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LEVITZKI RADICAL FOR CERTAIN VARIETIES

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Let A be a nonassociative algebra. We let A^n denote the subalgebra generated by all products of n elements from A. Inductively we define $A^{(0)} = A$ and $A^{(n+1)} = (A^{(n)})^2$. We say that A is *nilpotent* if, for some n, $A^n = \{0\}$. A is *solvable* if $A^{(n)} = \{0\}$ for some n. An algebra is *locally nilpotent* (*locally solvable*) if each finitely generated subalgebra is nilpotent (solvable). In this paper $\mathscr{V}$ will always be some variety of algebras defined by a set of homogeneous identities. We say that local nilpotence is a *radical property in* $\mathscr{V}$ if each $A \in \mathscr{V}$ contains a maximal locally nilpotent ideal L, and A/L has no non-zero locally nilpotent ideals. The ideal L is then called the *Levitzki radical* of A.

The symbols L_x and R_x will be used to denote the left and right multiplication operators on A $(a \rightarrow xa$, and $a \rightarrow ax)$.The letter S will be used to mean either L or R. Thus, S_xS_y could have four meanings. If B is any subset of an algebra A we let B^* be the algebra generated by all maps (on A) of the form L_b or R_b where $b \in B$.

Our purpose is to show that local nilpotence is a radical property in some particular varieties. Let us agree that a variety has property $*$ if

*For each finitely generated algebra $A \in \mathscr{V}$ there exists a natural number m such that $A^m \subseteq A^{(2)}$.

LEMMA 1. (Anderson) *Let* $\mathscr{V}$ *be a variety with property* $*$. *Then if* B *is any finitely generated algebra in* $\mathscr{V}$, $B^{(n)}$ *is also finitely generated for each* n.

Proof. See [**1**, p. 30].

LEMMA 2. *Let* $\mathscr{V}$ *be a variety with property* $*$. *Then for each finitely generated* $A \in \mathscr{V}$ *and for each integer* $n \geqq 1$, *there exists an integer* $p = p(A, n)$ *such that* $A^p \subseteq A^{(n)}$.

Proof. Consider the statement: For each finitely generated $B \in \mathscr{V}$ there is an integer $p = p(B)$ such that $B^p \subseteq B^{(n)}$. We will show by induction that the statement is valid for each n. The statement holds trivially for $n = 1$, and by property $*$ it is true for $n = 2$. We may assume then that it holds for some $n \geqq 3$. Let A be finitely generated. By Lemma 1 A^2 is also finitely generated. By our induction assumption we must have a number k such that $(A^2)^k \subseteq (A^2)^{(n)} = A^{(n+1)}$. Since A is finitely generated we may consider a finitely generated free algebra $F \in \mathscr{V}$ having A as a homomorphic image. By property $*$ there is an m such that $F^m \subseteq F^2F^2$. Let $p = 2^{k-2}m$. We will show $A^p \subseteq A^{(n+1)}$

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which will complete the induction. Let $x_1, x_2, \ldots, x_p \in A$ and let $x = x_1x_2 \ldots x_p$ be some product. Since $p \geqq m$ we have $x = \sum (ab)(cd)$ where a, b, c and d are words in the x_i's with $\deg(ab)(cd) = p$. Define $u_1 = ab$ or cd depending on which has the larger degree. (If the degrees are the same choose either one). We may write $x = \sum u_1S_{v_1}$ where $S = L$ or R, $\deg v_1 \geqq 2$, and $\deg u_1 \geqq p/2 = 2^{k-3}m \geqq m$. We may repeat this process to the term u_1. Continuing, we arrive at $x = \sum u_iS_{v_i}S_{v_{i-1}} \ldots S_{v_1}$ where each v_j has degree at least 2, and $\deg u_i \geqq 2^{k-i-2}m$. We continue this as long as $i \leqq k - 2$, for then $\deg u_i \geqq m$. When $i = k - 2$ we shall have $x = \sum u_{k-2}S_{v_{k-2}} \ldots S_{v_1}$ where $\deg u_{k-2} \geqq m$. A final application shows that

$$x = \sum u_{k-1}S_{v_{k-1}} \ldots S_{v_1} \in (A^2)^k \subseteq A^{(n+1)}.$$

The conclusions of the following theorem are the same as those of Theorems 1 and 2 in [**3**]. However, our hypotheses are weaker: we do not assume that if $A \in \mathscr{V}$ then $A^{(n)}$ is an ideal of A.

THEOREM 1. *Let $\mathscr{V}$ be a variety with property $*$. Then each finitely generated solvable algebra is nilpotent. Furthermore, local nilpotence is a radical property in $\mathscr{V}$.*

Proof. The first claim follows directly from Lemma 2. Using Lemma 1 we may mimic Lemma 70 in [**2**] to show that if I is an ideal of A then the local solvability of A follows from I and A/I being locally solvable. The proof of Theorem 49 [**2**] may be followed verbatim to show that each $A \in \mathscr{V}$ has a Levitzki radical.

Let $s \geqq 2$ be a natural number. $\mathscr{V}$ is called an *s-variety* if whenever J is an ideal in $A \in \mathscr{V}$, J^s is also an ideal in A.

COROLLARY. *An s-variety $\mathscr{V}$ has property $*$ if and only if local nilpotence is a radical property in $\mathscr{V}$.*

Proof. One direction of the statement is contained in Theorem 1. On the other hand assume $\mathscr{V}$ is an s-variety in which local nilpotence is a radical property. Let $A \in \mathscr{V}$ be finitely generated. Consider $\bar{A} = A/(A^s)^s$. It can be shown that $\bar{A}$ is solvable. As it is observed in [**1**, p. 31] whenever local nilpotence is a radical property in a variety, solvability implies local nilpotence. Then $\bar{A}$ must be locally nilpotent. Since $\bar{A}$ is finitely generated it must be nilpotent. Therefore there is an m such that $A^m \subseteq (A^s)^s \subseteq A^{(2)}$. $\mathscr{V}$ has property $*$.

Generalized alternative algebras I. In this section we will concern ourselves with a variety of algebras which satisfy the following identities:

(1) $([w, x], y, z) + (w, x, yz) = y(w, x, z) + (w, x, y)z$

(2) $(wx, y, z) + (w, x, [y, z]) = w(x, y, z) + (w, y, z)x$

(3) $(x, x, x) = 0.$

Such algebras were studied in [**5**, **6**] under the name *generalized alternative algebras I*. For convenience let us call algebras satisfying (1)–(3) GAI algebras. We mention that alternative algebras are GAI algebras. However, the variety of GAI algebras does not seem to be an s-variety. We will always assume that our algebras have characteristic $\neq 2$. A will denote a GAI algebra and B will denote the ideal A^2. We will use $\langle B^3 \rangle$ to mean the ideal generated by B^3. For elements $x, y, z \in A$ we define $J(x, y, z)$ to be the linear span in A^* of the set

$$\{S_{(\pi x \pi y)\pi z}, S_{\pi x(\pi y \pi z)}, S_{\pi x \pi y} S_{\pi z}, S_{\pi x} S_{\pi y \pi z}\},$$

where $S = L$ or R and π is a permutation of x, y, z. By expanding (1) it follows that

$$(4) \qquad L_a L_b L_c \equiv L_b L_c L_a \bmod J(a, b, c)$$

$$(5) \qquad R_a L_b L_c \equiv L_b L_c R_a \bmod J(a, b, c)$$

$$(6) \qquad R_a L_b R_c \equiv R_b R_a R_c + L_b R_c L_a - R_c L_b L_a \bmod J(a, b, c).$$

A linearization of (3) gives

$$(7) \qquad 0 = L_{ab} - L_b L_a + L_a R_b - R_b L_a + L_b R_a - R_a L_b + L_{ba} - L_a L_b + R_a R_b - R_{ab} + R_b R_a - R_{ba}.$$

We shall make use of the fact that identities (1)–(3) are "symmetric". Therefore if an identity holds in A^*, the L's and R's may be interchanged to produce another identity. The next lemma shows how a subscript may be moved to the left.

LEMMA 3. *Each word of the form $S_a S_b S_c$ may be rewritten as a sum of words each of the form $\pm S_b S_a S_c$ or $\pm S_b S_c S_a$ modulo $J(a, b, c)$.*

Proof. The eight cases to consider are: 1. $L_a L_b L_c$, 2. $R_a R_b R_c$, 3. $R_a L_b L_c$, 4. $L_a R_b R_c$, 5. $R_a L_b R_c$, 6. $L_a R_b L_c$, 7. $L_a L_b R_c$, and 8. $R_a R_b L_c$. Cases 1 and 3 are valid by relations (4) and (5). By symmetry cases 2 and 4 are also valid. Case 5 is handled as follows: $R_a L_b R_c$ can be replaced by the right hand side of (6) which contains three terms. The first two of these are of the desired form. The last term, $-R_c L_b L_a$, may be transformed to $-L_b L_a R_c$ by (5). Case 6 then follows by symmetry. Next we deal with case 7. By multiplying (7) on the right by R_c we get

$$L_a L_b R_c \equiv -L_b L_a R_c - R_b L_a R_c + L_b R_a R_c + R_b R_a R_c + L_a R_b R_c - R_a L_b R_c + R_a R_b R_c.$$

The last three terms are then transformed using cases 4, 5, and 2 respectively. By symmetry we also have case 8 and the proof is complete.

LEMMA 4. *Assume A has characteristic $\neq 2$. Then each map of the form $S_a S_a S_a$ is in $J(a, a, a)$.*

Proof. The following identities are valid in A (see equations 1.10, 1.7, and 1.8 respectively in [**5**]):

(8) $(a^2, a, b) = 2(a, a, ba)$

(9) $(a, a, ba) = (a, a, b)a$

(10) $(a, ab, a) = a(a, b, a)$.

By characteristic $\neq 2$, (8) implies $R_aL_aL_a \in J(a, a, a)$. Now (9) and symmetry imply $L_aR_aR_a$, $L_aL_aR_a$, $R_aR_aL_a \in J(a, a, a)$. Now (10) implies $L_aR_aL_a \in J(a, a, a)$ and so $R_aL_aR_a$ is in $J(a, a, a)$ as well. In (7) we let $a = b$ and multiply the equation by L_a. It follows that $L_aL_aL_a \in J(a, a, a)$. Symmetry now shows $R_aR_aR_a \in J(a, a, a)$ completing the proof.

LEMMA 5. *Any word of the form $S_aS_{x1} \dots S_{xk}S_a$ may be written as a sum of words each of the form $\pm S_{y_1}S_{y_2} \dots S_{yk}S_aS_a$ plus terms which contain factors from B^*.*

Proof. We induct on k, the case for $k = 0$ being trivial. When $k = 1$ the result follows by setting $a = c$ in Lemma 3. Now assume the result for all values less than k where $k \geqq 2$. If $T = S_aS_{x_1}S_{x_2} \dots S_{xk}S_a$ we may apply Lemma 3 to rewrite T as a sum of terms each of the form $\pm S_{x_1}S_aS_{x_2} \dots S_{xk}S_a$ or $\pm S_{x_1}S_{x_2}S_a \dots S_{xk}S_a$ plus terms involving factors from B^*. By our induction assumption we are done.

LEMMA 6. *Let A have characteristic $\neq 2$ and be generated by the set $\{x_i\}_{i=1}^n$. Then every product of $2n + 1$ S's is a sum of terms each containing a factor $S_b \in B^*$.*

Proof. Let $T = S_{a_1}S_{a_2} \dots S_{a_{2n+1}}$. Obviously we may assume each a_i is one of the generators. Then there is some $a \in \{x_i\}_{i=1}^n$ which must occur at least three times in $a_1, a_2, \dots, a_{2n+1}$. Hence T contains a subword of the form $S_a \dots S_a \dots S_a$. We may apply Lemma 5 to rewrite this as $\sum \pm S_a \dots S_aS_a$ modulo terms with factors from B^*. Another application of Lemma 5 shows that each $S_a \dots S_aS_a$ may be written as $\sum \pm S \dots S_aS_aS_a$ modulo the extra terms. By Lemma 4 each of these last terms is a sum of terms each containing a factor from B^*. This completes the proof.

LEMMA 7. *Let $T = S_{a_1}S_{a_2} \dots S_{a_k}$ where two of the a_i's are elements of B. Then $BT \subseteq \langle B^3 \rangle$.*

Proof. We first make some observations. It is shown in [**6**, p. 144] that $(B^2A)B$, $B(AB^2) \subseteq B^3$. By (1) we get that

$$(w(b_1b_2))b_3 = -(w, b_1, b_2)b_3 + ((wb_1)b_2)b_3 = b_2(w, b_1, b_3) - ([w, b_1], b_2, b_3) - (w, b_1, b_2b_3) + ((wb_1)b_2)b_3.$$

This shows that $(AB^2)B \subseteq B(AB^2) + \langle B^3 \rangle \subseteq \langle B^3 \rangle$. By symmetry we also get $B(B^2A) \subseteq \langle B^3 \rangle$.

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Since both B and $\langle B^3 \rangle$ are ideals it suffices to prove the lemma for the case where a_1 and a_k are the elements of B. Assume then that

$$T = S_b S_{a_2} S_{a_3} \ldots S_{b'}$$

where $b, b' \in B$. We shall induct on k, the number of S's appearing. When $k = 2$, $T = S_b S_{b'}$ and the result is obvious. Let $k = 3$ so that $T = S_b S_{a_2} S_{b'}$. By Lemma 3

$$T = \sum \pm S_{a_2} S_b S_{b'} + \sum + S_{a_2} S_{b'} S_b$$

plus terms in $J(b, a_2, b')$. It is clear that the terms $\pm S_{a_2} S_b S_{b'}$, $\pm S_{a_2} S_{b'} S_b$ all map B into $\langle B^3 \rangle$. The above paragraph shows that any term in $J(b, a_2, b')$ will map B into $\langle B^3 \rangle$. Now assume $k > 3$ and that the lemma holds for each value smaller than k. We have

$$T = S_b S_{a_2} S_{a_3} \ldots S_{b'}.$$

Applying Lemma 3 to the first 3 factors in T shows

$$BT \subseteq BS_{a_2} S_b S_{a_3} \ldots S_{b'} + BS_{a_2} S_{a_3} S_b \ldots S_{b'} + BW \ldots S_{b'}$$

where $W \in J(b, a_2, a_3)$. W is a sum of terms each having either 1 or 2 S's and at least one of the S's of the form $S_{b''}$, $b'' \in B$. By induction we are done.

THEOREM 2. *Let $\mathscr{V}$ be a variety of* GAI *algebras with characteristic $\neq 2$. Then local nilpotence is a radical property, and each finitely generated solvable $A \in \mathscr{V}$ is nilpotent.*

Proof. By Theorem 1 we need only show $\mathscr{V}$ has property $*$. So let A be any finitely generated algebra in $\mathscr{V}$, generated by $\{x_i\}_{i=1}^n$. By Lemma 6 each product of $2n + 1$ S's is a sum of terms with each term having a factor from B^*. Therefore each product of $4n + 2$ S's must be a sum with each term containing 2 factors from B^*. By Lemma 7 we have any product of $4n + 2$ S's maps B into $\langle B^3 \rangle$. This implies any product of $4n + 3$ S's maps A into $\langle B^3 \rangle$. By Lemma 7 in [**6**] we have $\langle B^3 \rangle \subseteq B^2 = A^{(2)}$. Setting $p = 4n + 3$ we have $A(A^*)^p \subseteq A^{(2)}$. Thus $A^{2p} \subseteq A^{(2)}$ completing the proof.

As a corollary we may generalize Theorem 1.3 in [**5**].

COROLLARY. *Let A be a* GAI *nilalgebra with characteristic $\neq 2, 3$. If A has the ascending chain condition on subalgebras then A is nilpotent.*

Proof. A is finitely generated. By Theorem 2 we need to show A is solvable. Letting J be the maximal solvable ideal in A, a result of [**4**] implies A/J is alternative. Since A/J is alternative, nil and has ACC on right ideals a result of [**9**] shows A/J is nilpotent. Hence A is solvable.

Algebras with $(a, b, c) = (c, a, b)$**.** In this section we consider another generalization of the alternative law by studying algebras of characteristic $\neq 2, 3$ which satisfy

(11) $(a, b, c) = (c, a, b)$.

From [**7**, eq. 4] we know such algebras satisfy

(12) $(((a, x, x), x, x), x, x) = (a, (x, x, x), (x, x, x))/9$.

THEOREM 3. *Let* $\mathscr{V}$ *be a variety of algebras satisfying* (11) *having characteristic* $\neq 2, 3$. *Then local nilpotence is a radical property in* $\mathscr{V}$ *and the finitely generated solvable algebras are nilpotent.*

Proof. We show $\mathscr{V}$ has a property $*$. Let A be finitely generated. $\mathscr{V}$ is a 2-variety [**8**]. Since $A^{(2)}$ is an ideal, by passing to the quotient ring it is sufficient to show that $(ab)(cd) = 0$ implies A is nilpotent. We assume then that A is generated by $\{x_i\}_{i=1}^n$ and $A^{(2)} = \{0\}$. Letting $B = A^2$, we claim any map of the form $T = S_{a_1}S_{a_2} \ldots S_{a_{11n+1}}$ is a sum of terms each containing a factor $S_b, b \in B$. To show this we may assume each a_i is a generator. Hence there is some x_j which occurs at least 12 times in the sequence $a_1, a_2, \ldots, a_{11n+1}$. Identity (11) easily implies the following relations, where $\equiv$ means "congruent modulo terms with factors from B^*":

$$\text{(i) } L_xL_y \equiv -R_yR_x, \text{ (ii) } R_xR_y \equiv -L_yL_x, \text{ (iii) } R_xL_y \equiv L_yR_x - R_xR_y \equiv L_yR_x + L_yL_x, \text{ (iv) } L_xR_y \equiv R_yL_x + R_yR_x.$$

From these relations it follows that we may "move" the factors S_{x_j} to the left to get

$$T = \sum S^1_{x_j}S^2_{x_j} \ldots S^{12}_{x_j}S \ldots S + \text{terms containing a } B^* \text{ factor.}$$

There are either at least 6 factors L_{x_j} or at least 6 factors R_{x_j} in the subword $S_{x_j}S_{x_j} \ldots S_{x_j}$. Without loss of generality assume R_{x_j} occurs 6 times. Using (iv) we see that

$$L_{x_j}R_{x_j} \equiv R_{x_j}L_{x_j} + R_{x_j}R_{x_j}.$$

Therefore we may move the R_{x_j}'s to the left and get

$$T = \sum R^6_{x_j}S \ldots S + \text{terms with } B^* \text{ factors.}$$

However (12) shows that even $R^6_{x_j}$ is a sum of terms with factors from B^*. We have shown then that every product of $11n + 1$ S's is a sum with each term having a factor from B^*. Since $A^{(2)} = \{0\}$ each product of $11n + 2$ S's is the zero map. Therefore A^* is nilpotent and A must also be nilpotent.

Sterling has shown that the semiprime algebras satisfying (11) are alternative. Using this fact and the same proof as in our previous corollary we generalize Theorem 2 in [**8**].

COROLLARY. *Let A be a nilalgebra satisfying* (11) *having characteristic* $\neq 2, 3$. *If A has the ascending chain condition on subalgebras then A is nilpotent.*

REFERENCES

1. T. Anderson, *The Levitzki radical in varieties of algebras*, Math. Ann. *194* (1971), 27–34.
2. N. Divinsky, *Rings and radicals* (University of Toronto Press, 1965).
3. G. V. Dorofeev, *A locally nilpotent radical of nonassociative rings*, Algebra and Logic *10* (1971), 219–224.
4. I. R. Hentzel and G. M. P. Cattaneo, *Semi-prime generalized right alternative rings*, J. of Algebra *43* (1976), 14–27.
5. H. F. Smith, *The Wedderburn principal theorem for a generalization of alternative algebras*, Trans. Amer. Math. Soc. *198* (1974), 139–154.
6. ——— *The Wedderburn principal theorem for generalized alternative algebras I*, Trans. Amer. Math. Soc. *212* (1975), 139–148.
7. N. J. Sterling, *Rings satisfying* $(x, y, z) = (y, z, x)$, Can. J. Math. *20* (1968), 913–918.
8. D. J. Pokrass, *Some radical properties of rings with* $(a, b, c) = (c, a, b)$, Pacific J. of Math. *76* (1978), 479–483.
9. K. A. Zevlakov, *Nil-ideals of an alternative ring satisfying a maximal condition*, Algebra i Logika *6* (1967), 19–26.

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