\ &= \tilde{\phi}(\gamma_1)\tilde{\phi}(\gamma_2)\tilde{w}\tilde{\phi}(\gamma_1)^{-1}\tilde{\phi}(\gamma_2)^{-1}\tilde{w}^{-1}\\ &= \tilde{\phi}(\gamma_1)\tilde{\phi}(\gamma_2)\tilde{w}\tilde{\phi}(\gamma_2)^{-1}\tilde{w}^{-1}\tilde{w}\tilde{\phi}(\gamma_1)^{-1}\tilde{w}^{-1}\\ &= \tilde{\phi}(\gamma_1)\langle w,\gamma_2\rangle_\phi\tilde{w}\tilde{\phi}(\gamma_1)^{-1}\tilde{w}^{-1}\\ &= \tilde{\phi}(\gamma_1)\tilde{w}\tilde{\phi}(\gamma_1)^{-1}\tilde{w}^{-1}\langle w,\gamma_2\rangle_\phi\\ &= \langle w,\gamma_1\rangle_\phi\langle w,\gamma_2\rangle_\phi.\end{aligned}$$

Therefore, $\langle\ ,\ \rangle_\phi$ is a homomorphism with respect to γ. Similarly, $\langle\ ,\ \rangle_\phi$ is a homomorphism with respect to w.

(iv) The left kernel of the pairing $\langle\ ,\ \rangle_\phi$ contains Z_ϕ°. Therefore $\langle\ ,\ \rangle_\phi$ factors through $A_\phi \times W_F$. Given $w \in Z_\phi^\circ$, we can choose a path f so that $f(0) = e$ and $f(1) = w$, where e is the identity element of ${}^L G^\circ$. We can lift f to a path $\tilde{f}$ on ${}^L\tilde{G}^\circ$ since ${}^L\tilde{G}^\circ$ is a covering. Let $\tilde{f}(0) = \tilde{e}$ and $\tilde{w} = \tilde{f}(1)$, where $\tilde{e}$ is the identity element of ${}^L\tilde{G}^\circ$. Then the map $t \mapsto \langle f(t), \gamma\rangle_\phi$ from $[0,1]$ to ${}^L\tilde{G}^\circ$ is a continuous map with values on a discrete group $\pi_1({}^L G^\circ)$. The image must be a point. Obviously $\langle f(0),\gamma\rangle_\phi = \langle e,\gamma\rangle_\phi \equiv \tilde{e}$. Hence, $\langle -,\gamma\rangle_\phi$ is trivial on Z_ϕ°.

Since $\pi_1({}^L G^\circ)$ is finite and W_F dense in Γ, we can extend $\langle\ ,\ \rangle_\phi$ to $A_\phi \times \Gamma$ uniquely (the "existence" part is from finiteness and the "unique" part is from density). Finally, we have a well-defined pairing, still denoted by $\langle\ ,\ \rangle_\phi$,

$$\langle\ ,\ \rangle_\phi \colon A_\phi \times \Gamma \to \pi_1({}^L G^\circ).$$

3.6 The Main Theorem

Now we have a pairing $\langle\ ,\ \rangle_\phi$ whose construction only involves ϕ. The main theorem of this paper is the following statement, which basically says that $\langle\ ,\ \rangle_\phi$ is the dual version of $\langle\ ,\ \rangle_\lambda$ defined in Section 2.3 (or see [10, Main Theorem]).

Theorem 3.4 *With notation as above, the homomorphism from A_ϕ to*

$$\mathrm{Hom}(\Gamma, \pi_1({}^L G^\circ))$$

defined by $\langle\ , -\rangle_\phi$ coincides with $\varrho^\vee$, the dual version of ϱ defined in Section 2.3*, under the identifications of A_ϕ with R_λ via Proposition* 3.3*, and $H^1(\Gamma, \pi_1({}^L G^\circ))$ with $H^1(\Gamma, Z(\overline{\mathrm{G}}))^\wedge$ via the Tate duality.*

4 A Proof of the Main Theorem

4.1 Dual Characters ϕ_w

To prove the Langlands version of ϱ, we need to build a bridge between the group side and the dual group side. Let ϕ be the homomorphism from W_F to the dual torus ${}^L T^\circ$ which is the Langlands dual of λ. We pick up an arbitrary lifting $\tilde{\phi}$ of ϕ from W_F to ${}^L \tilde{T}^\circ$. Given $w \in W$, we choose an element $\tilde{w}$ of ${}^L \tilde{G}^\circ$ which represents w. Define a map ϕ_w from W_F to ${}^L \tilde{T}^\circ$ by $\phi_w(\gamma) := \tilde{\phi}(\gamma)\tilde{w}\tilde{\phi}(\gamma)^{-1}\tilde{w}^{-1}$, where $\gamma \in W_F$.

Remark. (1) Note that the definition of ϕ_w is slightly different than $\langle w, -\rangle_\phi$; however, if $w \in W_\lambda$, then they are equal.

(2) ϕ_w is well-defined, *i.e.*, independent of the choices of $\tilde{w}$ and $\tilde{\phi}$ (since the different choices of $\tilde{w}$ and $\tilde{\phi}$ differ only by the elements of the abelian groups ${}^L \tilde{T}^\circ$ and $\pi_1({}^L G^\circ)$, respectively.)

4.2 An Inductive Lemma

The key connection between the group side and the dual group side is the following lemma.

Lemma 4.1 *Let α be a simple root and λ a character of T. Define a character λ_{s_α} of T_{ad} by $\lambda_{s_\alpha}(\bar{t}) = \lambda \circ \alpha^\vee(\overline{\alpha}(\bar{t}))$, where $\bar{t} \in T_{\mathrm{ad}}$, $\alpha^\vee$ the coroot of $\mathbf{G}$, and $\overline{\alpha}$ the root of $\mathbf{G}_{\mathrm{ad}}$. Define ϕ_{s_α} as in Section* 4.1*. Then ϕ_{s_α} is the Langlands dual of λ_{s_α},* i.e., $\mathfrak{L}(\lambda_{s_\alpha}) = \phi_{s_\alpha}$.

Proof We can write ϕ_{s_α} in a different way:

$$\phi_{s_\alpha}(\gamma) = \tilde{\phi}(\gamma)\tilde{s}_\alpha\tilde{\phi}(\gamma)^{-1}\tilde{s}_\alpha^{-1} = \tilde{\phi}(\gamma)s_\alpha(\tilde{\phi}(\gamma))^{-1}.$$

We know $s_\alpha(\tilde t) = \tilde t(\tilde\alpha \circ \tilde\alpha^\vee(\tilde t))^{-1}$, where $\tilde\alpha^\vee$ is the root of ${}^L\tilde T^\circ$, $\tilde\alpha$ is the coroot of ${}^L\tilde T^\circ$, and $\tilde\alpha^\vee \circ \tilde\phi = \alpha^\vee \circ \phi$. We get

$$\phi_{s_\alpha} = \tilde\phi(\tilde\phi(\tilde\alpha \circ \tilde\alpha^\vee \circ \tilde\phi)^{-1})^{-1} = \tilde\alpha \circ (\tilde\alpha^\vee \circ \tilde\phi) = \tilde\alpha \circ (\alpha^\vee \circ \phi).$$

We note a result in [8]: $\lambda \circ \alpha^\vee = (\alpha^\vee \circ \phi)^{-1}$. Therefore, $\phi_{s_\alpha} = \tilde\alpha \circ (\lambda \circ \alpha^\vee)^{-1}$. On the other hand, $\lambda_{s_\alpha} = (\lambda \circ \alpha^\vee) \circ \overline{\alpha}$. According to the Langlands correspondence in Section 3.2, ϕ_{s_α} is the Langlands dual of λ_{s_α}. ■

4.3 A Duality Theorem

Recall from Section 2.3 that $\langle\ ,\ \rangle_\lambda$ is a pairing from $W \times T_{\mathrm{ad}}$ to $\mathbb{C}^*$ as follows

$$\langle w, \overline{t}\rangle_\lambda = \prod_{i=1}^{n} (\lambda_i)_{\alpha_i}\left(\overline{\alpha}_i(\overline{t})\right).$$

Here $w \in R_\lambda, \overline{t} \in T_{\mathrm{ad}}$, and we factorize $w = s_{\alpha_n} s_{\alpha_{n-1}} \cdots s_{\alpha_1}$ in a reduced expression, where α_i are simple roots. For $i \geq 2$ (resp., $i = 1$), define $w_i' = s_{\alpha_{i-1}} \cdots s_{\alpha_1}$ (resp., w_1' is the identity), $\lambda_i = w_i'\lambda$ and $(\lambda_i)_{\alpha_i} = \lambda_i \circ \alpha_i^\vee$. For all $w \in W$, define a character λ_w of T_{ad} by $\lambda_w(\overline{t}) = \langle w, \overline{t}\rangle_\lambda$, $\overline{t} \in T_{\mathrm{ad}}$.

Theorem 4.2 *For all $w \in W$, the Langlands parameter ϕ_w of ${}^L\tilde T^\circ$ is the Langlands dual of the character λ_w of T_{ad}.*

Proof We prove it by induction on the length $l(w)$ of w. Lemma 4.2 takes care of the case of $l(w) = 1$. Assume it is true for $l(w') = n-1$, *i.e.,* for $w' = s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1}$ of length $n-1$, $n > 1$, we have $\mathfrak{L}(\lambda_{w'}) = \tilde\phi \tilde w' \tilde\phi^{-1} (\tilde w')^{-1}$. Let $l(w) = n$ and $w = s_{\alpha_n} s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1} = s_{\alpha_n} w'$, where $w' = s_{\alpha_{n-1}} s_{\alpha_{n-2}} \cdots s_{\alpha_1}$. Note that $\lambda_n = w'\lambda$ and $\mathcal{L}(\lambda_n) = w'(\phi)$.

We can choose $\tilde w' \tilde\phi (\tilde w')^{-1}$ as the lifting $w'(\phi)^\sim$ of $w'(\phi)$ and $\tilde w = \tilde s_{\alpha_n} \tilde w'$ as a representative of w. Then

$$\begin{aligned}
\mathfrak{L}(\lambda_w) &= \mathfrak{L}(\lambda_{w'})\mathfrak{L}((\lambda_n)_{\alpha_n}) \\
&= [\tilde\phi \tilde w' \tilde\phi^{-1} (\tilde w')^{-1}] \cdot [(w'(\phi)^\sim) \tilde s_{\alpha_n} (w'(\phi)^\sim)^{-1} \tilde s_{\alpha_n}^{-1}] \\
&= \tilde\phi (\tilde w' \tilde\phi (\tilde w')^{-1})^{-1} w'(\phi)^\sim \tilde s_{\alpha_n} (w'(\phi)^\sim)^{-1} \tilde s_{\alpha_n}^{-1} \\
&= \tilde\phi \tilde s_{\alpha_n} (\tilde w' \tilde\phi (\tilde w')^{-1}) \tilde s_{\alpha_n}^{-1} = \tilde\phi \tilde w \tilde\phi^{-1} \tilde w^{-1} \\
&= \phi_w.
\end{aligned}$$ ■

4.4 The Langlands Correspondence and the Tate Duality

The following lemma gives us a link between the Langlands correspondence and the Tate duality. We postpone the proof to the final section.

Lemma 4.3 *Let ξ be a character of T_{ad} whose kernel contains $p(T)$. Then ξ induces a character of $T_{\mathrm{ad}}/p(T)$, which is isomorphic to $H^1(\Gamma, Z(\overline{G}))$ (*cf.* Section 2.2). By the*

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Tate duality, it can be associated with an element of $H^1(\Gamma, \pi_1({}^LG^\circ))$. Since G is split, the Galois action is trivial. Therefore, we have a homomorphism ϕ

$$\xi \leftrightsquigarrow \phi\colon \Gamma \longrightarrow \pi_1({}^LG^\circ) \hookrightarrow {}^L\tilde{T}^\circ.$$

Then the restriction $\phi|_{W_F}$ of ϕ to the Weil group W_F is the same as the Langlands parameter $\mathfrak{L}(\xi)$ of ξ.

Proof See Section 5.3. ∎

4.5 The Proof of the Main Theorem

By Lemma 4.3 above, we have that the functor $\mathfrak{L}$ is the same isomorphism induced by the Tate duality. Theorem 4.2 tells us that $\mathfrak{L}(\lambda_w)$ is equal to ϕ_w. Now everything for λ_w can be translated in terms of ϕ_w. Then the theorem follows from the definition of $\varrho^\vee$ in Section 3.4. Now the proof is complete.

5 A Cohomological Lemma

In this section, we will give a proof of Lemma 4.3

5.1 The Langlands Correspondence on Arbitrary Tori

We have the following lemma from the elementary cohomology theory.

Lemma 5.1 (Kottwitz [9, Lemma 2.2]) *Let M be a Γ-module that is finitely generated as an abelian group. Let A be the diagonalizable $\mathbb{C}$-group $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{C}^*)$. Then $X(A) = M$ and*

$$\mathrm{Ext}^n_\Gamma(M, \mathbb{Z}) = \begin{cases} X(A)^\Gamma & n = 0, \\ \pi_0(A^\Gamma) & n = 1, \\ H^{n-1}(\Gamma, A) & n \geq 2. \end{cases}$$

Furthermore, if M is torsion free, then $H^n(\Gamma, X(A)) = \mathrm{Ext}^n_\Gamma(M, \mathbb{Z})$.

Using the previous lemma and the Tate duality, we get the following consequence of local dualities.

Proposition 5.2 (Kottwitz [9, 3.3.2] and Langlands [11, Theorem 1]) *The cohomology group $H^1(\Gamma, {}^LT^\circ)$ is canonically isomorphic to the group of continuous characters of finite order of T. This is a part of the Langlands correspondence on T.*

5.2 A Cohomological Lemma

There is a D-functor $*^D$ which induces an anti-equivalence between Galois modules whose underlying groups are finitely generated abelian groups without q-torsion and diagonalizable groups, where q is the characteristic of the ground field, which is 0 in our case (see [16, §13.1.5]).

Let $\mathbf{T} \to \mathbf{T}_{\mathrm{ad}}$ be an isogeny of split tori over F, and $\mathbf{Z}$ the kernel. The exact sequence

$$1 \longrightarrow \overline{Z} \longrightarrow \overline{T} \longrightarrow \overline{T}_{\mathrm{ad}} \longrightarrow 1$$

induces the following exact sequence, by the D-functor $*^D$

$$1 \longrightarrow \overline{Z}^D \longrightarrow \overline{T}^D \longrightarrow \overline{T}_{\mathrm{ad}}^D \longrightarrow 1.$$

By the Pontryagin duality, this gives an exact sequence

$$1 \longrightarrow \mathrm{Hom}(\overline{T}^D, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\overline{T}_{\mathrm{ad}}^D, \mathbb{Z}) \longrightarrow \mathrm{Ext}^1(\overline{T}^D, \mathbb{Z}) \longrightarrow 1.$$

Applying D-functor again, we have an exact sequence of tori over $\mathbb{C}$

$$1 \longrightarrow Z^\vee \longrightarrow {}^L\tilde{T}^\circ \longrightarrow {}^LT^\circ \longrightarrow 1,$$

where the group $Z^\vee = \mathrm{Hom}(\mathrm{Ext}^1(\overline{Z}^D, \mathbb{Z}), \mathbb{C}^*)$ and we can identify $Z^\vee$ with $\mathbf{Z}^D(\overline{F}) = \overline{Z}^D$ through the exact sequence $1 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \to 1$.

Now we consider the exact sequence

$$1 \longrightarrow Z \longrightarrow T \longrightarrow T_{\mathrm{ad}} \longrightarrow H^1(\Gamma, \overline{Z}) \longrightarrow 1.$$

It induces

$$1 \longrightarrow \mathrm{Hom}(H^1(\Gamma, \overline{Z}), \mathbb{C}^*) \longrightarrow \mathrm{Hom}(T_{\mathrm{ad}}, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(T, \mathbb{C}^*).$$

We use the Tate duality to rewrite the first term, and the Langlands correspondence to rewrite the second and third terms, and then we get

$$(\dagger) \qquad 1 \longrightarrow H^1(\Gamma, \overline{Z}^D) \longrightarrow H^1(W_F, {}^L\tilde{T}^\circ) \longrightarrow H^1(W_F, {}^LT^\circ).$$

The following lemma gives us a link between the Tate duality and the Langlands correspondence. Here we thank Jiu-Kang Yu for letting us use his formulation and proof.

Lemma 5.3 *After we identify $Z^\vee$ with $\overline{Z}^D$, the exact sequence $(\dagger)$ is compatible with a natural one obtained by applying $H^1(W_F, -)$ to $1 \to Z^\vee \to {}^L\tilde{T}^\circ \to {}^LT^\circ \to 1$.*

Proof By Proposition 5.2, it is harmless to replace W_F by Γ and $\mathrm{Hom}(T, \mathbb{C}^*)$ by $\mathrm{Hom}(T, \mathbb{Q}/\mathbb{Z})$. If we do the replacement and use Lemma 5.1, the exact sequences that we would like to analyze are

$$H^1(\Gamma, \overline{Z}^D) \xrightarrow{\delta} H^2(\Gamma, \overline{T}_{\mathrm{ad}}^D) \longrightarrow H^2(\Gamma, \overline{T}^D),$$

$$H^1(\Gamma, \overline{Z}) \xleftarrow{\delta} H^0(\Gamma, \overline{T}_{\mathrm{ad}}) \longleftarrow H^0(\Gamma, \overline{T}).$$

The terms in these two exact sequences are related pairwise by the Tate duality, which is just the cup product with values in $H^2(\Gamma, \overline{F}^*) = \mathbb{Q}/\mathbb{Z}$. Then the lemma comes down to whether the connecting homomorphism commutes with the cup product pairing, *i.e.*, the identity $\langle \delta a, b\rangle = \langle a, \delta b\rangle$, for all $a \in H^1(\Gamma, \overline{Z}^D)$ and $b \in H^0(\Gamma, \overline{T}_{\mathrm{ad}})$. This identity is indeed a general fact in group cohomology; it follows from the claim below. Therefore we are done. ■

Claim 5.4 *Assume that we have two exact sequences of Γ-modules*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \quad \textit{and} \quad 0 \longleftarrow A' \xleftarrow{f'} B' \xleftarrow{g'} C' \longrightarrow 0.$$

Furthermore, we assume that there is another Γ-module E and pairings of Γ-modules $A \times A' \to E$, $B \times B' \to E$, and $C \times C' \to E$ which are compatible with the exact sequences in the sense that

$$\langle fa, b' \rangle = \langle a, f'b' \rangle \quad \textit{and} \quad \langle gb, c' \rangle = \langle b, g'c' \rangle,$$

for all $a \in A, b' \in B', b \in B$, and $c' \in C'$. Then $\langle \delta\gamma, \alpha' \rangle = (-1)^{i+1} \langle \gamma, \delta' \alpha' \rangle$ in $H^{i+j+1}(\Gamma, E)$ for all $\gamma \in H^i(\Gamma, C)$ and $\alpha' \in H^j(\Gamma, A')$.

Proof We represent γ by a cochain c. Then $c = gb$ for some b and $\delta\gamma$ is represented by a cochain a such that $fa = d_B b$. Similarly, we represent α' by a cochain a'. Then $a' = f'b'$ for some b', and $\delta'\alpha'$ is represented by a cochain c' such that $g'c' = d_{B'} b'$. Now we have

$$\begin{aligned} \langle \delta\gamma, \alpha' \rangle &= \langle a, f'b' \rangle = \langle fa, b' \rangle = \langle d_B b, b' \rangle, && \text{for all } a \in A, b' \in B', \\ \langle \gamma, \delta\alpha' \rangle &= \langle gb, c' \rangle = \langle b, g'c' \rangle = \langle b, d_{B'} b' \rangle, && \text{for all } b \in B, c' \in C'. \end{aligned}$$

However, the well-known identity of cochains $d_E \langle b, b' \rangle = \langle d_B b, b' \rangle + (-1)^i \langle b, d_{B'} b' \rangle$ shows that $\langle b, d_{B'} b' \rangle = (-1)^{i+1} \langle d_B b, b' \rangle$ as cohomology classes. ∎

5.3 A Proof of Lemma 4.3

The short exact sequence $1 \longrightarrow Z(\overline{G}) \longrightarrow \overline{T} \longrightarrow \overline{T}_{\mathrm{ad}} \longrightarrow 1$ gives rise to a long exact sequence

$$1 \longrightarrow Z(\overline{G})^\Gamma \longrightarrow T \xrightarrow{p} T_{\mathrm{ad}} \longrightarrow H^1(\Gamma, Z(\overline{G})) \xrightarrow{\delta} H^1(\Gamma, T)(=1) \longrightarrow \cdots,$$

which induces

$$1 \longrightarrow \mathrm{Hom}(H^1(\Gamma, Z(\overline{G})), \mathbb{C}^*) \longrightarrow \mathrm{Hom}(T_{\mathrm{ad}}, \mathbb{C}^*) \longrightarrow \mathrm{Hom}(T, \mathbb{C}^*) \longrightarrow \cdots.$$

The map ξ in $\mathrm{Hom}(T_{\mathrm{ad}}, \mathbb{C}^*)$, which is trivial on $p(T)$, is from $\mathrm{Hom}(H^1(\Gamma, Z(\overline{G})), \mathbb{C}^*)$.

By Lemma 5.3, we have the following commutative diagram

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Hom}(H^1(\Gamma, Z(\overline{G})), \mathbb{C}^*) \arrow[r] \arrow[d, "\text{Tate duality}"'] & \mathrm{Hom}(T_{\mathrm{ad}}, \mathbb{C}^*) \arrow[r] \arrow[d, "\mathfrak{L}"'] & \mathrm{Hom}(T, \mathbb{C}^*) \arrow[d, "\mathfrak{L}"'] \\
1 \arrow[r] & H^1(W_F, \pi_1({}^L G^\circ)) \arrow[r] & H^1(W_F, {}^L \tilde{T}^\circ) \arrow[r] & H^1(W_F, {}^L T^\circ).
\end{tikzcd}$$

Here we use the Tate duality and Proposition 5.2 for the the first map (*cf.* Lemma 3.3) and the Langlands correspondence for second and third maps. Now our lemma follows from the first commutative square. ∎

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