Adv. Appl. Prob. **38,** 522–544 (2006)
Printed in Northern Ireland

DECAY RATES FOR QUASI-BIRTH-AND-DEATH PROCESSES WITH COUNTABLY MANY PHASES AND TRIDIAGONAL BLOCK GENERATORS

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Abstract

We consider the class of level-independent quasi-birth-and-death (QBD) processes that have countably many phases and generator matrices with tridiagonal blocks that are themselves tridiagonal and phase independent. We derive simple conditions for possible decay rates of the stationary distribution of the 'level' process. It may be possible to obtain decay rates satisfying these conditions by varying only the transition structure at level 0. Our results generalize those of Kroese, Scheinhardt, and Taylor, who studied in detail a particular example, the tandem Jackson network, from the class of QBD processes studied here. The conditions derived here are applied to three practical examples.

Keywords: Decay rate; QBD process; countable phase; tridiagonal block generator; stationary distribution; two-node Jackson network

2000 Mathematics Subject Classification: Primary 60K25
Secondary 60K20; 90B22

1. Introduction

A quasi-birth-and-death (QBD) process is a two-dimensional continuous-time Markov chain for which the generator has a block tridiagonal structure. The first variable of the QBD process is called the *level*, the second variable the *phase*.

The properties of QBD processes with finitely many possible values of the phase variable have been studied extensively. A comprehensive discussion can be found in the monographs of Neuts [17] and Latouche and Ramaswami [9]. It is well known that the level process of a positive-recurrent QBD process with finitely many possible values of the phase possesses a stationary distribution which decays geometrically as the level is increased. The decay rate is given by the spectral radius of Neuts' $\boldsymbol{R}$-matrix, which is strictly less than 1.

The situation is more complicated for a QBD process with *countably* many possible values of the phase. The $\boldsymbol{R}$-matrix is infinite-dimensional and its spectral properties are not obvious. Also, the relationship between various decay rates differs from that in the finite-dimensional case. It was shown in [8] and [19] that the decay rate of the level process for a countable-phase QBD process is not necessarily equal to the limiting value of the decay rate for finite-phase truncations of that process.

Received 8 March 2005; revision received 17 January 2006.
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https://doi.org/10.1239/aap/1151337083 Published online by Cambridge University Press

In this paper we make a contribution to the study of the behaviour of countable-phase QBD processes by considering a specific class of these processes where the phase variable also has the skip-free property. That is, the tridiagonal blocks in the generator matrix themselves each have a tridiagonal structure. We assume that the tridiagonal entries in the blocks are homogeneous (that is, phase independent). These processes are random walks in the quarter-plane. We restrict ourselves to this class of QBD processes because they allow us to take advantage of the properties of orthogonal polynomials. By using other methods to analyse (2.12), below, such as those, for example, in [13] and [14], it is likely that this restriction can be lifted. We generalize the analysis of [8], to which we will extensively refer.

The asymptotic behaviour of the stationary distribution for random walks in the quarter-plane has been studied extensively. See, for example, [3, pp. 145–148] and [15]. The results presented here differ from prior results in that we give the decay rates of the stationary distribution that may be obtained when the transition intensities along the level-boundary are varied.

Sufficient conditions under which the stationary distribution for the level process of a QBD process with countably many phases has a geometric tail were obtained in [21]. Similar conditions were obtained in [5] and [16]. The results obtained, however, did not provide a method of computation of the decay rate in question. Haque *et al.* [7] considered the class of processes studied here. They provided a method for computing the decay rate when certain restrictive conditions are met.

We show that for the class of QBD processes considered here there exist simple conditions on z for there to exist a z^{-1}-invariant measure of Neuts' $\boldsymbol{R}$-matrix which is positive and in ℓ^1 (the space of all complex sequences $\{x_n\}$ such that $\sum_{n=1}^{\infty} |x_n| < \infty$). These conditions can be expressed in terms of roots of polynomials in z and are of degree four at most. Depending on the transition structure at level 0, any of the z satisfying these conditions may give the decay rate of the stationary distribution of the level process (if it exists).

Haque *et al.* [7] stated sufficient conditions for the existence of an exact geometric tail asymptotic for the level process, equivalent to conditions in [21]. One of these conditions is z^{-1}-positivity of the $\boldsymbol{R}$-matrix. This is checked by solving a polynomial which is of degree four for our model. We observe that z^{-1}-positivity holds only for isolated values of z. In general, the conditions presented in this paper for finding possible decay rates z do not require z^{-1}-positivity. In this respect they generalise the previous work.

To illustrate our method, we apply the conditions obtained in this paper to find the possible decay rates for three examples that fall into the class of processes studied here: the two-demand model, where there are simultaneous arrivals to two independent queues; the two-node Jackson network; and a system of two queues each with its own server, which receives assistance from the other server when the latter is idle. The examples show that the set of possible decay rates may be either a single value, an interval of values, or a disjoint union of the two. Alternatively, the stationary distribution of the level process may not have a decay rate.

The rest of the paper is organized as follows. In Section 2, we summarise some general results for QBD processes with both finite and countable phase spaces and formulate the random walk in the quarter-plane as a QBD process. In Section 3, we present the main result and its proof. We provide conditions which must be satisfied by the decay rate for the stationary distribution of the level process. In Section 4, we investigate conditions for z^{-1}-positivity under which the results of [7] may be applied. In Section 5, we show how we can obtain different decay rates for the stationary distribution of the level process by controlling the transition structure at level 0. In Section 6, we apply the results of Section 3 to find a range of possible decay rates for the three examples mentioned above. In Section 7, we provide a concluding discussion and remarks.

2. Quasi-birth-and-death processes

A level-independent *quasi-birth-and-death* process (QBD process) is a continuous-time Markov chain $(Y_t, J_t : t \geq 0)$ on the state space $\{0, 1, \dots\} \times \{0, 1, \dots, m\}$ whose generator $\boldsymbol{Q}$ has a block tridiagonal representation

$$\boldsymbol{Q} = \begin{pmatrix} \tilde{\boldsymbol{Q}}_1 & \boldsymbol{Q}_0 & & & \\ \boldsymbol{Q}_2 & \boldsymbol{Q}_1 & \boldsymbol{Q}_0 & & \\ & \boldsymbol{Q}_2 & \boldsymbol{Q}_1 & \boldsymbol{Q}_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

The matrices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, and $\tilde{\boldsymbol{Q}}_1$ are of dimension $(m+1) \times (m+1)$. The random variable Y_t is called the *level* of the process at time t and the random variable J_t is called the *phase* of the process at time t. Thus, the parameter m gives the number of possible values of the phase variable, and may be either finite or infinite.

We assume that the QBD process is irreducible, aperiodic, and positive recurrent, and denote the limiting probabilities by $\pi_{kj} := \lim_{t\to\infty} \mathrm{P}(Y_t = k,\ J_t = j)$. Let $\boldsymbol{\pi}_k = (\pi_{k0}, \dots, \pi_{km})$, $k = 0, 1, \dots,$ and $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \dots)$. Then

$$\boldsymbol{\pi}_k = \boldsymbol{\pi}_0 \boldsymbol{R}^k, \qquad k \geq 0, \tag{2.2}$$

where $\boldsymbol{R}$ is the minimal nonnegative solution to the equation

$$\boldsymbol{Q}_0 + \boldsymbol{R}\boldsymbol{Q}_1 + \boldsymbol{R}^2\boldsymbol{Q}_2 = \boldsymbol{0}. \tag{2.3}$$

The following well-known ergodicity condition holds for both $m < \infty$ and $m = \infty$.

Theorem 2.1. *The QBD process is ergodic, that is, $\boldsymbol{\pi}$ is positive and has components which sum to unity, if and only if there exists a probability measure $\boldsymbol{y}_0$ such that*

$$\boldsymbol{y}_0(\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2) = \boldsymbol{0} \tag{2.4}$$

and

$$\boldsymbol{y}_0 \sum_{k=0}^{\infty} \boldsymbol{R}^k \boldsymbol{1} < \infty, \tag{2.5}$$

where $\boldsymbol{1}$ is a column vector of 1*s. In this case,*

$$\boldsymbol{\pi}_0 = \boldsymbol{y}_0 \Big(\boldsymbol{y}_0 \sum_{k=0}^{\infty} \boldsymbol{R}^k \boldsymbol{1} \Big)^{-1}.$$

The matrix $\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2$ in (2.4) is the generator of the process of (Y_t, J_t) filtered so that it is observed only when it is in level 0. Thus, the condition that there exists a probability measure satisfying (2.4) states that the filtered process at level 0 must be ergodic.

For $m < \infty$, (2.5) is satisfied if and only if

$$\mathrm{sp}(\boldsymbol{R}) < 1,$$

where $\mathrm{sp}(\boldsymbol{R})$ denotes the spectral radius of $\boldsymbol{R}$.

If $m < \infty$ and there exists a (row) vector $\boldsymbol{x}$ such that $\boldsymbol{x}\mathbf{1} = 1$,

$$\boldsymbol{x}(\boldsymbol{Q}_0 + \boldsymbol{Q}_1 + \boldsymbol{Q}_2) = \mathbf{0},$$

and

$$\boldsymbol{x}\boldsymbol{Q}_0\mathbf{1} < \boldsymbol{x}\boldsymbol{Q}_2\mathbf{1}, \tag{2.6}$$

then the QBD process is positive recurrent; see [17, pp. 82–83] and [11]. This was proved for $m = \infty$ in [22], under the additional assumption that $\tilde{\boldsymbol{Q}}_1 = \boldsymbol{Q}_1 + \boldsymbol{Q}_2$. Equation (2.6) can be interpreted as the requirement that 'the average drift of the level process is negative'.

We now consider the decay rate of the stationary distribution, assuming that the QBD process is ergodic. This decay rate is also known as the *caudal characteristic*. It is well known that for $m < \infty$ the geometric decay rate is given by the spectral radius of $\boldsymbol{R}$. In [9, pp. 204–205] it was shown that

$$\lim_{k\to\infty} \frac{\sum_i \pi_{ki}}{\mathrm{sp}(\boldsymbol{R})^k} = \kappa, \tag{2.7}$$

where κ is a constant. That is, the marginal stationary probability that the QBD process is in positive level k decays geometrically with rate $\mathrm{sp}(\boldsymbol{R})$.

There is no known 'infinite-dimensional' analogue of the limiting result (2.7) for $m = \infty$. In particular, the role of the spectral radius of $\boldsymbol{R}$ is not fully understood in this case.

We will require the following definitions.

Definition 2.1. Let $\boldsymbol{A}$ be a nonnegative, aperiodic, and irreducible square matrix of infinite or finite dimension. The power series

$$\sum_{k=0}^{\infty} A^k(i,j)\xi^k,$$

where $A^k(i,j)$ is the (i,j)th element of $\boldsymbol{A}^k$, has a convergence radius α, $0 \le \alpha \le \infty$, *independent* of i and j. This common convergence radius is called the *convergence parameter* of the matrix $\boldsymbol{A}$. When $\sum_{k=0}^{\infty} A^k(i,j)\alpha^k$ converges, the matrix is called *α-transient*; otherwise it is called *α-recurrent*. The α-recurrent case can further be split into the *α-null* and *α-positive* cases according to whether $\lim_{k\to\infty} A^k(i,j)\alpha^k$ is equal to 0 (α-null) or greater than 0 (α-positive).

The quantity $1/\alpha$ is called the *convergence norm* of $\boldsymbol{A}$. It can be shown to satisfy

$$1/\alpha = \lim_{k\to\infty} (A^k(i,j))^{1/k},$$

independently of i and j. This implies, in particular, that if the dimension of $\boldsymbol{A}$ is finite, then the convergence norm is exactly the Perron–Frobenius eigenvalue of $\boldsymbol{A}$ [20, pp. 200–201].

Definition 2.2. For $\beta > 0$, a nonnegative vector $\boldsymbol{x} \ne \mathbf{0}$ is called a *β-subinvariant measure* of $\boldsymbol{A}$ if $\beta\boldsymbol{x}\boldsymbol{A} \le \boldsymbol{x}$ and called a *β-invariant measure* of $\boldsymbol{A}$ if $\beta\boldsymbol{x}\boldsymbol{A} = \boldsymbol{x}$. Similarly, a nonnegative vector $\boldsymbol{y} \ne \mathbf{0}$ is called a *β-subinvariant vector* of $\boldsymbol{A}$ if $\beta\boldsymbol{A}\boldsymbol{y} \le \boldsymbol{y}$ and called a *β-invariant vector* of $\boldsymbol{A}$ if $\beta\boldsymbol{A}\boldsymbol{y} = \boldsymbol{y}$.

Definition 2.3. If

$$\lim_{k\to\infty} \frac{\boldsymbol{\pi}_k}{z^k} = \boldsymbol{w}, \tag{2.8}$$

elementwise, for some positive scalar z and positive row vector $\boldsymbol{w} \in \ell^1$, then we say that the decay rate of the stationary distribution $\boldsymbol{\pi}$ is z. If the limit in (2.8) does not exist for any positive z and $\boldsymbol{w} \in \ell^1$ then we say there is no decay rate.

Note that condition (2.8) differs from (2.7). However, if (2.8) holds then, by Fatou's lemma,

$$\liminf_{k\to\infty} \sum_i \frac{\pi_{ki}}{z^k} \geq \sum_i w_i,$$

and so, if it exists,

$$\lim_{k\to\infty} \frac{\sum_i \pi_{ki}}{z^k}$$

is positive and there will be a positive κ such that (2.7) is satisfied. Also, by using Fatou's lemma together with (2.2), we see that if $\boldsymbol{w}$ satisfies (2.8) then $\boldsymbol{w}$ must be a z^{-1}-subinvariant measure of $\boldsymbol{R}$. In this paper, we study conditions under which $\boldsymbol{w}$ is a z^{-1}-invariant measure of $\boldsymbol{R}$.

The following theorem was presented in [8].

Theorem 2.2. *Consider an irreducible QBD process with a finite or countable phase space. If there exist a nonnegative vector $\boldsymbol{w} \in \ell^1$ and a nonnegative number $z < 1$, such that*

$$\boldsymbol{w}(\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2) = \boldsymbol{0} \tag{2.9}$$

and

$$\boldsymbol{w}\boldsymbol{R} = z\boldsymbol{w}, \tag{2.10}$$

then the QBD process is ergodic, with $\boldsymbol{\pi}_0$ proportional to $\boldsymbol{w}$. Possibly by multiplying by a constant, we can choose $\boldsymbol{w}$ to satisfy

$$\frac{\boldsymbol{\pi}_k}{z^k} = \boldsymbol{w} \quad \text{for all } k.$$

It follows from Theorem 2.2 that if $\boldsymbol{\pi}_0$ is a z^{-1}-invariant measure of $\boldsymbol{R}$ for some z, then the stationary distribution of the QBD process possesses the *level–phase independence property* (see [10]) and decays at rate z.

From (2.3) we see that if the row vector $\boldsymbol{w}$ and scalar z satisfy $\boldsymbol{w}\boldsymbol{R} = z\boldsymbol{w}$, then for the general case, $m \leq \infty$, we have

$$\boldsymbol{w}(\boldsymbol{Q}_0 + z\boldsymbol{Q}_1 + z^2\boldsymbol{Q}_2) = \boldsymbol{0} \tag{2.11}$$

whenever the change of order of summation involved in using the associative law of matrix multiplication is permitted.

Under certain conditions the converse is also true for $m \leq \infty$. We will use the following theorem, Theorem 5.4 of [18], to determine z^{-1}-invariant measures of $\boldsymbol{R}$. For a more detailed analysis of the $m < \infty$ case, see [6].

Theorem 2.3. ([18, Theorem 5.4].) *Consider a continuous-time ergodic QBD process with generator of the form (2.1). Let $q_k = -Q_1(k,k)$. If the complex variable z and the vector $\boldsymbol{w} = \{w_k\}$ are such that $|z| < 1$ and $\sum_k |w_k| q_k < \infty$, then (2.11) implies (2.10).*

We now turn to a specific class of QBD processes where the number of phases is infinite, namely the random walks in the quarter-plane. We make the following assumption.

Assumption 2.1. *The tridiagonal blocks of the infinitesimal generator $\boldsymbol{Q}$ are themselves tridiagonal.*

https://doi.org/10.1239/aap/1151337083 Published online by Cambridge University Press

Thus, both variables, the level and the phase, possess the skip-free property. The transition rates are homogeneous if the level and phase are both greater than 0.

The infinite-dimensional matrices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_1$, $\boldsymbol{Q}_2$, and $\tilde{\boldsymbol{Q}}_1$ in (2.1) are given by

$$\boldsymbol{Q}_0 = \begin{pmatrix} \tilde{a}_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ & a_2 & a_1 & a_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \boldsymbol{Q}_1 = \begin{pmatrix} \tilde{b}_1 & b_0 & & & \\ b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\boldsymbol{Q}_2 = \begin{pmatrix} \tilde{c}_1 & c_0 & & & \\ c_2 & c_1 & c_0 & & \\ & c_2 & c_1 & c_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \qquad \tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} \bar{b}_1 & b_0 & & & \\ b_2 & \hat{b}_1 & b_0 & & \\ & b_2 & \hat{b}_1 & b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

where $a_0, a_1, \tilde{a}_1, a_2, b_0, b_2, c_0, c_1, \tilde{c}_1, c_2 \geq 0$ and

$$\begin{aligned} b_1 &= -(a_0 + a_1 + a_2 + b_0 + b_2 + c_0 + c_1 + c_2), \\ \tilde{b}_1 &= -(a_0 + \tilde{a}_1 + b_0 + c_0 + \tilde{c}_1), \\ \hat{b}_1 &= -(a_0 + a_1 + a_2 + b_0 + b_2), \\ \bar{b}_1 &= -(a_0 + \tilde{a}_1 + b_0). \end{aligned}$$

We assume that the process is ergodic. To study the decay rate of the QBD process we will make use of Theorem 2.3. Let

$$\begin{aligned} \gamma_i(z) &= a_i + b_i z + c_i z^2, \qquad i = 0, 1, 2, \\ \tilde{\gamma}_1(z) &= \tilde{a}_1 + \tilde{b}_1 z + \tilde{c}_1 z^2. \end{aligned}$$

For each z with $0 < |z| < 1$, let $\boldsymbol{Q}(z)$ be the infinite-dimensional tridiagonal matrix $\boldsymbol{Q}_0 + z\boldsymbol{Q}_1 + z^2\boldsymbol{Q}_2$, that is,

$$\boldsymbol{Q}(z) = \begin{pmatrix} \tilde{\gamma}_1(z) & \gamma_0(z) & & & \\ \gamma_2(z) & \gamma_1(z) & \gamma_0(z) & & \\ & \gamma_2(z) & \gamma_1(z) & \gamma_0(z) & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The significance of this matrix follows from Theorem 2.3. The infinite-dimensional row vector $\boldsymbol{w}$ satisfies $\boldsymbol{w}\boldsymbol{R} = z\boldsymbol{w}$, with $0 < |z| < 1$, if $\sum_k |w_k| q_k < \infty$ and $\boldsymbol{w}$ satisfies

$$\boldsymbol{w}\boldsymbol{Q}(z) = \boldsymbol{0}. \tag{2.12}$$

For our process, q_k is constant for $k \geq 1$ and, so, the condition that $\sum_k |w_k| q_k < \infty$ is equivalent to requiring that $\boldsymbol{w} \in \ell^1$. To avoid the trivial case in which $\boldsymbol{Q}(z)$ is a reducible matrix, we assume that both $\gamma_0(z) > 0$ and $\gamma_2(z) > 0$ for all $z > 0$.

3. Conditions for the decay rate

The following theorem provides some simple conditions that must be satisfied by any decay rate z for which there exists a positive z^{-1}-invariant measure of $\boldsymbol{R}$ in ℓ^1. The conditions are stated in terms of quadratic and quartic polynomial inequalities in the decay rate.

Let

$$\tau(z) = \gamma_1(z) + 2\sqrt{\gamma_0(z)\gamma_2(z)}, \tag{3.1}$$

$$\upsilon(z) = \gamma_2(z) + \min(\gamma_0(z), \gamma_2(z)), \tag{3.2}$$

$$\chi(z) = \tilde{\gamma}_1(z) + \frac{\gamma_0(z)\gamma_2(z)}{\tilde{\gamma}_1(z) - \gamma_1(z)}. \tag{3.3}$$

Theorem 3.1. *For the QBD process described in Section 2, the system of equations*

$$\boldsymbol{w}\boldsymbol{R} = z\boldsymbol{w} \tag{3.4}$$

has positive solutions $\boldsymbol{w} \in \ell^1$ for $0 < z < 1$ if and only if z is such that

(i) $\tau(z) \le 0$ *if* $(\tilde{\gamma}_1(z) - \gamma_1(z))^2 \le \gamma_0(z)\gamma_2(z)$ *and* $\chi(z) \le 0$ *otherwise, and either*

(ii) $\tau(z) \ge 0$ *and* $\gamma_0(z) < \gamma_2(z)$, *or*

(iii) $\tau(z) < 0$ *and*

$$\begin{aligned} \gamma_2(z) &> -\tilde{\gamma}_1(z) \quad \text{if } \chi(z) = 0, \\ \upsilon(z) &> -\gamma_1(z) \quad \text{otherwise.} \end{aligned}$$

For $0 < z < 1$, condition (i) of Theorem 3.1 is necessary and sufficient for (3.4) to have a positive solution, while this solution is in ℓ^1 if and only if either condition (ii) or (iii) is satisfied.

Theorem 3.1 follows from Theorem 2.3. We will defer its proof until after some preliminary analysis of the model, and some auxiliary lemmas. Our analysis closely follows that of [8, Section 4].

Consider the system of equations (2.12) for a fixed z such that $0 < |z| < 1$. Writing out the system, we have

$$w_0\tilde{\gamma}_1(z) + w_1\gamma_2(z) = 0, \tag{3.5}$$

$$w_k\gamma_0(z) + w_{k+1}\gamma_1(z) + w_{k+2}\gamma_2(z) = 0, \qquad k = 0, 1, \dots. \tag{3.6}$$

After substituting $w_k = u^k$ into (3.6), we derive the characteristic equation,

$$\gamma_2(z)u^2 + \gamma_1(z)u + \gamma_0(z) = 0. \tag{3.7}$$

The discriminant of (3.7) is $D(z) = \gamma_1^2(z) - 4\gamma_0(z)\gamma_2(z) = \tau(z)\sigma(z)$, where $\tau(z)$ is as defined in (3.1) and

$$\sigma(z) = \gamma_1(z) - 2\sqrt{\gamma_0(z)\gamma_2(z)}. \tag{3.8}$$

The discriminant $D(z)$ is, in general, a quartic in z and, so, has at most four roots. The solution to (3.6) takes one of three possible forms depending on whether $D(z)$ is positive, negative, or equal to 0. Only two of these three forms, however, may provide a positive solution and, thus, a z^{-1}-invariant measure of $\boldsymbol{R}$.

If $D(z) > 0$ then we obtain a solution to (3.6) of the form

$$w_k = c_1u_1^k + c_2u_2^k, \tag{3.9}$$

where

$$u_1 = \frac{1}{2\gamma_2(z)}(-\gamma_1(z) + \sqrt{D(z)}), \qquad u_2 = \frac{1}{2\gamma_2(z)}(-\gamma_1(z) - \sqrt{D(z)}).$$

The coefficients c_1 and c_2 can be derived from

$$c_1 + c_2 = 1, \tag{3.10}$$

$$c_1 u_1 + c_2 u_2 = -\frac{\tilde{\gamma}_1(z)}{\gamma_2(z)}, \tag{3.11}$$

where (3.10) follows from (arbitrarily) setting $w_0 = 1$ and (3.11) follows from the boundary condition (3.5). Thus, we find that

$$c_1 = \frac{1}{2} + \frac{\gamma_1(z) - 2\tilde{\gamma}_1(z)}{2\sqrt{D(z)}}, \qquad c_2 = \frac{1}{2} - \frac{\gamma_1(z) - 2\tilde{\gamma}_1(z)}{2\sqrt{D(z)}}. \tag{3.12}$$

If $D(z) = 0$ then we obtain a solution to (3.6) of the form

$$w_k = u^k(1 + ck), \tag{3.13}$$

where if $\tau(z) = 0$ then $u = \sqrt{\gamma_0(z)/\gamma_2(z)}$ and

$$c = -1 - \frac{\tilde{\gamma}_1(z)}{\sqrt{\gamma_0(z)\gamma_2(z)}},$$

and if $\sigma(z) = 0$ then $u = -\sqrt{\gamma_0(z)/\gamma_2(z)}$ and

$$c = -1 + \frac{\tilde{\gamma}_1(z)}{\sqrt{\gamma_0(z)\gamma_2(z)}}.$$

If $D(z) < 0$ then we obtain a solution to (3.6) of the form

$$w_k = (\cos(k\phi) + c\sin(k\phi))|u|^k, \tag{3.14}$$

where $|u| = \sqrt{\gamma_0(z)/\gamma_2(z)}$,

$$\phi = \arctan\left(\frac{\sqrt{-D(z)}}{-\gamma_1(z)}\right), \quad \text{and} \quad c = \frac{-\tilde{\gamma}_1(z) - \sqrt{\gamma_0(z)\gamma_2(z)}\cos\phi}{\sqrt{\gamma_0(z)\gamma_2(z)}\sin\phi}.$$

It is clear that a solution to (3.6) of the form (3.14) cannot be positive and, thus, cannot be a z^{-1}-invariant measure of $\boldsymbol{R}$. However, we will defer the question of whether $\boldsymbol{w}$ is positive, and consider solutions of all three forms, (3.9), (3.13), and (3.14), in determining whether $\boldsymbol{w}$ is an element of ℓ^1.

Lemma 3.1. *The vector $\boldsymbol{w}$ is an element of ℓ^1 for $z > 0$ if and only if*

(i) *$D(z) \leq 0$ and $\gamma_0(z) < \gamma_2(z)$, or*

(ii) *$D(z) > 0$ and*

  (a) *$\gamma_2(z) > |\tilde{\gamma}_1(z)|$ if $\chi(z) = 0$,*

  (b) *$\upsilon(z) > |\gamma_1(z)|$ otherwise.*

Proof. To prove (i), note that if $D(z) \leq 0$ then the form of (3.13) and (3.14) shows that $\boldsymbol{w} \in \ell^1$ if and only if $\gamma_0(z) < \gamma_2(z)$.

Now consider the $D(z) > 0$ case. If either of the coefficients, c_1 or c_2, are 0 (they cannot both be), then the root of (3.7) with nonzero coefficient is given by $-\tilde{\gamma}_1(z)/\gamma_2(z)$. Thus, for $\boldsymbol{w}$ to be in ℓ^1 it is necessary and sufficient that $|-\tilde{\gamma}_1(z)/\gamma_2(z)| < 1$.

From (3.12), we see that the coefficient c_1 is 0 if and only if

$$2\tilde{\gamma}_1(z) - \gamma_1(z) - \sqrt{D(z)} = 0. \tag{3.15}$$

The coefficient c_2 is 0 if and only if

$$2\tilde{\gamma}_1(z) - \gamma_1(z) + \sqrt{D(z)} = 0. \tag{3.16}$$

From (3.15) and (3.16), we find that one of the coefficients, c_1 or c_2, is 0 if and only if

$$\gamma_0(z)\gamma_2(z) + \tilde{\gamma}_1(z)(\tilde{\gamma}_1(z) - \gamma_1(z)) = 0. \tag{3.17}$$

Since $\gamma_0(z)\gamma_2(z) > 0$ for $z > 0$, (3.17) is equivalent to $\chi(z) = 0$.

Consider the case in which c_1 and c_2 are both nonzero. For $\boldsymbol{w}$ to be in ℓ^1 it is necessary and sufficient that both u_1 and u_2 be in $(-1, 1)$. To study when this is the case, denote by $f(u)$ the left-hand side of (3.7). Since the coefficient of u^2 in $f(u)$ is $\gamma_2(z) > 0$ for $z > 0$, the statement that both u_1 and u_2 are in $(-1, 1)$ is equivalent to saying that $f(-1) > 0$, $f(1) > 0$, $f'(-1) < 0$, and $f'(1) > 0$.

From $f(1) > 0$, we obtain

$$\gamma_0(z) + \gamma_1(z) + \gamma_2(z) > 0. \tag{3.18}$$

From $f(-1) > 0$, we obtain

$$\gamma_0(z) - \gamma_1(z) + \gamma_2(z) > 0. \tag{3.19}$$

From $f'(1) > 0$, we obtain

$$\gamma_1(z) + 2\gamma_2(z) > 0. \tag{3.20}$$

From $f'(-1) < 0$, we obtain

$$\gamma_1(z) - 2\gamma_2(z) < 0. \tag{3.21}$$

If $\gamma_0(z) < \gamma_2(z)$ then condition (3.20) follows immediately from condition (3.18). Likewise, condition (3.21) follows immediately from condition (3.19). If $\gamma_0(z) \geq \gamma_2(z)$ then conditions (3.18) and (3.19) follow from conditions (3.20) and (3.21), respectively. We can thus rewrite conditions (3.18)–(3.21) more compactly as

$$\upsilon(z) > |\gamma_1(z)|,$$

where $\upsilon(z)$ is as defined by (3.2), proving Lemma 3.1.

The vector $\boldsymbol{w}$ must be nonnegative in order for us to be able to apply Theorem 2.3. We investigate this by generalising equations (3.5) and (3.6) to

$$P_0(x; z) = 1, \tag{3.22}$$

$$\gamma_2(z)P_1(x; z) = x - \tilde{\gamma}_1(z), \tag{3.23}$$

$$\gamma_2(z)P_n(x; z) = (x - \gamma_1(z))P_{n-1}(x; z) - \gamma_0(z)P_{n-2}(x; z), \qquad n \geq 2. \tag{3.24}$$

Equations (3.22)–(3.24) define a sequence of *orthogonal polynomials* $P_n(x; z)$ for any given real and positive value of z. It is clear that $w_n = P_n(0; z)$. The $P_n(0; z)$ are positive for all n if and only if the zeros of all the $P_n(x; z)$ are less than 0. This enables us to study conditions for the positivity of $\boldsymbol{w}$ via the properties of the polynomials $P_n(0; z)$.

Lemma 3.2. *For $z > 0$, the sequence $\{P_n(x; z)\}$ satisfies the orthogonality relationship*

$$\int_{\mathrm{supp}(\psi)} P_n(x; z) P_m(x; z) \psi(\mathrm{d}x) = \left(\frac{\gamma_0(z)}{\gamma_2(z)}\right)^n \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta symbol and

$$\mathrm{supp}(\psi) = \begin{cases} [\sigma(z), \tau(z)] & \text{if } (\tilde{\gamma}_1(z) - \gamma_1(z))^2 \le \gamma_0(z)\gamma_2(z), \\ [\sigma(z), \tau(z)] \cup \{\chi(z)\} & \text{otherwise,} \end{cases}$$

and $\tau(z)$, $\sigma(z)$, and $\chi(z)$ are as given in (3.1), (3.8), and (3.3), respectively. The measure ψ is given by

$$\psi(\mathrm{d}x) = \frac{\sqrt{\gamma_0(z)\gamma_2(z)}\sqrt{4\gamma_0(z)\gamma_2(z) - (x - \gamma_1(z))^2}}{\pi(\gamma_0(z)\gamma_2(z) + (\tilde{\gamma}_1(z) - \gamma_1(z))(\tilde{\gamma}_1(z) - x))}\,\mathrm{d}x, \qquad \sigma \le x \le \tau,$$

$$\psi(\{\chi(z)\}) = 1 - \frac{\gamma_0(z)\gamma_2(z)}{(\tilde{\gamma}_1(z) - \gamma_1(z))^2} \quad \text{if } (\tilde{\gamma}_1(z) - \gamma_1(z))^2 > \gamma_0(z)\gamma_2(z).$$

Proof. For a fixed $z > 0$, let

$$T_n(x) = \left(\sqrt{\frac{\gamma_2(z)}{\gamma_0(z)}}\right)^n P_n(2x\sqrt{\gamma_0(z)\gamma_2(z)} + \gamma_1(z); z). \tag{3.25}$$

It follows that

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= 2x - b, \\ T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x), \qquad n \ge 2, \end{aligned}$$

where

$$b = \frac{\tilde{\gamma}_1(z) - \gamma_1(z)}{\sqrt{\gamma_0(z)\gamma_2(z)}}.$$

The T_n are *perturbed Chebyshev polynomials*, for which the orthogonalizing relationship is given (see [2, pp. 204, 205]) by

$$\begin{aligned} &\frac{2}{\pi}\int_{-1}^{1} T_n(x)T_m(x)\frac{\sqrt{1-x^2}}{1+b^2-2bx}\,\mathrm{d}x + 1_{\{|b|>1\}}T_n\left(\frac{b}{2}+\frac{1}{2b}\right)T_m\left(\frac{b}{2}+\frac{1}{2b}\right)(1-b^{-2}) \\ &\quad = \delta_{n,m}, \end{aligned}$$

where $1_{\{|b|>1\}} = 1$ if $|b| > 1$ and $1_{\{|b|>1\}} = 0$ otherwise. Substituting (3.25) into this identity yields the result.

We have the following lemma as a consequence.

Lemma 3.3. *For each value of $z > 0$, $P_n(x; z)$ has n distinct real zeros, $x_{n,1}, \dots, x_{n,n}$, $x_{n,1} < \cdots < x_{n,n}$, and these zeros interlace. That is, for all $n \ge 2$ and $i = 1, \dots, n-1$,*

$$x_{n,i} < x_{n-1,i} < x_{n,i+1}.$$

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Proof. The lemma follows from a well-known result on orthogonal polynomial sequences; see Theorem I.5.3 of [2].

The support of the measure ψ and the limiting behaviour of the zeros of the $P_n(x; z)$ are related. Some useful results are stated in the following lemma.

Lemma 3.4. *The sequences of smallest, second-largest, and largest zeros of the $P_n(x; z)$ possess the following properties.*

- $\{x_{n,1}\}_{n=1}^{\infty}$ *is a strictly decreasing sequence with limit $\sigma(z)$.*
- $\{x_{n,n-1}\}_{n=1}^{\infty}$ *is a strictly increasing sequence with limit $\tau(z)$.*
- $\{x_{n,n}\}_{n=1}^{\infty}$ *is a strictly increasing sequence with limit $\chi_1(z)$, where*

$$\chi_1(z) = \sup(\operatorname{supp}(\psi)) = \begin{cases} \tau(z) & \text{if } (\tilde{\gamma}_1(z) - \gamma_1(z))^2 \le \gamma_0(z)\gamma_2(z), \\ \chi(z) & \text{otherwise}. \end{cases}$$

Proof. See Section II.4 of [2].

Lemma 3.5. *Let $z > 0$. Then $P_n(x; z)$ is positive for all n if and only if $x \ge \chi_1(z)$.*

Proof. The leading coefficient of $P_n(x; z)$ is positive for all n (since $\gamma_2(z) > 0$ for $z > 0$). This implies that $P_n(x; z)$ is positive for $x > x_{n,n}$. Since $x_{n,n}$ is strictly increasing, we know that $P_n(x; z)$ is positive for all n if $x \ge \chi_1(z)$. Conversely, $P_k(x; z)$ is negative for $x \in (x_{k-1,k}, x_{k,k})$, so the interlacing property given in Lemma 3.3 implies that, for every $x < x_{n,n}$, $P_k(x; z)$ is less than 0 for at least one $k \in \{1, \dots, n\}$. Thus, if $x < \chi_1(z)$ then $P_k(x; z)$ is less than 0 for at least one $k \in \mathbb{Z}_+$.

We are now in a position to say when the vector $\boldsymbol{w}$ solving (3.5) and (3.6) is positive.

Lemma 3.6. *The vector $\boldsymbol{w}$ is positive if and only if $\chi_1(z) \le 0$.*

Proof. This follows immediately from Lemma 3.5 and the fact that $w_n = P_n(0; z)$ for a given z.

Having established conditions for the vector $\boldsymbol{w}$ to be in ℓ^1 and positive, we are in a position to provide a proof of Theorem 3.1.

Proof of Theorem 3.1. Theorem 3.1 follows from Theorem 2.3 by combining the conditions of Lemmas 3.1 and 3.6.

We have assumed that $\gamma_0(z), \gamma_2(z) > 0$ for $z > 0$. The positivity condition on $\boldsymbol{w}$ then implies that $\tilde{\gamma}_1(z), \gamma_1(z) < 0$, as follows. If we (arbitrarily) set $w_0 = 1$, then from (3.5) we have $w_1 = -\tilde{\gamma}_1(z)/\gamma_2(z)$. Thus, to have $w_1 > 0$ we require $\tilde{\gamma}_1(z) < 0$. From (3.6) we also have

$$w_2 = \frac{1}{\gamma_2(z)}\left(\frac{\tilde{\gamma}_1(z)\gamma_1(z)}{\gamma_2(z)} - \gamma_0(z)\right).$$

Thus, to have $w_2 > 0$ we require $\gamma_1(z) < 0$. This means we must have $\sigma(z) < 0$, which implies that $D(z) = 0$ is equivalent to $\tau(z) = 0$, that $D(z) > 0$ is equivalent to $\tau(z) < 0$, and that $D(z) < 0$ is equivalent to $\tau(z) > 0$. This concludes the proof of Theorem 3.1.

Theorem 3.1, along with Theorem 2.2, indicates that it might be possible for the QBD process to have level–phase-independent stationary distributions for a range of different values of z. For this to occur, the vector $\boldsymbol{w}$ that satisfies (2.10) must also satisfy (2.9).

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In Section 5, we show that it is possible to vary $\tilde{\boldsymbol{Q}}_1$ and, thus, (2.9), to obtain a vector $\boldsymbol{w}$ that satisfies both (2.9) and (2.10).

4. Conditions for z^{-1}-positivity

We now investigate conditions for z^{-1}-positivity of the $\boldsymbol{R}$-matrix for QBD processes studied in this paper. Our motivation stems from the technique of [7] that tests for z^{-1}-positivity. If the $\boldsymbol{R}$-matrix is not z^{-1}-positive then the technique of [7] does not tell us anything about the geometric decay rate.

First, we state the following theorem, which is Theorem 6.4 of [20].

Theorem 4.1. ([20, Theorem 6.4].) *Suppose that $\boldsymbol{x}$ is a β-invariant measure of $\boldsymbol{A}$ and $\boldsymbol{y}$ a β-invariant vector of $\boldsymbol{A}$. Then $\boldsymbol{A}$ is β-positive if*

$$\boldsymbol{x}\boldsymbol{y} := \sum_i x_i y_i < \infty.$$

Conversely, if $\boldsymbol{A}$ is β-positive and $\boldsymbol{x}$ and $\boldsymbol{y}$ are respectively a β-invariant measure and vector of $\boldsymbol{A}$, then $\boldsymbol{x}\boldsymbol{y} < \infty$.

We will use Theorem 4.1 directly to find conditions for z^{-1}-positivity of the $\boldsymbol{R}$-matrix.

Theorem 4.2. *The $\boldsymbol{R}$-matrix of the QBD process described in Section 2 is z^{-1}-positive, for $0 < z < 1$, if and only if*

$$\tau(z) < 0, \tag{4.1}$$

$$2\tilde{\gamma}_1(z) - \gamma_1(z) - \sqrt{\gamma_1^2(z) - 4\gamma_0(z)\gamma_2(z)} = 0. \tag{4.2}$$

If (4.1) and (4.2) hold then the z^{-1}-invariant measure of $\boldsymbol{R}$ is an element of ℓ^1 if and only if $\gamma_2(z) > -\tilde{\gamma}_1(z)$.

Proof. The characteristic equations of the z^{-1}-invariant measure $\boldsymbol{w}$ and vector $\boldsymbol{v}$ of $\boldsymbol{R}$ are given by (3.7) and

$$\gamma_0(z)u^2 + \gamma_1(z)u + \gamma_2(z) = 0, \tag{4.3}$$

respectively. The measure $\boldsymbol{w}$ and vector $\boldsymbol{v}$ also satisfy the boundary equations (3.5) and

$$v_0\tilde{\gamma}_1(z) + v_1\gamma_0(z) = 0,$$

respectively. Clearly, (3.7) and (4.3) have the same discriminant $D(z)$, and the roots of (3.7) are the reciprocals of the roots of (4.3). As mentioned in Section 3, the solutions for $\boldsymbol{w}$ and $\boldsymbol{v}$ may take one of three possible forms depending on $D(z)$, two of which may provide a nonnegative solution. We (arbitrarily) assume that $w_0 = v_0 = 1$.

If $D(z) < 0$ then $\boldsymbol{w}$ and $\boldsymbol{v}$ are both of the form (3.14). Let

$$w_k = (\cos(k\phi_1) + c_1\sin(k\phi_1))|u_1|^k,$$
$$v_k = (\cos(k\phi_2) + c_2\sin(k\phi_2))|u_2|^k,$$

with $|u_1| = 1/|u_2|$, $\phi_1 = \phi_2$, and $c_1 = c_2$. It is clear that neither $\boldsymbol{w}$ nor $\boldsymbol{v}$ is a nonnegative vector, so we do not have z^{-1}-positivity. It can also be seen that

$$\boldsymbol{w}\boldsymbol{v} = \sum_{k=0}^{\infty}(\cos(k\phi_1) + c_1\sin(k\phi_1))^2 = \infty.$$

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Similarly, if $D(z) = 0$ then $\boldsymbol{w}$ and $\boldsymbol{v}$ are both of the form (3.13). Let

$$w_k = (1 + c_1 k)u_1^k, \qquad v_k = (1 + c_2 k)u_2^k,$$

with $u_1 = 1/u_2$ and $c_1 = c_2$. Then

$$\boldsymbol{w}\boldsymbol{v} = \sum_{k=0}^{\infty}(1 + c_1 k)^2 = \infty.$$

Now, if $D(z) > 0$ (that is, $\tau(z) < 0$), then $\boldsymbol{w}$ and $\boldsymbol{v}$ are both of the form (3.9). Let

$$w_k = c_1 u_1^k + c_2 u_2^k, \qquad v_k = c_3 u_3^k + c_4 u_4^k,$$

with $u_1 = 1/u_4$, $u_2 = 1/u_3$, $c_1 = c_3$, and $c_2 = c_4$. Then

$$\begin{aligned} w_k v_k &= (c_1 u_1^k + c_2 u_2^k)(c_3 u_3^k + c_4 u_4^k) \\ &= 2c_1 c_2 + c_1^2 (u_1/u_2)^k + c_2^2 (u_2/u_1)^k, \end{aligned}$$

with $u_1/u_2 > 1$, and, thus, $\boldsymbol{w}\boldsymbol{v} < \infty$ if and only if $c_1 = 0$. From (3.12) it follows that $c_1 = 0$ if and only if (4.2) holds.

Equation (4.2) ensures that $\chi(z) = 0$. Thus, by Lemma 3.6, (4.1) and (4.2) ensure that $\boldsymbol{w}$ and $\boldsymbol{v}$ are positive. Furthermore, by Lemma 3.1, $\boldsymbol{w} \in \ell^1$ if and only if

$$\gamma_2(z) > -\tilde{\gamma}_1(z).$$

Remark 4.1. Theorem 4.2 tells us that $\boldsymbol{R}$ is z^{-1}-positive only for isolated values of z. This also follows from the theory of nonnegative matrices [20, pp. 206–207]. The $\boldsymbol{R}$-matrix can be z^{-1}-positive if and only if z is the convergence norm of $\boldsymbol{R}$.

The authors of [7] considered a model that is essentially the same as the model presented in Section 3 (although it is in discrete time and the transition rates at the boundary may differ). They stated conditions for $\boldsymbol{\pi}_k$ to have an exact geometric tail asymptotic. These conditions are met only if there exists a z^{-1}-invariant measure $\boldsymbol{w}$ and vector $\boldsymbol{v}$ of $\boldsymbol{R}$ with $\boldsymbol{w}\boldsymbol{v} < \infty$ and $\boldsymbol{w} \in \ell^1$, for some decay rate z with $0 < z < 1$. A polynomial of degree eight must be solved in order to determine the values of the decay rate for which the conditions of [7] are met. When the transition rates at the boundary are the same as for our model in Section 3, the polynomial reduces to a quartic. By uniformizing our model to discrete time, we see that this quartic equation (Equation (9) of [7]) is

$$\gamma_0(z)\gamma_2(z)(\gamma_0(z)\gamma_2(z) + \tilde{\gamma}_1(z)[\tilde{\gamma}_1(z) - \gamma_1(z)]) = 0. \tag{4.4}$$

For $z > 0$, (4.4) is equivalent to $\chi(z) = 0$.

By Theorem 4.2, it is clear that the conditions of [7] are met only if (4.1), (4.2), and

$$\gamma_2(z) > -\tilde{\gamma}_1(z)$$

are satisfied. The roots of (4.2) are roots of (4.4), but not all roots of (4.4) are roots of (4.2). This makes sense, since in [7] the decay rate z must not only satisfy (4.4), but also a further condition (Equation (11) of [7]).

5. Varying the decay rate

In Section 3, we saw that there may be a range of possible values of z for which (2.10) has a positive solution, $\boldsymbol{w}$, in ℓ^1. Theorem 2.2 suggests that it may be possible to change the transition intensities at level 0, that is, change the entries in $\tilde{\boldsymbol{Q}}_1$ and, thus, (2.9), to ensure that the stationary distribution decays at a rate that is given by any of the possible values of z. We do this by finding the vector $\boldsymbol{w}$ that satisfies (2.10) and then adjusting $\tilde{\boldsymbol{Q}}_1$ so that this same $\boldsymbol{w}$ satisfies (2.9). Below we present two examples of how we may go about changing $\tilde{\boldsymbol{Q}}_1$ to achieve this. The examples below are generalizations of the examples presented in [8] for the two-node tandem Jackson network. In each of these, the modification to $\tilde{\boldsymbol{Q}}_1$ preserves the fact that it is tridiagonal. Other modifications are of course possible.

Example 5.1. We wish to have a decay rate z satisfying the conditions of Theorem 3.1. We replace each b_0 in $\tilde{\boldsymbol{Q}}_1$ by a phase dependent $b_0^{(k)}$. That is, $\tilde{\boldsymbol{Q}}_1$ now has the structure

$$\tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} b_1^{(0)} & b_0^{(0)} & & & \\ b_2 & b_1^{(1)} & b_0^{(1)} & & \\ & b_2 & b_1^{(2)} & b_0^{(2)} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We define the $b_0^{(k)}$ recursively by

$$b_0^{(0)} = z\tilde{c}_1 - a_0 - \tilde{a}_1 + \frac{w_1}{w_0}(b_2 + zc_2), \tag{5.1}$$

$$b_0^{(k)} = zc_1 - a_0 - a_1 - a_2 - b_2 + \frac{w_{k-1}(b_0^{(k-1)} + zc_0) + w_{k+1}(b_2 + zc_2)}{w_k}, \qquad k = 1, 2, \dots. \tag{5.2}$$

Changing $\tilde{\boldsymbol{Q}}_1$ in this way is appropriate only if the sequence $\{b_0^{(k)}\}_{k=0}^{\infty}$ is nonnegative. It can easily be seen that this is the case if

$$a_0 + a_1 + a_2 \le z(c_0 + c_1 + c_2), \qquad a_0 + \tilde{a}_1 \le z(c_0 + \tilde{c}_1),$$

and b_2 is sufficiently large. It is possible, however, for the sequence $\{b_0^{(k)}\}_{k=0}^{\infty}$ to be nonnegative even if these simple conditions are not met, as was shown for the case of the two-node tandem Jackson network in [8].

The recursion equations (5.1) and (5.2) ensure that

$$\boldsymbol{w}(\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2) = \boldsymbol{0},$$

whence, by (2.4) and Theorem 2.1, it follows that the stationary distribution, $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \dots)$, of (Y_t, J_t) is given by

$$\boldsymbol{\pi}_n = c\boldsymbol{w}\boldsymbol{R}^n = z^n c\boldsymbol{w}, \qquad n \ge 0,$$

for some normalizing constant c. It is clear that the decay rate is now z.

This example shows that by changing the transition intensities at level 0 such that they become dependent on the phase, it may be possible to produce a different decay rate satisfying the conditions of Theorem 3.1.

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Example 5.2. Again, we wish to have a decay rate satisfying the conditions in Theorem 3.1. We now leave the b_0 in $\tilde{\boldsymbol{Q}}_1$ unchanged, but replace each b_2 by a phase dependent $b_2^{(k)}$. That is, $\tilde{\boldsymbol{Q}}_1$ now has the structure

$$\tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} \bar{b}_1 & b_0 & & & \\ b_2^{(1)} & b_1^{(1)} & b_0 & & \\ & b_2^{(2)} & b_1^{(2)} & b_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We define the $b_2^{(k)}$ recursively by

$$b_2^{(1)} = \frac{w_0}{w_1}(a_0 + \tilde{a}_1 + b_0 - z\tilde{c}_1) - zc_2, \tag{5.3}$$

$$b_2^{(k+1)} = \frac{w_k(a_0 + a_1 + a_2 + b_0 + b_2^{(k)} - zc_1) - w_{k-1}(b_0 + zc_0)}{w_{k+1}} - zc_2, \qquad k = 1, 2, \dots. \tag{5.4}$$

As before, it is appropriate to change $\tilde{\boldsymbol{Q}}_1$ in this way only if the sequence $\{b_2^{(k)}\}_{k=1}^{\infty}$ is nonnegative. This is clearly the case if

$$a_0 + a_1 + a_2 \geq z(c_0 + c_1 + c_2), \qquad a_0 + \tilde{a}_1 \geq z(c_0 + \tilde{c}_1),$$

and b_0 is sufficiently large, but again may also be true if these conditions do not hold, as was shown for the two-node tandem Jackson network in [8].

The recursion equations (5.3) and (5.4) ensure that $\boldsymbol{w}(\tilde{\boldsymbol{Q}}_1 + \boldsymbol{R}\boldsymbol{Q}_2) = \boldsymbol{0}$, whence the stationary distribution of (Y_t, J_t) is given by

$$\boldsymbol{\pi}_n = c\boldsymbol{w}\boldsymbol{R}^n = z^n c\boldsymbol{w}, \qquad n \geq 0,$$

for some normalizing constant c. Thus, it is clear that z is the decay rate in this model.

We have again shown how it may be possible to change $\tilde{\boldsymbol{Q}}_1$ so as to produce a different decay rate satisfying the conditions of Theorem 3.1. The forms of the recursion equations in Examples 5.1 and 5.2 indicate that if changing $\tilde{\boldsymbol{Q}}_1$ as in Example 5.1 does not produce proper (nonnegative) transition intensities, then it may be possible to do so by changing $\tilde{\boldsymbol{Q}}_1$ as in Example 5.2, and vice versa.

6. Examples

We now demonstrate the application of Theorem 3.1 by considering some specific models which can be formulated as QBD processes with a generator of the form specified in Section 2. In each case, we investigate possible decay rates, z, given by Theorem 3.1. We also investigate the z^{-1}-positivity of each of the models by applying Theorem 4.2.

Example 6.1. (*Two-demand model.*) We consider the two-demand model [4], also considered in [7]. A double queue arises when customers arriving at the system simultaneously place two demands on two different servers working independently. The customer arrivals form a Poisson process with rate 1, and the service times of the two servers are independent and exponential with rates α and β, respectively. Let $X_1(t)$ and $X_2(t)$ represent the number of customers waiting or in service at time t in queues 1 and 2, respectively. We consider the two-dimensional Markov chain $(X_1(t), X_2(t))$ with state space $\mathbb{Z}_+ \times \mathbb{Z}_+$, viewed as a QBD process in which $X_1(t)$ represents the level and $X_2(t)$ represents the phase. Thus, the decay rate we are seeking is that experienced by the number of customers in the first queue. The Markov chain is stable if and only if $\alpha > 1$ and $\beta > 1$, which we assume to hold.

The tridiagonal blocks of the generator $\boldsymbol{Q}$ are given by the infinite-dimensional matrices

$$\boldsymbol{Q}_0 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix}, \qquad \boldsymbol{Q}_2 = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \alpha & \\ & & & \ddots \end{pmatrix},$$

$$\boldsymbol{Q}_1 = \begin{pmatrix} -(1+\alpha) & & & \\ \beta & -(1+\alpha+\beta) & & \\ & \beta & -(1+\alpha+\beta) & \\ & & \ddots & \ddots \end{pmatrix},$$

$$\tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} -1 & & & \\ \beta & -(1+\beta) & & \\ & \beta & -(1+\beta) & \\ & & \ddots & \ddots \end{pmatrix}.$$

We have

$$\gamma_0(z) = 1, \qquad \gamma_1(z) = \alpha z^2 - (1+\alpha+\beta)z,$$
$$\gamma_2(z) = \beta z, \qquad \tilde{\gamma}_1(z) - \gamma_1(z) = \beta z.$$

Thus, we obtain

$$\tau(z) = \alpha z^2 - (1+\alpha+\beta)z + 2\sqrt{\beta z},$$
$$\chi(z) = \alpha z^2 - (1+\alpha)z + 1, \qquad z \neq 0,$$
$$\upsilon(z) = \min(\beta z + 1, 2\beta z).$$

By applying Theorem 3.1, we obtain the following result for the possible decay rate. The details of the derivation are presented in Appendix A.

Proposition 6.1. *If $\alpha < \beta$ then the only value of $z \in (0, 1)$ for which (2.10) is satisfied by a positive vector $\boldsymbol{w} \in \ell^1$ is $z = 1/\alpha$. If $\alpha \geq \beta$ then there are no values of $z \in (0, 1)$ for which (2.10) is satisfied by a positive vector $\boldsymbol{w} \in \ell^1$.*

By Theorem 4.2, it is possible for the $\boldsymbol{R}$-matrix to be z^{-1}-positive if and only if $z = 1/\alpha$ and $\alpha > \beta$. Taking into account Proposition 6.1, we see there are no possible decay rates, z, for which the $\boldsymbol{R}$-matrix is z^{-1}-positive. This agrees with the results of [7] and [4].

Example 6.2. (*Two-node Jackson network.*) We consider the two-node Jackson network. The effect of a finite buffer truncation in the two-node Jackson network was studied in [19]. There it was found that the decay rate of the stationary distribution of the two-node Jackson network may not be well approximated by using a finite buffer truncation. These results generalized the results of [8] for the two-node tandem Jackson network.

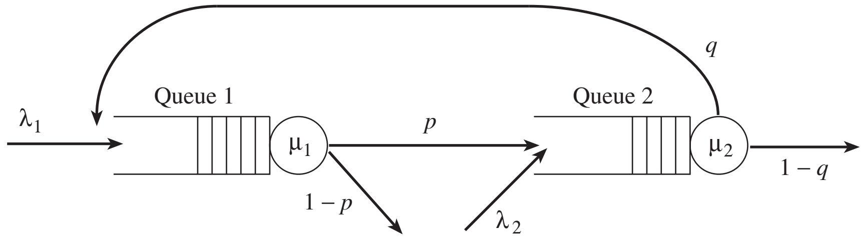

FIGURE 1: The two-node Jackson network.

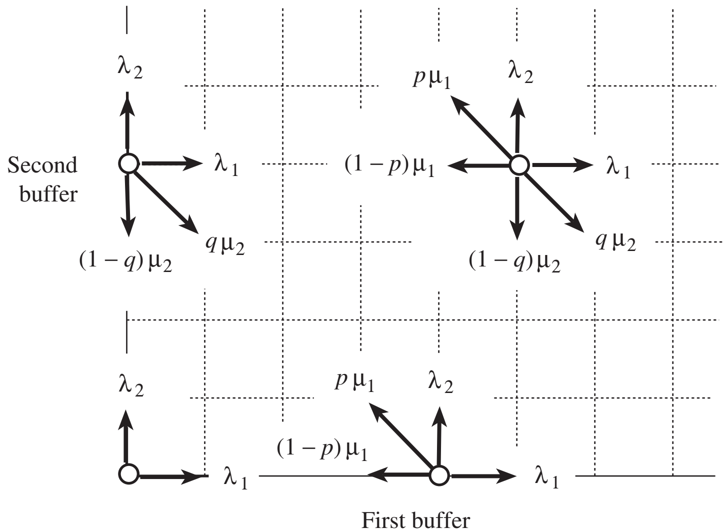

FIGURE 2: Transition intensities for the two-node Jackson network.

Customers arrive at nodes 1 and 2 according to independent Poisson processes with rates λ_1 and λ_2, respectively. Customers at each node are served according to a first-come–first-served discipline. The service times of customers at nodes 1 and 2 are independent and exponentially distributed with means $1/\mu_1$ and $1/\mu_2$, respectively. After completing service at node 1, customers enter node 2 with probability p or leave the system with probability $1 - p$, where $0 \leq p \leq 1$. (See Figures 1 and 2.) After completing service at node 2, customers enter node 1 with probability q or leave the system with probability $1 - q$, where $0 \leq q \leq 1$.

The traffic intensities, ρ_1 and ρ_2, are

$$\rho_1 = \frac{\lambda_1 + q\lambda_2}{(1 - pq)\mu_1}, \qquad \rho_2 = \frac{p\lambda_1 + \lambda_2}{(1 - pq)\mu_2}.$$

It is required for stability that $\rho_1 < 1$ and $\rho_2 < 1$.

Let Y_t and J_t denote the numbers of customers in queues 1 and 2 at time t, respectively. We consider the two-dimensional Markov chain (J_t, Y_t) with state space $\mathbb{Z}_+ \times \mathbb{Z}_+$, viewed as a QBD process in which J_t represents the level and Y_t represents the phase.

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The tridiagonal blocks of the generator $\boldsymbol{Q}$ are given by the infinite-dimensional matrices

$$\boldsymbol{Q}_0 = \begin{pmatrix} \lambda_2 & & & & \\ p\mu_1 & \lambda_2 & & & \\ & p\mu_1 & \lambda_2 & & \\ & & \ddots & \ddots & \end{pmatrix},$$

$$\boldsymbol{Q}_2 = \begin{pmatrix} (1-q)\mu_2 & q\mu_2 & & & \\ & (1-q)\mu_2 & q\mu_2 & & \\ & & (1-q)\mu_2 & q\mu_2 & \\ & & & \ddots & \ddots \end{pmatrix},$$

$$\boldsymbol{Q}_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+\mu_2) & \lambda_1 & & & \\ (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & & \\ & (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & & & \\ (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1) & \lambda_1 & & \\ & (1-p)\mu_1 & -(\lambda_1+\lambda_2+\mu_1) & \lambda_1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

The Markov chain (J_t, Y_t) has the well-known stationary distribution $\boldsymbol{\pi}$ given by

$$\pi_{n_1,n_2} = (1-\rho_1)(1-\rho_2)\rho_1^{n_1}\rho_2^{n_2}, \qquad n_1, n_2 \geq 0.$$

That is, the decay rate for the level process is ρ_2 with $\tilde{\boldsymbol{Q}}_1$ as specified. We will now use Theorem 3.1 to determine the set of possible decay rates that might be obtained by varying $\tilde{\boldsymbol{Q}}_1$.

We have

$$\begin{aligned} \gamma_0(z) &= (\lambda_1 + q\mu_2 z)z, \\ \gamma_1(z) &= \lambda_2 - (\lambda_1+\lambda_2+\mu_1+\mu_2)z + (1-q)\mu_2 z^2, \\ \gamma_2(z) &= p\mu_1 + (1-p)\mu_1 z, \\ \tilde{\gamma}_1(z) - \gamma_1(z) &= \mu_1 z. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \tau(z) &= \lambda_2 - (\lambda_1+\lambda_2+\mu_1+\mu_2)z + (1-q)\mu_2 z^2 + 2\sqrt{\mu_1 z(\lambda_1 + q\mu_2 z)(p + (1-p)z)}, \\ \chi(z) &= p\lambda_1 + \lambda_2 - (p\lambda_1 + \lambda_2 + (1-pq)\mu_2)z + (1-pq)\mu_2 z^2. \end{aligned}$$

The conditions of Theorem 3.1 are polynomial inequalities in z of degree at most four, so it is possible to obtain the possible decay rates in closed form. The closed-form solution, however, is complicated. For some fixed parameter values, we obtained the possible decay rates for which a positive z^{-1}-invariant measure of $\boldsymbol{R}$ exists in ℓ^1. The cases considered are presented in Table 1. The possible decay rates and the values of z for which each of the conditions of Theorem 3.1 are satisfied are presented in Table 2.

In each of the numerical cases considered, ρ_2 is a value of z for which (2.10) is satisfied, as it must be, since ρ_2 is the decay rate of the stationary distribution of the level process for the

TABLE 1: Parameter values of numerical examples for the two-node Jackson network.

Case	λ_1	λ_2	μ_1	μ_2	p	q	ρ_1	ρ_2
1	1	0	1.5	2	1	0	0.6667	0.5000
2	1	0	2	1.5	1	0	0.5000	0.6667
3	0	1	1.5	2	0	1	0.6667	0.5000
4	0	1	2	1.5	0	1	0.5000	0.6667
5	1	1	2	2	0.1	0.8	0.9783	0.5978
6	1	1	2	2	0.8	0.1	0.5978	0.9783
7	1	1	2	2	0.4	0.4	0.8333	0.8333
8	1	1	10	10	0.5	0.5	0.2000	0.2000
9	1	5	10	15	0.4	0.9	0.8594	0.5625
10	5	1	15	10	0.9	0.4	0.5625	0.8594

TABLE 2: Intervals and values of decay rate for which the conditions (i), (ii), and (iii) of Theorem 3.1 are satisfied, for the two-node Jackson network.

Case	(i)	(ii)	(iii)	Possible decay rates	z^{-1}-positive value
1	[0.4774, 1)	(0, 0.4774]	(0.4774, 0.7500)	[0.4774, 0.7500)	—
2	[0.6667, 1)	(0, 0.6365]	(0.6365, 1)	[0.6667, 1)	0.6667
3	[0.5000, 1)	(0, 0.4713]	(0.4713, 0.5000]	{0.5000}	0.5000
4	[0.6667, 1)	(0, 0.4713]	(0.4713, 0.6667]	{0.6667}	0.6667
5	[0.5971, 1)	(0, 0.5971]	(0.5971, 0.6000)	[0.5971, 0.6000)	—
6	[0.9783, 1)	(0, 0.8239]	(0.8239, 1)	[0.9783, 1)	0.9783
7	[0.8333, 1)	(0, 0.8193]	(0.8193, 0.9000)	[0.8333, 0.9000)	0.8333
8	[0.2000, 1)	(0, 0.1859]	(0.1859, 0.6000)	[0.2000, 0.6000)	0.2000
9	[0.5578, 1)	(0, 0.5578]	(0.5578, 0.6000)	[0.5578, 0.6000)	—
10	[0.8594, 1)	(0, 0.7154]	(0.7154, 1)	[0.8594, 1)	0.8594

two-node Jackson network. There may or may not be values of z other than ρ_2 for which (2.10) is satisfied. If there are other such values, then for the cases considered they form an interval, rather than isolated values. Also, ρ_2 may or may not be on the boundary of the interval, if one exists. It may be possible to obtain decay rates given by these values of z by modifying $\tilde{\boldsymbol{Q}}_1$, as discussed in Section 5.

We also observe that for each of the numerical cases considered, the right-hand end-point of the interval satisfying condition (ii) coincides with the left-hand end-point of the interval satisfying condition (iii). We have found nonnegative parameter values for which this is not the case. However, these values lead to non-ergodic systems. It is an open question as to whether this property must hold in the ergodic case.

The case in which $\lambda_1 \equiv \lambda$, $\lambda_2 = 0$, $p = 1$, and $q = 0$, known as the two-node tandem Jackson network, was considered in [8]. Theorem 3.1 was used to verify Proposition 6.2 below, which is Theorem 4.9 of [8]. The details of the derivation appear in Appendix B.

Proposition 6.2. *If $\mu_1 < \mu_2$ then the values of $z \in (0, 1)$ for which (2.10) is satisfied by a positive vector $\boldsymbol{w} \in \ell^1$ for the two-node tandem Jackson network are those in the interval $[\eta, \mu_1/\mu_2)$, where η is the root of $\tau(z)$ in $(0, 1)$. If $\mu_1 \geq \mu_2$ then the values of $z \in (0, 1)$ for which (2.10) is satisfied by a positive vector $\boldsymbol{w} \in \ell^1$ are those in the interval $[\rho_2, 1)$.*

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By Theorem 4.2, the only value of z for which the $\boldsymbol{R}$-matrix may be z^{-1}-positive is ρ_2. The cases in which the $\boldsymbol{R}$-matrix is z^{-1}-positive for $z = \rho_2$ are listed in the right-most column of Table 2. We observe that for the fixed parameter value cases considered that are z^{-1}-positive, ρ_2 is on the lower boundary of the range of possible decay rates. These are the cases in which ρ_2 is the convergence norm of $\boldsymbol{R}$.

Example 6.3. (*Assistance from idle server.*) Here we consider another specific class of QBD processes with infinitely many phases. It models a system where two servers each serve queues containing customers of a certain type. When one of the queues is empty, the server for that queue assists the other server.

Arrivals to queues 1 and 2 occur as independent Poisson processes with parameters λ_1 and λ_2, respectively. The service times of servers 1 and 2 are exponentially distributed with parameters μ_1 and μ_2, respectively.

Each server serves its own queue according to a first-come–first-served discipline. If one of the queues is empty, the server for that queue assists the other server, doubling the latter's service rate. If there is an arrival to a queue while its server is assisting the other queue, the server immediately ceases assisting and serves its own queue.

By applying the ergodicity conditions of [3, p. 3, p. 142], we see that the QBD process is ergodic if and only if $\rho_1 + \rho_2 < 2$, where $\rho_i = \lambda_i/\mu_i$, $i = 1, 2$. We assume that this condition holds.

Let J_t and Y_t denote the numbers of customers in queues 1 and 2 at time t, respectively. We investigate the behaviour of the two-dimensional Markov chain (J_t, Y_t), viewed as a QBD process in which J_t represents the level and Y_t represents the phase.

The phase space of this QBD process is infinite. The tridiagonal blocks of the generator $\boldsymbol{Q}$ are given by the infinite-dimensional matrices

$$\boldsymbol{Q}_0 = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_1 & & \\ & & \lambda_1 & \\ & & & \ddots \end{pmatrix}, \qquad \boldsymbol{Q}_2 = \begin{pmatrix} 2\mu_1 & & & \\ & \mu_1 & & \\ & & \mu_1 & \\ & & & \ddots \end{pmatrix},$$

$$\boldsymbol{Q}_1 = \begin{pmatrix} -(\lambda_1+\lambda_2+2\mu_1) & \lambda_2 & & & \\ \mu_2 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_2 & & \\ & \mu_2 & -(\lambda_1+\lambda_2+\mu_1+\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix},$$

$$\tilde{\boldsymbol{Q}}_1 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_2 & & & \\ 2\mu_2 & -(\lambda_1+\lambda_2+2\mu_2) & \lambda_2 & & \\ & 2\mu_2 & -(\lambda_1+\lambda_2+2\mu_2) & \lambda_2 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

We thus have

$$\gamma_0(z) = \lambda_2 z, \qquad \gamma_1(z) = \lambda_1 - (\lambda_1+\lambda_2+\mu_1+\mu_2)z + \mu_1 z^2,$$
$$\gamma_2(z) = \mu_2 z, \qquad \tilde{\gamma}_1(z) = \lambda_1 - (\lambda_1+\lambda_2+2\mu_1)z + 2\mu_1 z^2.$$

As in the previous examples, it is possible to derive closed-form expressions for the possible decay rates. However, we do not reproduce them here, due to their length. We determined possible decay rates numerically for the fixed parameter cases displayed in Table 3.

TABLE 3: Numerical examples for Example 6.3.

Case	λ_1	λ_2	μ_1	μ_2	ρ_1	ρ_2
1	1	1	2	2	0.5000	0.5000
2	1	1	1	2	1.0000	0.5000
3	1	1	2	1	0.5000	1.0000
4	1	2	1	3	1.0000	0.6667
5	2	1	3	1	0.6667	1.0000
6	3	1	2	4	1.5000	0.2500
7	1	3	4	2	0.2500	1.5000

TABLE 4: Possible decay rates for which the conditions (i), (ii), and (iii) of Theorem 3.1 are satisfied, for Example 6.3.

Case	(i)	(ii)	(iii)	Possible decay rates	z^{-1}-positive value
1	[0.4342, 1)	(0, 0.4342]	(0.4342, 0.5000)	[0.4342, 0.5000)	—
2	[0.7071, 1)	(0, 0.6628]	(0.6628, 1)	[0.7071, 1)	0.7071
3	[0.5000, 1)	(0, 0.5000]	—	{0.5000}	—
4	[0.7808, 1)	(0, 0.7287]	(0.7287, 1)	[0.7808, 1)	0.7808
5	[0.2397, 0.3333) ∪ [0.6667, 1)	(0, 0.6667]	—	[0.2397, 0.3333) ∪ {0.6667}	—
6	[0.8660, 1)	(0, 0.6340]	(0.6340, 1)	[0.8660, 1)	0.8660
7	[0.2419, 1)	—	—	—	—

The possible decay rates and the values of z for which each of the conditions of Theorem 3.1 are satisfied are presented in Table 4. In the cases considered, the possible decay rates may be either an isolated point, form an interval, or be the union of an isolated point and an interval. We make a similar observation as in Example 6.2 about the intervals satisfying conditions (ii) and (iii) sharing a common point. When both these intervals exist and the system is ergodic, this observation appears to be valid. However, the intervals do not necessarily share a common end-point when the system is non-ergodic. The values of z for which the $\boldsymbol{R}$-matrix is z^{-1}-positive are listed in the right-most column of Table 4.

7. Discussion and remarks

The anonymous referee observed that it is a consequence of our Assumption 2.1 that the matrix $\boldsymbol{Q}(z)$ is tridiagonal. This enabled us to use orthogonal polynomial techniques to ascertain whether (2.12) has a positive solution.

The referee further observed that if Assumption 2.1 does not hold, it may be possible to employ some of the results of the calculation of quasistationary distributions for continuous-time Markov chains with generator matrices with structures that are not tridiagonal (see, for example, [1] and [12]) to find a vector $\boldsymbol{w}$ that solves (2.12). This is a very interesting suggestion, which we plan to investigate in future work.

Appendix A. The two-demand model

The following results (stated without proof) apply to the two-demand model considered in Example 6.1, and allow the conditions of Theorem 3.1 to be expressed in closed form.

Proposition A.1. *If* $\alpha < \beta$ *then condition (i) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $[z_1, 1/\beta] \cup [1/\alpha, 1)$.

Proposition A.2. *If* $\alpha \geq \beta$ *then condition (i) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $[z_1, 1/\beta] \cup (1/\beta, 1)$.

Proposition A.3. *Condition (ii) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* $1/\beta < z_1$ *and* z *is in the interval* $(1/\beta, z_1]$.

Proposition A.4. *If* $\alpha < \beta$ *then condition (iii) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $(z_1, 1) \cap [1/\beta, 1/\alpha]$.

Proposition A.5. *If* $\alpha \geq \beta$ *then condition (iii) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $(z_1, 1) \cap ((1 + \alpha - \beta)/\alpha, 1/\beta]$.

From these results we obtain two further propositions.

Proposition A.6. *If* $\alpha < \beta$ *and* $z_1 \geq 1/\beta$ *then the only value of* $z \in (0, 1)$ *for which the conditions of Theorem 3.1 are satisfied is* $1/\alpha$. *If* $\alpha \geq \beta$ *and* $z_1 \leq 1/\beta$ *then there are no values of* $z \in (0, 1)$ *for which the conditions of Theorem 3.1 are satisfied.*

Proposition A.7. *If* $\alpha < \beta$ *then* $z_1 > 1/\beta$, *and if* $\alpha \geq \beta$ *then* $z_1 \leq 1/\beta$.

Combining Propositions A.6 and A.7 gives Proposition 6.1.

Appendix B. The two-node Jackson network

The following results (stated without proof) apply to the two-node Jackson network considered in Example 6.2 for the case considered in [8], in which $p = 1$, $q = 0$, $\lambda_1 \equiv \lambda$, and $\lambda_2 = 0$. We now have $\rho_1 = \lambda/\mu_1$, $\rho_2 = \lambda/\mu_2$, and

$$\begin{aligned}
\gamma_0(z) &= \lambda z, \\
\gamma_1(z) &= -(\lambda + \mu_1 + \mu_2)z + \mu_2 z^2, \\
\gamma_2(z) &= \mu_1, \\
\tilde{\gamma}_1(z) - \gamma_1(z) &= \mu_1 z, \\
\tau(z) &= -(\lambda + \mu_1 + \mu_2)z + \mu_2 z^2 + 2\sqrt{\lambda_1 \mu_1 z}, \\
\chi(z) &= (z - 1)(\mu_2 z - \lambda), \\
\upsilon(z) &= \lambda z + \mu_1.
\end{aligned}$$

Proposition B.1. *If* $\mu_1 < \mu_2$ *then condition (i) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $[z_1, 1)$. *If* $\mu_1 \geq \mu_2$ *then condition (i) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $[\rho_2, 1)$.

This condition is the same as that of [8] for the vector $\boldsymbol{w}$ to be positive.

Proposition B.2. *Condition (ii) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $(0, z_1]$.

Proposition B.3. *Condition (iii) of Theorem 3.1 is satisfied for* $z \in (0, 1)$ *if and only if* z *is in the interval* $(z_1, \min(\mu_1/\mu_2, 1))$.

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We see that either condition (ii) or (iii) is satisfied if and only if $z \in (0, \min(\mu_1/\mu_2, 1))$. This is equivalent to the conditions of [8] for the vector $\boldsymbol{w}$ to be in ℓ^1.

Combining the above results gives Proposition 6.2.

Acknowledgements

The first author would like to thank the Australian Research Council Centre of Excellence for Mathematics and Statistics of Complex Systems for providing a scholarship. The second author would like to thank the Australian Research Council for support through Discovery Grant number DP0209921. Both authors thank the anonymous referee for suggesting the ideas related in Section 7.

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