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# ON PRIMITIVE ABUNDANT NUMBERS

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**Abstract**

Let $n$ be a natural number with largest component $s^d$. We prove that if $x\sigma(n) = yn + z$ ($x, y, z$ given positive integers), $n$ is not primitive $(y/x)$-abundant and $n/s$ is not $(y/x)$-perfect, then $n < 4(z + \frac{1}{2})^3/27y$ (if $z \geqslant 175$). All solutions are tabled for the equation $x\sigma(n) = yn + z$ when $x = 1$, $y \geqslant 2$, $1 \leqslant z \leqslant 210$, and $n$ is not primitive $y$-abundant. We also prove that if $n$ is primitive $(y/x)$-abundant, then $s^{3d} < (yn/2)^2$. A number of results are proved concerning the range of $\sigma(n)/n$ when $n$ is primitive $\alpha$-abundant, for any real number $\alpha \geqslant 1$. For example, then $\sigma(n)/n < \alpha + \min\{\frac{1}{2}, 3\alpha e^{-5\alpha/9}/2\}$ and $\sigma(n)/n < \alpha + 1.6\alpha/\log n$. All primitive abundant numbers $n$ with $\sigma(n)/n \geqslant 2.05$ are listed.

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## 1. Introduction

Let $n$ be a natural number, $\sigma(n)$ the sum of its positive divisors, $\alpha \geqslant 1$ a real number. We call $n$ $\alpha$-deficient, $\alpha$-perfect or $\alpha$-abundant, according as $\sigma(n) < \alpha n$, $\sigma(n) = \alpha n$ or $\sigma(n) > \alpha n$, respectively; $n$ is $\alpha$-nondeficient if $\sigma(n) \geqslant \alpha n$ and in this case we call $\alpha$ a coefficient of abundance of $n$. When $\alpha = 2$, the prefix will be omitted. An $\alpha$-nondeficient number is said to be primitive if all its proper divisors are $\alpha$-deficient.

The basic result of this paper is a lemma giving new necessary and sufficient conditions for an $\alpha$-abundant number to be primitive. We shall give many consequences of this lemma and shall also prove a theorem relating to a result of Pomerance (1975).

Pomerance showed that for given integers $y$ and $z \geqslant 0$, the equation

$$(1) \qquad \sigma(n) = yn + z$$

has only finitely many solutions, which are not primitive $y$-nondeficient, apart from some solutions of an explicit form. We shall consider instead the equation $x\sigma(n) = yn + z$, where $x$, $y$ and $z$ are positive integers, and will show that if $n$ is not primitive $(y/x)$-abundant then $n < 4(z + \frac{1}{2})^3/27y$ (when $z \geqslant 175$) apart from a possibility similar to that in Pomerance's result. Robbins (1980) found all non-primitive $y$-abundant $n$ satisfying (1) for $y = 2$ and $0 < z \leqslant 100$. We shall find all such $n$ for $y \geqslant 2$ and $0 < z \leqslant 210$.

A number of results will be given concerning the range of $\sigma(n)/n$ when $n$ is primitive $\alpha$-abundant. The main one is that then $\sigma(n)/n < \alpha + \min\{\frac{1}{2}, 3\alpha e^{-\rho\alpha}/2\}$, where $\rho = e^{-\gamma}$ and $\gamma$ is Euler's constant. We shall also show that the only limit point of the set $\{\sigma(n)/n\colon n \text{ primitive } \alpha\text{-abundant}\}$ is $\alpha$ and will list all primitive abundant numbers $n$ with $\sigma(n)/n \geqslant 2.05$.

All Roman letters in this paper (except $e$) denote nonnegative integers, with $xy > 0$, $n > 1$, and $p$, $q$, $r$, $s$ primes. We always use $p$ for the smallest prime factor of $n$ and $s^d$ for the largest component (or prime power) in $n$. As usual, $q^b \| n$ means that $q^b \mid n$ and $q^{b+1} \nmid n$; we allow $b = 0$. By $\omega(n)$, we mean the number of distinct prime factors of $n$.

## 2. The basic result

The "lemma" referred to in the Introduction is Theorem 1, below. We need

LEMMA 1. *If $m$ is a proper divisor of $n$, then $\sigma(m)/m < \sigma(n)/n$.*

PROOF. This is a consequence of the equation

$$\frac{\sigma(n)}{n} = \sum_{d \mid n} \frac{1}{d}.$$

From this follows the classical result that all $\alpha$-perfect numbers are primitive.

THEOREM 1. *Let $n$ be an $\alpha$-abundant number.* (i) *If*

$$\sigma(n) - \alpha n \leqslant \frac{p-1}{p} \cdot \frac{\alpha n}{s^d},$$

*then $n$ is primitive.* (ii) *If $n$ is primitive, then*

$$\sigma(n) - \alpha n \leqslant \frac{\alpha n}{s^d}.$$

PROOF. (i) Suppose $n$ is not primitive. Then $n$ has an $\alpha$-nondeficient proper divisor and, by virtue of Lemma 1, we may write

$$n = mq \quad \text{where } \sigma(m) \geqslant \alpha m.$$

Suppose $q^b \| n$. Then

$$\begin{aligned}\sigma(n) &= \sigma(m)\frac{\sigma(q^b)}{\sigma(q^{b-1})} \geqslant \alpha m \frac{\sigma(q^b)}{\sigma(q^{b-1})} \\ &= \alpha n + \alpha m\left(\frac{\sigma(q^b)}{\sigma(q^{b-1})} - q\right) \\ &= \alpha n + \frac{\alpha m}{\sigma(q^{b-1})}.\end{aligned}$$

Since

$$\sigma(q^{b-1}) = \frac{q^b - 1}{q - 1} < \frac{q}{q-1} \cdot q^{b-1} \leqslant \frac{p}{p-1} \cdot q^{b-1},$$

we thus have

$$\begin{aligned}\sigma(n) &> \alpha n + \alpha \cdot \frac{p-1}{p} \cdot \frac{m}{q^{b-1}} = \alpha n + \frac{p-1}{p} \cdot \frac{\alpha n}{q^b} \\ &\geqslant \alpha n + \frac{p-1}{p} \cdot \frac{\alpha n}{s^d},\end{aligned}$$

and the result is clear.

(ii) Suppose $\sigma(n) - \alpha n > \alpha n / s^d$. Then

$$\begin{aligned}\sigma\left(\frac{n}{s}\right) &= \sigma\left(\frac{n}{s^d}\right)\sigma(s^{d-1}) = \frac{\sigma(n)}{\sigma(s^d)}\sigma(s^{d-1}) \\ &> \left(\alpha n + \frac{\alpha n}{s^d}\right)\frac{\sigma(s^{d-1})}{\sigma(s^d)} = \frac{\alpha n}{s} \cdot \frac{s^d + 1}{s^{d-1}} \cdot \frac{\sigma(s^{d-1})}{\sigma(s^d)} \geqslant \frac{\alpha n}{s}.\end{aligned}$$

Thus $n$ has an $\alpha$-abundant proper divisor, contrary to hypothesis. This completes the proof.

REMARK 1. From this proof, we can also deduce the stronger result: *An $\alpha$-nondeficient number $n$ is primitive if and only if for each component $q^b$ of $n$*

$$\sigma(n) - \alpha n < \frac{\alpha n}{\sigma(q^b) - 1}.$$

For our applications, Theorem 1 is more convenient.

It follows from Theorem 1 that if $\sigma(n) = yn + 1$ $(y \geqslant 2)$, then $n$ is primitive $y$-abundant. (We need only observe that $\omega(n) \geqslant 2$.) For $y = 2$, this was first

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noticed by Cattaneo (1951). Our extension of the work of Robbins (1980), summarized in Table 2, will show that if $\sigma(n) = yn + z$ $(y \geqslant 2)$, then $n$ is primitive $y$-abundant for $z = 1, 2, 5, 6, 9, \ldots, 207, 209, 210$.

## 3. On integers with rational coefficient of abundance

We now give the main result of this paper.

THEOREM 2. *Suppose* $x\sigma(n) = yn + z$. *If* $n$ *is not primitive* $(y/x)$*-abundant and* $n/s$ *is not* $(y/x)$*-perfect, then*

$$n < \max\left\{\frac{2(z-1)^{5/2}}{y}, \frac{4\left(z+\frac{1}{2}\right)^3}{27y}\right\}.$$

(*The* max *is assumed by the second function when* $z \geqslant 175$.)

PROOF. Suppose $y = 1$ and, first, that $x = 1$. If $n$ is prime, then $n/s = 1$ is 1-perfect, so $n$ is composite. Then $n + z = \sigma(n) > n + \sqrt{n}$, so $n < z^2$. But $z^2 < 2(z-1)^{5/2}$ for $z \geqslant 3$. If $y = 1$ and $x \geqslant 2$, then $xn < x\sigma(n) = n + z$ so that $n < z/(x-1) \leqslant z$. We may assume now that $y \geqslant 2$.

Suppose there is a prime $q$ with $q \neq s$, $q^b \| n$ and $n/q$ not $(y/x)$-deficient. Then

$$\sigma(n) = \sigma\left(\frac{n}{q}\right)\frac{\sigma(q^b)}{\sigma(q^{b-1})} \geqslant \frac{yn}{xq}\left(q + \frac{1}{\sigma(q^{b-1})}\right) > \frac{yn}{x} + \frac{yn}{2xq^b} > \frac{yn}{x} + \frac{y\sqrt{n}}{2x},$$

where we use $n \geqslant q^b s^d > q^{2b}$ for the last inequality. Thus, since $z = x\sigma(n) - yn$, we have

$$n < \frac{4z^2}{y^2} \leqslant \frac{2z^2}{y}.$$

Note that $z^2 < (z-1)^{5/2}$ for $z \geqslant 5$. For $z \leqslant 4$,

$$4 \geqslant x\sigma(n) - yn = \left(x\sigma\left(\frac{n}{q}\right) - \frac{yn}{q}\right)q + x\sigma\left(\frac{n}{q^b}\right) \geqslant x\left(\frac{n}{q^b} + 1\right),$$

so that $x = 1$ and $n/q^b \leqslant 3$. Thus $s^d = 3$, $q^b = 2$ and $n = 6$. But 6 is not $y$-abundant for $y \geqslant 2$.

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So we may assume for each prime $q \mid n$ with $q \neq s$ that $n/q$ is $(y/x)$-deficient. Thus by hypothesis, $n/s$ is $(y/x)$-abundant. Then, setting $m = n/s^d$, we have

$$0 < x\sigma\left(\frac{n}{s}\right) - \frac{yn}{s} = x\sigma(n)\frac{\sigma(s^{d-1})}{\sigma(s^d)} - \frac{yn}{s} = \frac{x\sigma(n)}{\sigma(s^d)} \cdot \frac{\sigma(s^d) - 1}{s} - \frac{yn}{s}$$
$$= \frac{x\sigma(n) - yn}{s} - \frac{x\sigma(n)}{s\sigma(s^d)} = \frac{z - x\sigma(m)}{s}$$

so that $x\sigma(m) < z$. But from the equation $x\sigma(m)\sigma(s^d) = yms^d + z$, we obtain the equation

$$s^d(xs\sigma(m) - yms + ym) = x\sigma(m) + z(s-1),$$

so that

$$s^d \leqslant x\sigma(m) + z(s-1) < z + z(s-1) = zs.$$

Hence $s^{d-1} \leqslant z - 1$, so $s^d \leqslant (z-1)^{d/(d-1)}$. Since

$$\frac{y}{x} < \frac{\sigma(n)}{n} = \frac{\sigma(m)}{m} \cdot \frac{\sigma(s^d)}{s^d} < \frac{2\sigma(m)}{m},$$

we thus have

$$n = ms^d < \frac{2x}{y}\sigma(m)s^d \leqslant \frac{2}{y}(z-1)(z-1)^{d/(d-1)} \leqslant \frac{2(z-1)^{5/2}}{y}$$

for $d \geqslant 3$.

Suppose now that $d = 1$ or $2$, and note that

$$x\sigma(n) = x\sigma\left(\frac{n}{s}\right)\frac{\sigma(s^d)}{\sigma(s^{d-1})} \geqslant \left(\frac{yn}{s} + 1\right)\left(s + \frac{1}{\sigma(s^{d-1})}\right) > yn + s + \frac{yn}{s\sigma(s^{d-1})}.$$

Thus

$$z = x\sigma(n) - yn > s + \frac{yn}{s\sigma(s^{d-1})}.$$

If $d = 1$, then

$$z > s + \frac{yn}{s} \geqslant 2\sqrt{yn},$$

so $n < z^2/4y$. If $d = 2$, then, since $\xi + \eta/\xi^2 \geqslant 3 \cdot 2^{-2/3}\eta^{1/3}$ for $\xi > 0$ and fixed $\eta$, we have

$$z > s + \frac{yn}{(s+\frac{1}{2})^2} \geqslant 3 \cdot 2^{-2/3}(yn)^{1/3} - \frac{1}{2},$$

so $n < 4(z + \frac{1}{2})^3/27y$.

This completes the proof.

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COROLLARY. *If $\sigma(n) = yn + z$, $n$ is not primitive $y$-abundant and $n/s$ is not $y$-perfect, then $n < 4(z + \frac{1}{2})^3/27y$.*

PROOF. From Theorem 2, this is true when $z \geqslant 175$. Using Table 2, below, it may be verified also when $z \leqslant 174$ and $y \geqslant 2$. Suppose $y = 1$ and refer to the first paragraph of the proof of Theorem 2. We have $z^2 < 4(z + \frac{1}{2})^3/27$ for $z \geqslant 6$. The conditions of the Corollary exclude the possibility $z = 1$, and $z = 2$ and $z = 5$ are always impossible. If $z = 3$ then $n = 4$ and if $z = 4$ then $n = 9$; in these cases the result is seen to be true.

REMARK 2. Pomerance (1975) showed that the equation $\sigma(n) = yn + z$ has only finitely many solutions $n$ which are not primitive $y$-abundant, provided $n \neq qm$ where $q \nmid m$ and $\sigma(m) = z \equiv 0 \pmod m$. (A glance at Table 2, say for $z = 12$, allows this to be contrasted with the corollary.) In particular, Pomerance includes the possibility that $n/s$ is $y$-perfect if $s^2 \mid n$, and we do not. However, Theorem 2 can be adjusted to take this into account, with a bound of the same order on $n$.

We give next a necessary condition for a number with rational coefficient of abundance to be primitive.

THEOREM 3. *If $n$ is primitive $(y/x)$-nondeficient, then*

$$s^d < (ym/2)^2, \quad \textit{where } m = n/s^d.$$

PROOF. We may write $x\sigma(n) = yn + z$ for some $z \geqslant 0$. Since $n$ is primitive, we have, by Theorem 1(ii),

$$z = x\sigma(n) - yn \leqslant ym.$$

From the proof of Theorem 2, we have $s^d \leqslant x\sigma(m) + z(s-1)$, so

$$s^d \leqslant x\sigma(m) + ym(s-1) < yms,$$

since $\sigma(m)/m < y/x$. Thus $s^{d-1} < ym$ so that, if $d \geqslant 3$ (and $ym \geqslant 16$),

$$s^d < (ym)^{d/(d-1)} \leqslant (ym)^{3/2} \leqslant (ym/2)^2.$$

(The case studies required to verify the theorem where $ym \leqslant 15$ will be omitted.) The possibilities $d = 1$ and $d = 2$ are treated separately.

Since $n$ is $(y/x)$-nondeficient,

$$\frac{y}{x} \leqslant \frac{\sigma(n)}{n} = \frac{\sigma(m)}{m} \cdot \frac{s - s^{-d}}{s-1} < \frac{\sigma(m)}{m} \cdot \frac{s}{s-1},$$

so that $s(ym - x\sigma(m)) < ym$. But $ym - x\sigma(m) \geqslant 1$, so $s < ym$. In particular, this proves the theorem when $d = 1$ (unless $ym \leqslant 3$, in which case the verification of the theorem is again omitted).

Suppose $d = 2$. Then we cannot have $x\sigma(m) = ym - 1$: if this were the case, then

$$x\sigma(ms) = (ym - 1)(s + 1) = yms - (s - ym + 1) \geqslant yms,$$

since $s \leqslant ym - 1$, and so $n$ is not primitive. Thus $x\sigma(m) \leqslant ym - 2$, and, since $n$ is $(y/x)$-nondeficient,

$$x\sigma(m)(s^2 + s + 1) = x\sigma(ms^2) \geqslant yms^2 \geqslant (x\sigma(m) + 2)s^2,$$

so that

$$ym - 2 \geqslant x\sigma(m) \geqslant \frac{2s^2}{s + 1} > 2s - 2.$$

Thus $s < ym/2$, and the proof is finished.

REMARK 3. The result in Theorem 3 is best possible in that, if $n = 2$, $y = 3$ and $x = 2$ (and only in this case), $n$ is primitive $(y/x)$-nondeficient and $4s^d = y^2m^2 - 1$.

## 4. On the range of $\sigma(n)/n$

The theorems in this section give information on the range of $\sigma(n)/n$ when $n$ is primitive $\alpha$-abundant.

THEOREM 4. *Let $\pi$ denote the set of all primitive $\alpha$-nondeficient numbers. Then*

$$\lim_{n \in \pi} \frac{\sigma(n)}{n} = \alpha.$$

PROOF. Certainly, $\sigma(n)/n \geqslant \alpha$ for $n \in \pi$. Suppose

$$\limsup_{n \in \pi} \frac{\sigma(n)}{n} = \alpha + \beta, \qquad \beta > 0.$$

Then for infinitely many members $n$ of $\pi$ we have

$$\alpha + \frac{\beta}{2} \leqslant \frac{\sigma(n)}{n} \leqslant \alpha + \frac{\alpha}{s^d},$$

using Theorem 1(ii). Thus the largest component $s^d$ of any such $n$ satisfies $s^d \leqslant 2\alpha/\beta$. This restricts the possible values of $n$ to a finite set, giving a contradiction and proving the theorem.

We remark that if we do not insist that the $\alpha$-abundant numbers $n$ be primitive, then the method of Somayajulu (1977) may be adapted to show that any $\delta$,

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$\alpha \leqslant \delta \leqslant \infty$, is a limit point for the set of values of $\sigma(n)/n$. It follows from Theorem 4 that there are only finitely many primitive $\alpha$-abundant numbers $n$ such that $\sigma(n)/n \geqslant \alpha + \varepsilon$ for any given $\varepsilon > 0$. The next two theorems find values for $\varepsilon$ such that the set $\{n\colon n \text{ primitive } \alpha\text{-abundant}, \sigma(n)/n \geqslant \alpha + \varepsilon\}$ is empty. Theorem 5 is proved in Section 5.

THEOREM 5. *If $n$ is primitive abundant, then*

$$\frac{\sigma(n)}{n} \leqslant \frac{832}{385} < 2.16104.$$

*Equality occurs only for $n = 3^2 5 \cdot 7 \cdot 11$.*

THEOREM 6. *If $n$ is primitive $\alpha$-abundant, then*

$$\frac{\sigma(n)}{n} < \alpha + \min\{\tfrac{1}{2}, 3\alpha e^{-\rho\alpha}/2\},$$

*where $\rho = e^{-\gamma}$ and $\gamma$ is Euler's constant. The number $\frac{1}{2}$ cannot be replaced by a smaller constant.*

PROOF. Let $P_i$ be the $i$th prime, and write

$$\theta_i = \prod_{j=1}^{i} \frac{P_j}{P_j - 1}, \qquad \theta_0 = 1.$$

Let $i \geqslant 1$ be such that $\theta_{i-1} \leqslant \alpha < \theta_i$. We show first that $\omega(n) \geqslant i$. Suppose $\omega(n) \leqslant i - 1$ (if $i \geqslant 2$) and let the prime factor decomposition of $n$ be $\prod_{j=1}^{u} q_j^{b_j}$, where $q_1 < \cdots < q_u$. Then $u = \omega(n) \leqslant i - 1$, and

$$\frac{\sigma(n)}{n} = \prod_{j=1}^{u} \frac{q_j - q_j^{-b_j}}{q_j - 1} < \prod_{j=1}^{u} \frac{q_j}{q_j - 1} \leqslant \prod_{j=1}^{i-1} \frac{P_j}{P_j - 1} = \theta_{i-1} \leqslant \alpha,$$

contradicting the fact that $n$ is $\alpha$-abundant. Thus $\omega(n) \geqslant i$, so $s^d \geqslant P_i$ and, from Theorem 1(ii),

$$\frac{\sigma(n)}{n} \leqslant \alpha + \frac{\alpha}{s^d} \leqslant \alpha + \frac{\alpha}{P_i}.$$

From Rosser and Schoenfeld (1962), we know that

$$\theta_i < e^{\gamma} \log P_i \left(1 + \frac{1}{\log^2 P_i}\right),$$

so, if $i \geqslant 6$,

$$\alpha < \theta_i < e^{\gamma} \log P_i + \frac{e^{\gamma}}{\log 13}$$

and so

$$P_i > \exp\left(e^{-\gamma}\alpha - (\log 13)^{-1}\right) > 2e^{\rho\alpha}/3.$$

Thus, when $\alpha \geqslant \theta_5 = 4.8125$, we have

$$\frac{\sigma(n)}{n} \leqslant \alpha + \frac{\alpha}{P_i} < \alpha + 3\alpha e^{-\rho\alpha}/2.$$

(Note: a useful under-estimate for $\rho$ is $5/9$.)

The function $3\xi e^{-\rho\xi}/2$ is decreasing for $\xi > 1/\rho$ and is less than $\frac{1}{2}$ for $\xi > 4.8$. Hence, if $i \geqslant 6$, so that $\alpha \geqslant \theta_5$, we also have $\sigma(n)/n < \alpha + \frac{1}{2}$. Suppose $i = 5$. Then $\alpha/s^d \leqslant \alpha/P_5 < \theta_5/P_5 = 0.4375 < 3\alpha e^{-\rho\alpha}/2$. Thus, using Theorem 1(ii), Theorem 6 has now been proved whenever $\alpha \geqslant \theta_4 = 4.375$.

Suppose $i = 3$ or $4$, so that $\theta_2 \leqslant \alpha < \theta_4$. We may assume that $s^d \leqslant 2\alpha$, so $s^d < 2\theta_4$, giving $s^d \leqslant 8$. Listing all such $n$, we find in all cases that $\sigma(n)/n < 3.5 \leqslant \alpha + \frac{1}{2}$.

Suppose $i = 2$, so that $2 \leqslant \alpha < 3$. We may assume that $s^d \leqslant 5$. We find that $\sigma(n)/n < 2.5 \leqslant \alpha + \frac{1}{2}$, except if $n = n_1 = 2^2 3 \cdot 5$. But we require $\alpha > \sigma(n_1/2)/(n_1/2) = 2.4$ in this case, so $\sigma(n_1)/n_1 = 2.8 < \alpha + \frac{1}{2}$.

Suppose finally that $1 \leqslant \alpha < 2$. Assuming $s^d \leqslant 2\alpha$, we have $s^d \leqslant 3$, so $n = 2, 3$ or $6$. Certainly, $\sigma(3)/3 < \alpha + \frac{1}{2}$. Writing $\alpha = 1 + \varepsilon/2$ or $\alpha = 1.5 + \varepsilon/2$ $(0 < \varepsilon < 1)$ with $n = 2$ or $n = 6$, respectively, we see that in these cases $n$ is primitive $\alpha$-abundant and $(\alpha + \frac{1}{2})$-deficient, but not $(\alpha + \frac{1}{2} - \varepsilon)$-deficient.

With the observation that $\frac{1}{2} < 3\alpha e^{-\rho\alpha}/2$ when $1 \leqslant \alpha < \theta_4$, the proof of Theorem 6 is now complete.

THEOREM 7. *If $n$ is primitive $\alpha$-abundant, then*

$$\frac{\sigma(n)}{n} < \alpha + \frac{\alpha e}{(e-1)\log n}.$$

PROOF. The result follows from Theorem 1(ii) once we have shown that $s^d > \tau \log n$, where $\tau = 1 - 1/e$. This is easily checked for $n \leqslant 15$, so we shall assume $n \geqslant 16$.

Suppose $s^d \leqslant \tau \log n$. Then $n \leqslant (\tau \log n)^{\omega(n)} < (\log n)^{\omega(n)}$, so that

$$\omega(n) > \frac{\log n}{\log\log n}.$$

If $P_i$ denotes the $i$th prime, then $P_i > i \log i$, and

$$\tau \log n \geqslant s^d \geqslant P_{\omega(n)} > \omega(n) \log \omega(n) > \frac{\log n}{\log\log n} \log \frac{\log n}{\log\log n}.$$

Thus $\tau \log\log n > \log\log n - \log\log\log n$, from which

$$\log\log n < \frac{1}{1-\tau} \log\log\log n = e \log\log\log n.$$

However, the function $\log\log \xi / \log\log\log \xi$ ($\xi > e^e$) has the minimum value $e$. We have a contradiction, so the theorem is proved.

REMARK 4. By a similar method to that above, we can show also that

$$s^d > \left(1 - \frac{\log\log\log n}{\log\log n}\right) \log n$$

when $n \geqslant 7$. When $n$ is squarefree, Theorem 6* of Schoenfeld (1976) implies that $s^d > \log n / 1.001093$.

## 5. Computational results

In Table 1 we give the 91 primitive abundant numbers $n$ such that $\sigma(n)/n \geqslant 2.05$, in decreasing order of $\sigma(n)/n$. Values of $\sigma(n)/n$ are rounded to the given number of decimal places. Theorem 5 is an obvious consequence of Table 1.

To construct Table 1, we noted, from Theorem 1(ii), that we require $2.05 \leqslant \sigma(n)/n \leqslant 2 + 2/s^d$, from which $s^d \leqslant 37$. A straightforward algorithm, requiring only a few hours' work with a calculator, was used to produce all primitive abundant numbers $n$ with largest component not exceeding 37. Those with $\sigma(n)/n \geqslant 2.05$ were then ranked.

We come finally to our extension of the work of Robbins (1980). For brevity, by a solution of

$$(2) \qquad \sigma(n) = yn + z, \qquad y \geqslant 2, 0 < z \leqslant 210,$$

we shall mean a triple $(n, z, y)$, where $n$ is not primitive $y$-abundant, which satisfies (2). Robbins gave all solutions $(n, z, 2)$ with $z \leqslant 100$; his Table 5 contains three misprints, in the values of $n$ for $z = 31$, 84 and 86. Our Table 2 gives all solutions of (2).

We used only a hand calculator in obtaining Table 2, making use of ideas to be described briefly below. It follows from Theorem 2, however, that if $(n, z, y)$ is a solution of (2) and $n/s$ is not $y$-perfect, then $n < 690912$, and it is not difficult to write a computer program to find these solutions. When $n/s$ is $y$-perfect, we find all solutions of (2) as follows. We will see here the reason for choosing the range $0 < z \leqslant 210$.

Suppose $(n, z, y)$ is a solution of (2) and that $\omega(n) \geqslant 5$. Then

$$\sigma(n) - yn = z \leqslant 210 \doteq \tfrac{1}{2} \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \leqslant \frac{p-1}{p} \cdot y \cdot \frac{n}{s^d},$$

so that, by Theorem 1(i), $n$ is primitive $y$-abundant. This contradiction shows that $\omega(n) \leqslant 4$.

We can show now that $y = 2$ or 3. For suppose $y \geqslant 4$. Then $\omega(n) = 4$, since otherwise, letting $\prod_{j=1}^{u} q_j^{b_j}$ be the prime factor decomposition of $n$, we have

$$4 < \frac{\sigma(n)}{n} = \prod_{j=1}^{u} \frac{q_j - q_j^{-b_j}}{q_j - 1} < \prod_{j=1}^{u} \frac{q_j}{q_j - 1} \leqslant \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{5}{4} < 4.$$

Then we must have $2^3 \mid n$, since otherwise

$$4 < \frac{\sigma(n)}{n} < \frac{\sigma(2^2)}{2^2} \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} < 4.$$

If $s = 2$, then $\sigma(n) - yn \leqslant 210 = \frac{1}{2} \cdot 4 \cdot 3 \cdot 5 \cdot 7 \leqslant (p-1)yn/ps^d$; if $s > 2$, then $\sigma(n) - yn \leqslant 210 < \frac{1}{2} \cdot 4 \cdot 2^3 3 \cdot 5 \leqslant (p-1)yn/ps^d$. Either way, by Theorem 1(i), $n$ is primitive.

Now suppose $(n, z, y)$ is a solution of (2) and that $n/s$ is $y$-perfect. As in the proof of Theorem 2, we then have

$$\sigma(n) - yn = \frac{yn}{s\sigma(s^{d-1})}.$$

Suppose $y = 3$. Since $\omega(n) \leqslant 4$, $n/s$ is either $2^3 3 \cdot 5 = 120$, $2^5 3 \cdot 7$ or $2^9 3 \cdot 11 \cdot 31$ (Dickson (1966), page 37). If $d = 1$, then

$$\sigma(n) - 3n = \frac{3n}{s} \geqslant 3 \cdot 120 > 210;$$

if $d \geqslant 2$, then we find only the following four solutions of (2):

$$(2^4 3 \cdot 5, 24, 3), \quad (2^3 3^2 5, 90, 3), \quad (2^3 3 \cdot 5^2, 60, 3), \quad (2^6 3 \cdot 7, 32, 3).$$

Suppose $y = 2$. If $d = 1$, then $\sigma(n) - 2n = 2n/s > 210$ unless $n/s$ is 6 or 28. This gives the solutions

$(6q, 12, 2)$, where 6 and $q$ are relatively prime,

$(28q, 56, 2)$, where 14 and $q$ are relatively prime and $q > 7$.

If $d \geqslant 2$ then, since $n$ is even (for an odd perfect number has more than four, in fact more than seven, distinct prime factors), we have either $n = 2^{a-1}(2^a - 1)^2$ or $n = 2^a(2^a - 1)$ (where $2^a - 1$ is prime), and $\sigma(n) - 2n = 2^a - 1$ or $2^a$, respectively. Hence $\sigma(n) - 2n > 210$ if $a \geqslant 8$ and we quickly find the only eight such solutions.

The rest of our manual approach to finding solutions of (2) relied in the first place on lists of primitive $y$-nondeficient numbers, with $y = 2$ or 3, since if $(n, z, y)$ is a solution of (2) then $n$ has a $y$-nondeficient divisor. The relevant numbers with $y = 3$ we found for ourselves. A purportedly complete list of odd primitive nondeficient numbers with at most four distinct prime factors was

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TABLE 1

All primitive abundant $n$ such that $\sigma(n)/n \geqslant 2.05$

| $n$ | $\sigma(n)/n$ | $n$ | $\sigma(n)/n$ |
|---|---|---|---|
| $3^2 5\cdot7\cdot11$ | 2.16104 | $3\cdot5\cdot7\cdot11\cdot13$ | 2.14825 |
| $3^2 5\cdot7\cdot13$ | 2.13333 | $2\cdot5\cdot11\cdot13$ | 2.11469 |
| $3\cdot5\cdot7\cdot11\cdot17$ | 2.11215 | $2^2 13\cdot17\cdot19$ | 2.10050 |
| $2^2 5$ | 2.10000 | $3\cdot5\cdot7\cdot11\cdot19$ | 2.09979 |
| $3^2 5\cdot7\cdot17$ | 2.09748 | $3\cdot5\cdot11\cdot13\cdot17\cdot19$ | 2.09504 |
| $2^3 17\cdot19$ | 2.08978 | $3^2 5\cdot7\cdot19$ | 2.08521 |
| $3\cdot5\cdot7\cdot13\cdot17$ | 2.08507 | $2\cdot7\cdot11\cdot17\cdot19$ | 2.08436 |
| $2^2 13\cdot17\cdot23$ | 2.08223 | $3\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23$ | 2.08203 |
| $3\cdot5\cdot7\cdot11\cdot23$ | 2.08154 | $3^2 5\cdot13\cdot17\cdot19$ | 2.08050 |
| $2\cdot5\cdot11\cdot17$ | 2.07914 | $3^2 5\cdot11\cdot19\cdot23$ | 2.07697 |
| $3\cdot5\cdot11\cdot13\cdot17\cdot23$ | 2.07683 | $3\cdot5\cdot7\cdot13\cdot19$ | 2.07287 |
| $2^3 17\cdot23$ | 2.07161 | $2^2 13\cdot19\cdot23$ | 2.07006 |
| $3^2 7\cdot13\cdot17\cdot19\cdot23$ | 2.06757 | $3^2 5\cdot7\cdot23$ | 2.06708 |
| $2\cdot5\cdot11\cdot19$ | 2.06699 | $2\cdot7\cdot11\cdot17\cdot23$ | 2.06623 |
| $3^2 5^2 17\cdot19\cdot29$ | 2.06512 | $3\cdot5\cdot11\cdot13\cdot19\cdot23$ | 2.06468 |
| $3\cdot7\cdot11\cdot17\cdot19\cdot23\cdot29\cdot31$ | 2.06450 | $2^2 13\cdot17\cdot29$ | 2.06428 |
| $3\cdot7\cdot11\cdot13\cdot17\cdot19\cdot29$ | 2.06408 | $3\cdot5\cdot7\cdot11\cdot29$ | 2.06359 |
| $3^2 5\cdot17\cdot19\cdot29\cdot31$ | 2.06297 | $2\cdot7\cdot17\cdot19\cdot23\cdot29$ | 2.06248 |
| $3^2 5\cdot13\cdot17\cdot23$ | 2.06240 | $3\cdot5^2 11\cdot17\cdot23\cdot29$ | 2.06148 |
| $3\cdot5^2 7\cdot11$ | 2.06130 | $2^2 11\cdot23\cdot29$ | 2.06079 |
| $3^2 5^2 17\cdot19\cdot31$ | 2.06068 | $2^2 13\cdot17\cdot31$ | 2.05985 |
| $3^2 5^2 13\cdot29\cdot31$ | 2.05977 | $3\cdot7\cdot11\cdot13\cdot17\cdot19\cdot31$ | 2.05964 |
| $2^3 19\cdot23$ | 2.05950 | $3\cdot5\cdot11\cdot17\cdot23\cdot29\cdot31$ | 2.05934 |

published by Dickson (1913a). The list has many errors, as pointed out and corrected by Ferrier (1950) and Herzog (1980). Dickson (1913b) listed primitive abundant numbers of the form $2t$ with $\omega(t) \leqslant 3$ and errors in this list do not appear to have been previously noticed. From that list, the entry $2 \cdot 5^3 17^4 181$ should be deleted and to it should be added $2 \cdot 5^4 23 \cdot 43$, $2 \cdot 7 \cdot 11^2 17^2$, $2 \cdot 7^2 11 \cdot 17$, $2 \cdot 7^2 11 \cdot 19$ and $2 \cdot 7^2 11^2 23$.

Essentially, for each primitive $y$-abundant number we added 1 to the exponent of each component in turn and calculated $\sigma(n) - yn$ for the resulting number $n$, repeating the process for $n$ if necessary. Restrictions on the primitive $y$-abundant numbers to be used were obtained through Theorem 1(i), as used above, and other considerations. Stops on the process were provided by the following results, which are of independent interest.

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TABLE 1 (continued)

| $n$ | $\sigma(n)/n$ | $n$ | $\sigma(n)/n$ |
|---|---|---|---|
| $3\cdot5\cdot7\cdot11\cdot31$ | 2.05915 | $3^2 5\cdot11\cdot19\cdot29$ | 2.05907 |
| $3\cdot5\cdot11\cdot13\cdot17\cdot29$ | 2.05892 | $3^2 7\cdot11\cdot17\cdot23\cdot29$ | 2.05831 |
| $2\cdot7\cdot17\cdot19\cdot23\cdot31$ | 2.05805 | $2\cdot7\cdot13\cdot17\cdot19$ | 2.05763 |
| $3\cdot5^2 7\cdot19\cdot29$ | 2.05756 | $2\cdot5\cdot17\cdot23\cdot29$ | 2.05732 |
| $2\cdot5\cdot7$ | 2.057143 | $2\cdot7\cdot13\cdot23\cdot29\cdot31$ | 2.057135 |
| $3\cdot5^2 11\cdot17\cdot23\cdot31$ | 2.05705 | $3^2 5^2 11\cdot19$ | 2.05678 |
| $3\cdot5^2 11\cdot13\cdot17$ | 2.05664 | $2^2 11\cdot23\cdot31$ | 2.05636 |
| $2^2 11\cdot13$ | 2.05594 | $3\cdot5\cdot7\cdot19\cdot29\cdot31$ | 2.05542 |
| $2\cdot5^2 17\cdot23$ | 2.05504 | $3\cdot5\cdot7\cdot13\cdot23$ | 2.05485 |
| $3^2 5\cdot11\cdot19\cdot31$ | 2.05464 | $3\cdot5\cdot11\cdot13\cdot17\cdot31$ | 2.05450 |
| $2\cdot7\cdot11\cdot19\cdot23$ | 2.05415 | $3\cdot7\cdot11\cdot17\cdot19\cdot23\cdot29\cdot37$ | 2.05403 |
| $3^2 7\cdot11\cdot17\cdot23\cdot31$ | 2.05389 | $2^3 17\cdot29$ | 2.05375 |
| $3^2 7\cdot11\cdot13\cdot17$ | 2.05348 | $3\cdot5^2 17\cdot19\cdot23\cdot29\cdot31$ | 2.05331 |
| $3\cdot5^2 7\cdot19\cdot31$ | 2.05313 | $3\cdot5^2 13\cdot17\cdot19\cdot29$ | 2.052903 |
| $2\cdot5\cdot17\cdot23\cdot31$ | 2.052900 | $5^2 7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot31\cdot37$ | 2.05277 |
| $2^2 19\cdot23\cdot29\cdot31$ | 2.05262 | $3\cdot5\cdot7\cdot17\cdot31\cdot37$ | 2.05261 |
| $3^2 5\cdot17\cdot19\cdot29\cdot37$ | 2.05252 | $2\cdot5\cdot13\cdot17$ | 2.05249 |
| $3^2 7\cdot13\cdot19\cdot29\cdot31\cdot37$ | 2.05233 | $2^2 13\cdot19\cdot29$ | 2.05221 |
| $2\cdot11\cdot13\cdot19\cdot23\cdot31\cdot37$ | 2.05208 | $2^4 17$ | 2.05147 |
| $3^3 7\cdot13\cdot17\cdot29\cdot37$ | 2.05117 | $2\cdot7\cdot11\cdot29\cdot31\cdot37$ | 2.05100 |
| $3\cdot5\cdot13\cdot17\cdot19\cdot29\cdot31$ | 2.05077 | $3^2 5\cdot13\cdot19\cdot23$ | 2.05034 |
| $3^2 5^2 17\cdot19\cdot37$ | 2.05024 | $3^2 7\cdot17\cdot19\cdot23\cdot29\cdot31$ | 2.05016 |
| $3^3 5^2 19\cdot31\cdot37$ | 2.05005 | | |

LEMMA 2. *If $m$ is a proper divisor of $n$ and $\sigma(m) \geqslant ym$, then $\sigma(m) - ym < \sigma(n) - yn$.*

LEMMA 3. *Let $q$, $r$ and $n$ be given ($q \neq r$) and suppose that $q^b r^c \| n$ ($bc \geqslant 0$). If $u$ and $v$ are positive integers such that*

$$q^u < r^v \quad \text{and} \quad \sigma(q^b r^{v-1}) \geqslant \sigma(q^{u-1} r^c), \tag{3}$$

*and if $\sigma(n) \geqslant yn$, then*

$$\sigma(q^u n) - yq^u n \leqslant \sigma(r^v n) - yr^v n.$$

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TABLE 2
All solutions of $\sigma(n) = yn + z$, $y \geqslant 2$, $0 < z \leqslant 210$, for which $n$ is not primitive $y$-abundant

y = 3 if n is in italics; otherwise, y = 2

| z | n | z | n | z | n |
|---|---|---|---|---|---|
| 3 | $2\cdot 3^2$ | 71 | $2^3 7^2$ | 144 | $2^3 3\cdot 7, 2^2 3\cdot 29, 2\cdot 3\cdot 11^2$ |
| 4 | $2^2 3$ | 72 | $2^2 3\cdot 11, 2\cdot 3\cdot 5^2, 2\cdot 3^2 11,$ | 146 | $2^7 109$ |
| 7 | $2^2 7^2$ | | $\mathit{2^5 3\cdot 5}$ | 148 | $2^2 5\cdot 53, 2^7 107$ |
| 8 | $2^3 7$ | 74 | $2^6 53$ | 150 | $2\cdot 3^2 37$ |
| 10 | $2^3 5$ | 76 | $2^2 5\cdot 17$ | 152 | $2^2 3\cdot 31, 2^7 103$ |
| 12 | $2^3 3, 2\cdot 3^3, 2\cdot 3q$* | 78 | $2\cdot 3^2 13$ | 154 | $2^7 101$ |
| 17 | $2^2 5^2$ | 80 | $2^2 3\cdot 13, 2^2 5\cdot 19, 2^6 47$ | 156 | $2\cdot 3\cdot 5\cdot 7$ |
| 18 | $2^4 13$ | 84 | $2^6 43$ | 158 | $2^7 97$ |
| 19 | $2^2 3^2$ | 86 | $2^6 41$ | 160 | $2^3 5\cdot 7, 2^2 5\cdot 59$ |
| 20 | $2^4 11$ | 88 | $2^2 5\cdot 23$ | 161 | $2^4 5^2$ |
| 24 | $2^4 7, \mathit{2^4 3\cdot 5}$ | 89 | $2^4 29^2$ | 162 | $2\cdot 3^2 41$ |
| 26 | $2^4 5$ | 90 | $2\cdot 3^2 17, 2^6 37, \mathit{2^3 3^2 5}$ | 164 | $2^2 5\cdot 61$ |
| 28 | $2^4 3$ | 92 | $2^2 5^3$ | 166 | $2^7 89$ |
| 31 | $2^4 31^2$ | 96 | $2^2 3\cdot 17, 2\cdot 3\cdot 7^2, 2\cdot 3^2 19,$ | 168 | $2^3 3^3, 2\cdot 3^2 43, 2\cdot 3\cdot 13^2,$ |
| 32 | $2^5 31, \mathit{2^6 3\cdot 7}$ | | $2^6 31, \mathit{2^7 3\cdot 7}$ | | $\mathit{2^6 3\cdot 5}$ |
| 34 | $2^5 29$ | 98 | $2^6 29$ | 172 | $2^7 83$ |
| 39 | $2\cdot 3^4$ | 100 | $2^2 5\cdot 29$ | 176 | $2^2 3\cdot 37, 2^2 5\cdot 67, 2^7 79$ |
| 40 | $2^5 23$ | 104 | $2^2 3\cdot 19, 2^2 5\cdot 31, 2^6 23$ | 180 | $2\cdot 3^3 5, 2\cdot 3^2 47$ |
| 41 | $2^3 13^2$ | 108 | $2\cdot 3^2 23, 2^6 19$ | 182 | $2^7 73$ |
| 44 | $2\cdot 5^2 7, 2^5 19$ | 110 | $2^6 17$ | 184 | $2^2 5\cdot 71, 2^7 71$ |
| 46 | $2\cdot 5\cdot 7^2, 2^5 17$ | 114 | $2^6 13$ | 185 | $2^5 61^2$ |
| 48 | $2^2 3\cdot 5$ | 115 | $2^4 3^2$ | 186 | $2^2 3^2 5$ |
| 50 | $2^5 13$ | 116 | $2^6 11, 2^2 5\cdot 37, 2\cdot 5^3 11$ | 188 | $2\cdot 5\cdot 7\cdot 11, 2^2 5\cdot 73, 2^7 67$ |
| 51 | $2^3 3^2$ | 119 | $2\cdot 5^2 13^2$ | 192 | $2^3 3\cdot 11, 2^2 3\cdot 41$ |
| 52 | $2^5 11, 2\cdot 5^3 13$ | 120 | $2^3 3\cdot 5, 2^2 3\cdot 23, 2^6 7, 2\cdot 3^5$ | 194 | $2^7 61$ |
| 54 | $2\cdot 3^2 5$ | 121 | $\mathit{2^2 3^2 5^2}$ | 196 | $2\cdot 5\cdot 7\cdot 13, 2^7 59$ |
| 56 | $2^5 7, 2^2 7^3, 2^2 7r$* | 122 | $2^6 5$ | 198 | $2\cdot 3^2 53$ |
| 58 | $2^5 5$ | 124 | $2^6 3, 2^2 5\cdot 41$ | 199 | $2^2 3^4, 2^4 7^2$ |
| 59 | $2^3 11^2$ | 126 | $2\cdot 3^2 29$ | 200 | $2^3 5\cdot 11, 2^2 3\cdot 43, 2^2 5\cdot 79,$ |
| 60 | $2^5 3, 2\cdot 3^2 7, \mathit{2^2 3^3 5},$ | 127 | $2^6 127^2$ | | $\mathit{2^4 3^2 7}$ |
| | $\mathit{2^3 3\cdot 5^2}$ | 128 | $2^2 5\cdot 43, 2^7 127$ | 202 | $2^7 53$ |
| 64 | $2^2 3^3, 2^2 5\cdot 11$ | 132 | $2\cdot 3^2 31$ | 204 | $2\cdot 3\cdot 5\cdot 11, 2\cdot 3^3 7$ |
| 65 | $2^3 5^2$ | 136 | $2^2 5\cdot 47$ | 208 | $2^3 7\cdot 11, 2^2 5\cdot 83, 2^7 47,$ |
| 66 | $2^6 61$ | 138 | $3^4 5\cdot 7$ | | $\mathit{2^4 3\cdot 7^3}$ |
| 68 | $2^2 5\cdot 13, 2^6 59$ | 142 | $2^7 113$ | | |

* $(q, 6) = (r, 14) = 1$ ($q$, $r$ prime)

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Lemma 2 follows easily from Lemma 1 (and generalises a result of Robbins (1980)). Lemma 3 is a consequence of the identity

$$(\sigma(r^v n) - yr^v n) - (\sigma(q^u n) - yq^u n)$$
$$= (r^v - q^u)(\sigma(n) - yn) + \left(\frac{\sigma(r^{v-1})}{\sigma(r^c)} - \frac{\sigma(q^{u-1})}{\sigma(q^b)}\right)\sigma(n).$$

In applications of Lemma 3, we always had $b = c = 0$, $u = v$ and $q < r$, so that (3) was always satisfied.

We mention finally that the only solution of (2) with $n$ odd is $(2835, 138, 2)$. In fact, we found that the only other solution of (1) in which $n$ is odd and not primitive $y$-abundant, $y \geqslant 2$ and $0 < z \leqslant 420$ is $n = 15435$, $z = 330$, $y = 2$.

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## References

P. Cataneo (1951), 'Sui numeri quasiperfetti', *Boll. Un. Mat. Ital.* (3) **6**, 59–62.

L. E. Dickson (1913a), 'Finiteness of the odd perfect and primitive abundant numbers with $n$ distinct prime factors', *Amer. J. Math.* **35**, 413–422.

L. E. Dickson (1913b), 'Even abundant numbers', *Amer. J. Math.* **35**, 423–426.

L. E. Dickson (1966), *History of the theory of numbers*, Vol. 1 (Chelsea, New York).

A. Ferrier (1950), Table errata 176, *M.T.A.C.* **4**, 222.

Fritz Herzog (1980), Table errata 571, *Math. Comp.* **34**, 652.

C. Pomerance (1975), 'On the congruences $\sigma(n) \equiv a (\mathrm{mod}\ n)$ and $n \equiv a\ (\mathrm{mod}\ \varphi(n))$', *Acta Arith.* **26**, 265–272.

Neville Robbins (1980), 'A class of solutions of the equation $\sigma(n) = 2n + t$', *Fibonacci Quart.* **18**, 137–147.

J. B. Rosser and L. Schoenfeld (1962), 'Approximate formulas for some functions of prime numbers', *Illinois J. Math.* **6**, 64–94.

L. Schoenfeld (1976), 'Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II', *Math. Comp.* **30**, 337–360.

B. S. K. R. Somayajulu (1977), 'The sequence $\sigma(n)/\varphi(n)$', *Math. Student* **45**, 52–54.

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