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BOOK REVIEWS

JENSEN, K. K. and THOMSEN, K. *Elements of KK-theory* (Mathematics: Theory and Applications, Birkhauser, Basel-Boston-Berlin 1991), viii+202 pp. 3 7643 3496 7, sFr. 98.

C*-algebras are nowadays often regarded as "noncommutative locally compact spaces". An important tool in the theory of C*-algebras in recent years has been the use of the two functors K_0, K_1 from C*-algebras to abelian groups, analogous to the usual K-theory of locally compact spaces. The use of K-theory stems from extension theory, index theory and the study of dimension groups of C*-algebras. $K_0(A)$ is generated by stable equivalence classes of projections in the C*-algebra A. $K_1(A)$ is defined by means of unitaries rather than projections and one obtains a sequence of abelian groups $K_0(A)$, $K_1(A)$, $K_2(A), \ldots$ satisfying (by Bott periodicity) $K_n(A) \simeq K_{n+2}(A)$. The K-groups of many C*-algebras have been calculated and, in addition to motivating new results on C*-algebras, they have also provided solutions to important problems in operator algebras. An early high point was the solution by Pimsner and Voiculescu of the long standing conjecture that the reduced C*-algebra of the free group on two generators is projectionless. Another very elegant proof of this was subsequently given by Cuntz, using KK-theory.

Kasparov's KK-theory added some completely new aspects and techniques to K-theory. The idea was to view K-theory as a functor $KK(A,B)$ of two variables A, B, where A and B are C*-algebras. The first variable represents K-homology in the sense of the theory of extensions developed by Brown, Douglas and Fillmore and the second represents ordinary K-theory. One of the features of KK-theory is the great generality of the definitions. This means that it can be applied in a broad variety of situations. It also makes it very hard to learn, because it spans so many different areas of mathematics.

The authors of the present book have provided an invaluable service to the mathematical community by expounding the basics of KK-theory using an absolute minimum of raw material. The prerequisites consist of elementary C*-algebra theory, tensor products, completely positive maps and Fredholm theory.

Two approaches to KK-theory are given. Firstly that adopted by Kasparov himself, based on what are now called Kasparov A-B modules. These consist of triples (E,ϕ,F) where E is a graded Hilbert B-module, ϕ is a * homomorphism from A into the set of bounded operators on E commuting with the action of B and satisfying a "Fredholm" condition relative to the operator F on E. The group operation is introduced and basic properties including functorality are proved. The power of KK-theory comes from the existence of the so called Kasparov product $KK(A,B)\times KK(B,C)\rightarrow KK(A,C)$. The existence of this is first proved using Kasparov's original approach involving a complicated tensor product-like construction. Then a more intuitive proof is given using the approach of Cuntz. In this, the second approach to KK, an element of $KK(A,B)$ is viewed as a generalized homomorphism from A to B (suitably stabilized). Here the Kasparov product corresponds to composition. Also included is a chapter on the connection of KK-theory with the theory of C*-algebraic extensions, which provided one of the first notable successes in applying algebraic topology to a concrete problem in operator theory.

The authors give the best available proofs of results, often including simplifications of their own, notably in the nice treatment of the generalized homomorphism approach to KK. There is no mention of the close connection between KK-theory and K-theory or of the wider applications of KK-theory. There are however already excellent informal and well motivated outlines of KK in the literature. The aim of the authors is to provide a clear and detailed exposition of the basic theory in compact form. In this they have succeeded admirably. After reading this book, any reader would surely be well equipped to attack the large and complex literature which has now arisen in this area.

A. GUYAN ROBERTSON

DONALDSON, S. K. and KRONHEIMER, P. B. *The geometry of four-manifolds* (Oxford Mathematical Monographs, Clarendon Press, Oxford 1990), x+440 pp. 0 19 853553 8, £35.

Analysis has played an important role in several of the great developments of geometry over the last 150 years. Complex analysis was an important motivating force behind Riemann's study of manifolds; the geometric theory of ordinary differential equations and dynamical systems led Poincaré to develop the basic ideas of algebraic topology; Hodge made fundamental use of the Laplacian in his deep study of the topology of manifolds. Ten years ago, Simon Donaldson utilised basic theorems on the space of solutions of certain non-linear partial differential equations for the resolution of what had seemed to be very intractable problems concerning the topology of four-dimensional spaces. The book under review is a polished account of Donaldson's pioneering work; also included are several of the consequent developments in which he has played the leading role.

The understanding of the topology of two-dimensional compact manifolds was complete by the end of the last century. During the present century considerable attention has been devoted to three-dimensional topology particularly through knot theory and attempts to solve the Poincaré conjecture. A recurring technical point in the study of higher-dimensional manifolds is the need for enough room to move lower-dimensional pieces around within the manifolds themselves. Two basic problems arise: the first is the need to find embedded two-dimensional discs in the manifolds, the second occurs in the "middle dimension", and is the need to separate two k-dimensional manifolds within a $2k$-dimensional one. H. Whitney and J. Milnor respectively understood how to treat these two problems in high dimensions; in four dimensions the two problems reinforce each other and how to deal with them in this case had been a puzzle for a considerable period. Donaldson's work proves that it is impossible to circumvent these problems in four dimensions because he shows that topology in four dimensions behaves in a fundamentally different way. A striking example of this contrast is the following: if $n \neq 4$, then any smooth manifold that is homeomorphic to $\mathbb{R}^n$ is diffeomorphic (i.e. smoothly homeomorphic) to $\mathbb{R}^n$ (a result proved for $n \geqslant 5$ during the considerable developments in higher-dimensional topology around 1960 and known previously for $n \leqslant 3$); however, there is an uncountable family of smooth 4-manifolds M^4 (including $\mathbb{R}^4$) all of which are homeomorphic to $\mathbb{R}^4$ but no two of which are diffeomorphic. Many of these examples can be exhibited as open sets in $\mathbb{R}^4$ itself.

Donaldson's method is to study the space of all anti-self-dual connections for a bundle over a Riemannian four-manifold; there are invariants of this space that turn out to be independent of the choice of Riemannian metric but vary for the different smooth structures on the manifold. The anti-self-dual connections correspond to absolute minima of the Yang–Mills functional and are therefore solutions of the corresponding Euler–Lagrange equations. The smooth structure on the manifold manifests itself through these differential equations; it turns out, in a later analysis, that only a Lipschitz or a quasiconformal structure is relevant. At about the same time as Donaldson made his initial breakthrough, the purely topological theory of four dimensions was clarified enormously by Michael Freedman. When the two pieces of work are put together there is a great simplification in the statements of a number of results, such as those made above about manifolds homeomorphic to $\mathbb{R}^4$.

For someone with a good working knowledge of a range of graduate level courses, this book is self-contained. The background is in algebraic and differential topology, differential geometry, algebraic geometry and global analysis; some specific topics are reviewed in the early chapters and the necessary background from analysis (e.g. Sobolev spaces) is given in the Appendix. The work