FINITE SOLUBLE GROUPS WITH CERTAIN SPLITTING PROPERTIES

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Abstract. In a recent paper, Bechtell obtained detailed structure theorems for finite supersoluble groups with the property that every minimal supplement for a non-Frattini normal subgroup is a complement. We consider finite soluble groups with this property. The situation is rather different to the supersoluble case, and the information we obtain is not as complete, though for such groups with non-trivial Frattini subgroup, some of the results are analogous.

1. Introduction

In a recent paper, Bechtell [1] studies finite supersoluble groups G with the properties:

$\mathscr{P}$: if N is a normal subgroup of G not in the Frattini subgroup ΦG of G, then N is complemented in G,

$\mathscr{P}^*$: if N is a normal subgroup of G not in ΦG, then every minimal supplement for N in G is a complement.

He finds detailed structure theorems for such groups, the interesting case for supersoluble groups being groups with non-trivial Frattini subgroups. In the present paper, we shall be mainly concerned with finite soluble $\mathscr{P}^*$ groups—not necessarily supersoluble ones (and from now on all groups will be assumed to be finite and soluble). The situation here is quite different to the supersoluble case, and few of Bechtell's results go over, though there are some analogues for the case of non-trivial Frattini subgroup.

A well known result of Gaschütz ([5] Satz 8) is that in a group with elementary abelian Sylow subgroups every normal subgroup has a complement: it follows easily that such a group is a $\mathscr{P}^*$ group. The converse of this theorem is false: witness S_4. However, S_4 is not a $\mathscr{P}^*$ group, and one is tempted to conjecture that being $\mathscr{P}^*$ and Φ-free ensures elementary abelian Sylow subgroups. We shall show that this is indeed so for groups of nilpotent length three, and give an example (of nilpotent length four) to show that it is not true in general. Side by side with this example, we construct another group which is not a $\mathscr{P}^*$ group, but is similar enough in structure to make it seem unlikely that an easily applicable structure theorem can be found which will be fine enough to distinguish between these two groups. We can obtain a structure theorem which places some limitations on the stucture of certain Sylow subgroups, and it turns out to be quite useful.

It is known (Christensen [3]) that the class of Φ-free $\mathscr{P}$ groups is a formation. The same is not true for the class of Φ-free $\mathscr{P}^*$ groups, and we show that this class is not even direct product closed. One might then ask whether the formation generated by a Φ-free $\mathscr{P}^*$ group consists of $\mathscr{P}^*$ groups: the answer is again no but it is somewhat harder to settle: the example is fairly complicated, and we shall not write it down. We shall prove however that if H has elementary abelian Sylow subgroups, and G is a Φ-free $\mathscr{P}^*$ group with elementary abelian Sylow p-subgroups for those primes p dividing the order of H, then $G \times H$ is a $\mathscr{P}^*$ group.

Finally, we consider $\mathscr{P}^*$ groups with non-trivial Frattini subgroup. Here we can obtain a certain amount of information on the Fitting subgroup, the most general result being for a $\mathscr{P}^*$ group G, with $\Phi G \neq 1$, $F(G)/\Phi G$ has at most two irreducible components.

2. Notation and Preliminaries

For a group G, ΦG denotes the *Frattini subgroup* of G, and $F(G)$ the *Fitting subgroup* of G. If $\Phi G = 1$, we say G is *Φ-free*. We shall use, often without comment, a number of facts about the Frattini and Fitting subgroups and the relation between them, and so we assume familiarity with the results of Gaschütz [6]. Notation is usually standard, or explained as it is introduced.

If π is a set of primes, then for a group G, G_π will denote a *Hall π-subgroup* of G, and $\mathscr{E}_\pi$ the class of *π-groups with elementary abelian Sylow subgroups*. We shall need the following result:

2.1 *If G is a group with $G_\pi \in \mathscr{E}_\pi$, and N is a normal π-subgroup of G, then N is complemented in G.*

This result is easily deduced from Gaschütz [5] Reduktionssatz 2.

We shall need several elementary facts about $\mathscr{P}$ groups and $\mathscr{P}^*$ groups, and we list them (without proof) below. The proofs of those for which no reference is given are easy.

2.2 *Each homomorphic image of a group $G \in \mathscr{P}$ (or $\mathscr{P}^*$) has property $\mathscr{P}$ (or $\mathscr{P}^*$).*

2.3 *If $G \in \mathscr{P}$, and N is normal subgroup of G, then either $N \leqq \Phi G$, or $\Phi G < N$.*

2.4 *If $G \in \mathscr{P}$, and $\Phi G \neq 1$, $F(G)$ is a p-group for some prime p.*

2.5 *If $G \in \mathscr{P}$, and $Z(G)$ is the centre of G, then*

(i) $Z(G) \leqq \Phi G$, *or*
(ii) $G = H \times Z(G)$, *where* $Z(H) = \Phi H = 1$.

2.6 *If* $G \in \mathscr{P}$, *then either* $Z(F(G)) \leqq \Phi G$ *or* $F(G)$ *is abelian.*

2.7 *If* $G \in \mathscr{P}$, $\Phi G = 1$, *and* N *is a normal subgroup of* G, *then* $\Phi(G/N) = 1$.

2.8 (Christensen [2] Theorem 3.5) *If* $G \in \mathscr{P}$, $\Phi G = 1$, *and* N *is a normal subgroup of* G, *then* $N \in \mathscr{P}$.

It is often convenient to regard chief factors and elementary abelian normal subgroups of a group as modules for the group: we shall do so without comment whenever it seems convenient.

3. Two examples

In this section we construct two very similiar groups, one of which is a $\mathscr{P}^*$ group, the other not.

Both groups have a common start to their construction. We begin with S_3, which we take to be generated by elements a, b with $a^2 = b^3 = 1$. S_3 has a faithful irreducible representation of degree two over $GF(7)$: let M be a module affording this representation. The following facts about M are easily deduced: M is induced from a one-dimensional faithful module for $\langle b \rangle$ over $GF(7)$ (use e.g. Blichfeldt's Theorem [4] 50.7), and $M_{\langle a \rangle}$ is the regular representation of $\langle a \rangle$ (use e.g. the Mackey Subgroup Theorem [4] 44.2). Thus we can choose generators u, v for M such that $u^a = u$, $v^a = v^{-1}$. Now let H be the splitting extension of M by S_3, and note that M is then the unique minimal normal subgroup of H.

We now construct two different faithful irreducible modules for H over $GF(3)$, U_1 and U_2, such that U_1H is $\mathscr{P}^*$ and U_2H is not $\mathscr{P}^*$. Both are induced from modules for $M\langle a \rangle$. Let V_1 be an irreducible module over $GF(3)$ for $M\langle a \rangle$ with kernel $\langle v, a \rangle$, and V_2 an irreducible module over $GF(3)$ for $M\langle a \rangle$ with kernel $\langle v \rangle$. It is easy to check that V_1 and V_2 have dimension six. Put $U_1 = V_1^H$, $U_2 = V_2^H$. Then we claim that V_1 and V_2 are faithful irreducible modules for H. That they are faithful is clear. To see that they are irreducible, observe that if W is a faithful irreducible module for H over $GF(3)$, and K an algebraically closed field of characteristic 3, we have $W^K = \sum_{i=1}^{n} W^i$, where the W^i are all faithful and absolutely irreducible of the same dimension, and $n \geqq 6$ (from [4] 70.15). Also, a athful irreducible module for H over an algebraically closed field has dimension at least 3 (this comes from applying Clifford's Theorem [4] 49.2 to $W^i_{M\langle b \rangle}$ and then Blichfeldt's Theorem to an irreducible component of $W^i_{M\langle b \rangle}$). It follows that W has dimension at least 18, and then since H has a unique minimal normal subgroup and U_1, U_2 have dimension 18, that U_1 and U_2 are irreducible.

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Using the Mackey subgroup Theorem, we get U_{1S_3} and U_{2S_3} are both direct sums of principal indecomposable modules for S_3 over $GF(3)$, and that for U_{1S_3} they are all isomorphic to P_1, and for U_{2S_3} they are all isomorphic to P_2. Here P_1 is the principal indecomposable for S_3 over $GF(3)$ whose socle is the trivial module for S_3, and P_2 the principal indecomposable whose socle is a non-trivial module for S_3.

Now put $G_1 = U_1H$, and $G_2 = U_2H$. Then we claim that G_1 is a $\mathscr{P}^*$ group. Consider them separately.

1) The non-trivial normal subgroups of G_1 are U_1, U_1M, $U_1M\langle b\rangle$. Clearly for U_1, $U_1M\langle b\rangle$ any minimal supplement is a complement. So we are left with U_1M. It is also easy to see that any minimal supplement for U_1M, L say, has order prime to 7. Hence (replacing L by a conjugate if necessary) we may assume that L is a subgroup of U_1S_3, and that $a \in L$. The Sylow 3-subgroup S of L is normal in L and is a subgroup of $U_1\langle b\rangle$. Consider $S/\Phi S$: as an $\langle a\rangle$-module, it is completely reducible, say $S/\Phi S = K_1/\Phi S \times \cdots \times K_n/\Phi S$. Now not all K_i can lie in U_1: suppose K_1 is not contained in U_1. Then $K_1\langle a\rangle$ is a supplement for U_1M and hence $L = K_1\langle a\rangle$. Thus the Sylow 3-subgroup of L is cyclic, and so L is isomorphic to either S_3 or D_{18}, the dihedral group of order 18. But in $U_1\langle b\rangle$, all third powers are acted on trivially by a, since third powers lie in the socle of U_{1S_3}. Hence L cannot be D_{18}, and so $L \cong S_3$, giving that L is a complement for U_1M.

2) We shall find a minimal supplement for U_2M in G_2 which is not a complement. For this, it is enough to show that P_2S_3 contains a subgroup isomorphic to D_{18}. To do this, observe that we can choose a basis for P_2, x, y, z, such that

$$x^a = x^{-1} \qquad x^b = xyz^{-1}$$

$$y^a = y \qquad y^b = yz$$

$$z^a = z^{-1} \qquad z^b = z$$

Now the claim is that $w = bxy^{-1}$ has order 9 and is inverted by a: this is just a matter of checking. We then have $\langle w, a\rangle \cong D_{18}$.

Observe that the subgroup $U_1M\langle b\rangle$ of G_1 is a normal subgroup of nilpotent length three, and has non-abelian Sylow 3-subgroup, and hence is not a $\mathscr{P}^*$ group by Theorem 4.6. The class of Φ-free $\mathscr{P}^*$ groups is therefore not normal subgroup closed; cf. 2.8.

4. Structure Theorems

In this section, we shall attemt to shed some light on the structure of Sylow subgroups of Φ-free $\mathscr{P}^*$ groups. Although we are mainly interested in $\mathscr{P}^*$ groups, the crucial lemma of this section is a result about $\mathscr{P}$ groups, and yields some information about the structure of Sylow subgroups in Φ-free $\mathscr{P}$ groups as well as Φ-free $\mathscr{P}^*$ groups.

LEMMA 4.1. *Let G be a Φ-free $\mathscr{P}$ group, M/N a p chief factor of G, U a normal subgroup of G, of index p, containing the centraliser of M/N in G. Then there is an element $y \in G$ with y of order p^2 and $y \notin U$.*

PROOF. This is a fairly standard Hall Higman type argument: we merely sketch it.

Suppose the lemma is false: let G be a minimal counterexample. It is easy to see that G has a unique minimal normal subgroup M, with p dividing the order of M, that a complement C for M in G acts faithfully and irreducibly on M, and that there is a normal subgroup V of C of index p, $U = MV$. If X is a complement for V in C, let $X = \langle x \rangle$. Since C is Φ-free $\sigma C = F(C)$, and so x cannot centralise the whole of σC: in particular, there is a minimal normal subgroup N of C such that $[N, x] \neq 1$. Since the order of N is prime to p, there is a subgroup N_0 of N on which X acts faithfully and irreducibly : put $D = N_0 X$. Consider M_D. Since D acts faithfully on M_D, there is a composition factor W on which D acts faithfully and irreducibly. Let K be an algebraically closed field of characteristic p. Using [4] 70.15 and Blichfeldt's Theorem, we get thet W_X^K contains the regular module for X, and hence so does W_X, and then M_X. But than MX contains an element of order p^2, y say. Then y is an element of order p^2 not in $U = MV$, contradicting the choice of G.

COROLLARY 4.2 *Let G be a Φ-free $\mathscr{P}$ group, and suppose that a Sylow p-subgroup of G has exponent p. Then the Sylow p-subgroups are elementary abelian.*

PROOF. By induction on the order of G. Clearly true if $|G| = 1$, so suppose true for all Φ-free $\mathscr{P}$ groups of order less than $|G|$. If either G has a normal subgroup of order prime to p or a normal subgroup of index prime to p, invoke 2.2 or 2.8 to obtain the result. If G is abelian there is nothing to prove. Thus $\sigma G = F(G)$ is a p-group, and there is a non-central minimal normal subgroup N. If U is a maximal normal subgroup of G containing the centraliser in G of N, U has index p, and we can apply Lemma 4.1 to deduce that G has elements of order p^2, a contradiction.

COROLLARY 4.3 *Let G, U, be as in the theorem. Then G is not a $\mathscr{P}^*$ group.*

PROOF. Put $Y = \langle y \rangle$. Then Y is a minimal supplement for U which is not a complement.

The next result is our main structural result for Φ-free $\mathscr{P}^*$ groups.

THEOREM 4.4 *Let G be a Φ-free $\mathscr{P}^*$ group, and U a normal subgroup of G of index p. Then the Sylow p-subgroups of G are elementary abelian.*

PROOF. Suppose the result is false, and let G be a minimal counterexample. The minimality of G gives immediately that σG is a p-group, and it then follows from Lemma 4.1 that G has no normal subgroups of index p containing σG. Thus $G = \sigma G.U$. Let V be a normal complement for $\sigma G \cap U$ in σG: then $G = U \times V$, and $|V| = p$. Now U cannot have elementary abelian Sylow subgroups, and so by Corollary 4.2, U contains an element x of order p^2. If $V = \langle v \rangle$, hen $\langle vx \rangle$ is a minimal supplement for U which is not a complement, a contradiction.

COROLLARY 4.5 *$\mathscr{P}^*$ is not direct product closed.*

PROOF. Consider the group G_1 of §3, and let C_3 be the cyclic group of order 3. By Theorem 4.4, $G_1 \times C_3$ is not a $\mathscr{P}^*$ group, though G_1 and C_3 are.

THEOREM 4.6. *Let G be a Φ-free $\mathscr{P}^*$ group, and suppose that G has nilpotent length 3: then G has elementary abelian Sylow subgroups.*

PROOF. Again, suppose that the result is false, and let G be a minimal counterexample, and let p be a prime for which G has a non-abelian Sylow p-subgroup. Then $\sigma G = F(G)$ is a p-group, and $F(G/\sigma G) = F_2/\sigma G$ is a p'-group. But then G/F_2 is nilpotent, and since a Sylow p-subgroup of G was assumed non-abelian, p divides $|G/F_2|$, whence G has a normal subgroup of index p, contradicting Theorem 4.4.

5. Direct Products

We have seen that the direct product of two Φ -free $\mathscr{P}^*$ groups need not be a $\mathscr{P}^*$ group. The best we have been able to do in this direction is the following theorem.

THEOREM 5.1. *Let G be a Φ-free $\mathscr{P}^*$ group, π a set of primes such that $G_\pi \in \mathscr{E}_\pi$, and $H \in \mathscr{E}_\pi$. Then $G \times H$ is a $\mathscr{P}^*$ group.*

PROOF. Put $D = G \times H$. We work by induction on the order of D. Let $1 \neq N$ be a normal subgroup of D, and S a minimal supplement for N in D, that is, $NS = D$, $S \cap N \leqq \Phi S$. Suppose that $N \cap S \neq 1$.

We first show $H \cap N = H \cap S = 1$. Observe that $H \cap N$ is normal in D, and $S(H \cap N)/(H \cap N)$ is a minimal supplement for $N/(H \cap N)$ in $D/(H \cap N)$. If $H \cap N \neq 1$, we conclude by induction that $S \cap N \leqq H \cap N$. But then $S \cap N$ is a normal π-subgroup of S, and so by 2.1 has a complement C_0 in S. C_0 is then a complement for N in D, giving $S = C_0$, contradicting $S \cap N \neq 1$. Hence $H \cap N = 1$. If $H \cap S \neq 1$, then $H \cap S$ is normal in D, and, by induction, there is a complement $C_1/(S \cap H)$ in $D/(S \cap H)$ for $N(S \cap H)/(S \cap H)$ (with $C_1 \leqq S$). But then $NC_1 = D$, and

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$$N \cap C_1 \leqq (S \cap H) \cap N = 1.$$

Hence $S = C_1$, again a contradiction, and so $S \cap H = 1$ also.

Next note that $N \cap G \neq 1$. For, it not, N is a π-group, and 2.1 gives us a contradiction again. Also, arguing as in the previous paragraph, if M is a minimal normal subgroup of G, $M \leqq N \cap G$, then $S \cap N \leqq M$. Then, using 2.1 again, $S \cap N$ is a p-group for some prime p not in π.

Let σ be the projection of D onto G. Since $N\sigma S\sigma = G$, and $G \in \mathscr{P}^*$, there is a subgroup C of S such that $C\sigma$ is a complement for $N\sigma$ in G (observe that σ is a monomorphism on S and N.) Let B be the normal subgroup of S defined by $B\sigma = N\sigma \cap S\sigma$: then C is a complement for B in S. Since $(S \cap N)\sigma = S \cap N$, we have

$$S \cap N \leqq B \cap N \leqq S \cap N,$$

giving $B \cap N = S \cap N$. Further, $B \cap N = B_{\pi'}$, where π' is the set of primes not in π. This comes from the fact that $B_{\pi'} = B_{\pi'}\sigma \leqq N\sigma$, whence $B_{\pi'} \leqq N$, and on the other hand $B \cap N = S \cap N \leqq B_{\pi'}$.

Now the usual Frattini argument, with S acting on the conjugates of B_0, a complement of $S \cap N$ in B, gives

$$S = (S \cap N)N_S(B_0).$$

But $S \cap N \leqq \Phi S$, and so $S = N_S(B_0)$, giving $S = B_0 C$. Then $S \cap N = 1$, a contradiction.

6. $\mathscr{P}^*$ groups with non-trivial Frattini subgroup

Recall that if G is a finite soluble group, $F(G)$ properly contains ΦG, and $F(G)/\Phi G$ is completely reducible as a module for G, and also if G is a $\mathscr{P}^*$ group with $\Phi G \neq 1$, $F(G)$ is a p-group for some prime p. We start by proving

THEOREM 6.1. *Let G be a $\mathscr{P}^*$ group with $\Phi G \neq 1$. Then $F(G)/\Phi G$ has at most two irreducible components, and if $F(G)$ is abelian, $F(G)/\Phi G$ is irreducible.*

PROOF. Since $G \in \mathscr{P}^*$, $F(G)$ has a complement, C say, in G. Suppose that

$$F(G)/\Phi G = T_1/\Phi G \times T_2/\Phi G \times T_3/\Phi G$$

is a decomposition of $F(G)/\Phi G$, with each $T_i \neq \Phi G$, $i = 1,2,3$. Then T_i is a non-Frattini normal subgroup of G, and $T_j T_k C$ (i,j,k distinct) is a supplement for T_i, and so contains a complement X_i for T_i in G, $i = 1,2,3$. Since

$$X_i \cong G/T_i \cong (G/\Phi G)/(T_i/\Phi G)$$

it follows that $X_i \cap F(G) = U_{ij} \times U_{ik}$, and $X_i \cap F(G)$ has a complement D_i in X_i which normalises U_{ij} and U_{ik}, and $T_l = U_{il}\Phi G$, $l = j,k$. Note also that U_{ij}, U_{ik} are elementary abelian and D_i is a complement for $F(G)$ in G.

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Consider first the case in which ΦG is central in $F(G)$. Then each T_l is abelian, $[U_{ij}, U_{ik}] = 1$, and hence $[T_j, T_k] = 1$. It follows that $F(G)$ is abelian. Since each U_{ij} admits some complement (viz. D_i) of $F(G)$ in G, it admits C, and hence $U_{21} \times U_{12} \times U_{13}$ is normal in G,

$$(U_{21} \times U_{12} \times U_{13})C = G$$

(since it generates G modulo ΦG), and

$$F(G) = U_{21} \times U_{12} \times U_{13}.$$

But then $F(G)$ is a direct sum of completely reducible modules and so is completely reducible. Thus $\Phi G = 1$, a contradiction.

If ΦG is not central in $F(G)$, put $N = [F(G), \Phi G]$, and consider G/N. Since $N < \Phi G$

$$\Phi(G/N) = \Phi G/N \neq 1, \qquad F(G/N) = F(G)/N, \qquad G/N \in \mathscr{P}^*,$$

and we are in the same situation as in the last paragraph.

If $F(G)$ is abelian, we assume a decomposition

$$F(G)/\Phi G = T_1/\Phi G \times T_2/\Phi G,$$

and proceed much as above.

For $G \in \mathscr{P}^*$ with $\Phi G \neq 1$, we can say a little more about the structure of $F(G)$, mainly for the case $F(G)/\Phi G$ reducible. If $G \in \mathscr{P}^*$, $\Phi G \neq 1$, and $F(G)/\Phi(G)$ is reducible, we have, as in the proof of Theorem 6.1, that there exist elementary abelian p-subgroups of $F(G)$, U_1, U_2, such that

$$U_i \cap \Phi G = U_1 \cap U_2 = 1,$$

$U_i \Phi G/\Phi G$ is irreducible, $i = 1, 2$, and

$$U_1 U_2 \Phi G = F(G).$$

THEOREM 6.2. *With G as above, we may choose the U_1 and U_2 to be normalised by the same complement C of $F(G)$ in G. Further, $\Phi(F(G)) = \Phi G = F(G)'$.*

PROOF. The proof separates into two cases, depending on whether one or none of U_1, U_2 is a trivial $G/F(G)$-module

i) Suppose that U_2 is a trivial $G/F(G)$-module. Then $G/\Phi G$ is a Φ-free $\mathscr{P}^*$ group, and has a quotient group of order p. Hence the Sylow p-subgroup of $G/\Phi G$ is elementary abelian (Theorem 4.4), and we deduce that $F(G)/\Phi G$ is the Sylow p-subgroup of $G/\Phi G$, and hence $F(G)$ is the Sylow p-subgroup of G. Now, as in the proof of Theorem 6.1, we see that there are complements C_1 and C_2 of $F(G)$ in G normalising U_1 and U_2 respectively. By the Schur-Zassenhaus Theorem, they are conjugate: say $C_1 = C_2^g$, $g \in G$. Then U_1, U_2^g both admit C_1, and have the same properties as U_1, U_2.

For this case, it is clear that $\Phi(F(G)) = \Phi G = F(G)'$.

ii) Hence we may suppose that both U_1 and U_2 are nontrivial. Let C be a complement for $F(G)$ which normalises U_1: we will show that we can choose an elementary abelian p-subgroup of $F(G)$ which complements $U_1\Phi G$ in $F(G)$, and is normalised by C. We proceed by induction on the class of $F(G)$. By 2.6 we have $Z(F(G)) = N \leqq \Phi G$. Consider G/N: if the class of $F(G) = 2$, $\Phi(G/N) = 1$, and there is a complement V/N for $U_1\Phi G/N$ in $F(G)/N$ which is normalised by CN/N: if the class of $F(G)$ is greater than 2, then by the induction hypothesis, there is a complement V/N for $U_1\Phi G/N$ in $F(G)/N$ normalised by CN/N. Consider VC: this is a supplement for $U_1\Phi G$ in G, and hence contains a complement, the compelment having the form U_2D, where U_2 is a complement for $U_1\Phi G$ in $F(G)$, and D is a complement for $F(G)$ in G, $U_2 \cap D = 1$. Now $U_2 \leqq V$, $N \leqq V$, $[U_2, N] = 1$, and $V = U_2N$, giving V abelian. It follows that U_2 is normalised by C, and U_1, U_2, C satisfy the requirement of the Theorem.

Next observe that $U_i[U_1, U_2]$ is a non-Frattini normal subgroup, $i = 1,2$, and hence $\Phi G \leqq U_i[U_1, U_2]$ (by 2.3). But

$$U_1[U_1, U_2] \cap U_2[U_1, U_2] = [U_1, U_2],$$

and so we have

$$\Phi(F(G)) \leqq \Phi G \leqq [U_1, U_2] \leqq F(G)' \leqq \Phi(F(G)),$$

and hence $\Phi(F(G)) = \Phi G = F(G)'$.

If, in the set up of Theorem 6.2, $F(G)$ has class c, form the $\mathfrak{N}_c$-product F^* of U_1 and U_2, and, letting C act on F^* in the natural way, form the split extension $F^*C = G^*$. Clearly G is a homomorphic image of G^*, and one might hope that a G^* formed with appropriate U_1, U_2, C is a $\mathscr{P}^*$ group. This is not in general true, and it is not difficult to find examples.

As an example, let V be the elementary abelian group of order 4, acted on non-trivially by C_3 (the cyclic group of order 3), and C_2, acted on trivially by C_3. Let F be the free $\mathfrak{N}_2$-product of V and C_2, and $G = F{\cdot}C_3$ then $\langle \Phi F, V, C_3 \rangle = N$ is a normal subgroup of index 2 in G and there are elements of order 4 not in N, whence G is not a $\mathscr{P}^*$ group. This example is not atypical of the situation when one of U_1 and U_2 is trivial: we can show that $F(G)$ cannot be a 2-group, or more generally, if $F(G)$ is a p-group of class p, it must have exponent p. It is for this case that the analogy with Bechtell's results is closest.

Finally, if $F(G)/\Phi G$ is irreducible, we can say nothing more in general: however, we can prove

THEOREM 6.3. *Let G be a group with the following properties*:
(i) *$G/\Phi G$ is a $\mathscr{P}^*$ group*
(ii) *$F(G)$ is a p-group for some prime p, and $F(G)/\Phi G$ is irreducible, and*
(iii) $(p, |G/F(G)|) = 1$.
Then G is a $\mathscr{P}^*$ group.

PROOF. From (iii) we deduce $\Phi(F(G)) = \Phi G$: from (ii) that $G/\Phi G$ has a unique minimal normal subgroup $F(G)/\Phi G$.

Let N be a non-Frattini normal subgroup of G. Then $N\Phi G \geqq F(G)$, and hence $N \geqq F(G)$. Let S be a minimal supplement for N: then $S\Phi G/\Phi G$ is a minimal supplement for $N/\Phi G$, and since $G/\Phi G$ is a $\mathscr{P}^*$ group, $S \cap N \leqq \Phi G$. But then from (iii)

$$(|S/S \cap N|, |S \cap N|) = 1,$$

and so by the Schur Zassenhaus Theorem $S \cap N$ has a complement C in S: and clearly $NC = G$, giving $S = C$, $S \cap N = 1$, i.e. S is a complement for N in G.

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