Appendix B

Algebraic topology

Having established the necessary background for integration on real and complex manifolds in Appendix A, we now move on to the topological results that allow us to manipulate these integrals in order to derive asymptotics. A differential form ω is said to be a ***closed form*** if $d\omega = 0$ and an ***exact form*** if $\omega = d\tau$ for some form τ. Many of the forms we care about are closed, for instance if ω is any holomorphic n-form in $\mathbb{C}^n$ then $\overline{\partial}\omega$ vanishes by holomorphicity and $\partial\omega$ vanishes because there are no holomorphic $(n+1)$-forms, hence ω is closed. A chain C is called a ***cycle*** if $\partial C = 0$ and a ***boundary*** if $C = \partial\mathcal{D}$ for some chain $\mathcal{D}$. The boundaries form a subset (in fact, a sub-vector space) of the cycles because $\partial^2 = 0$.

By the same reasoning as our proof of Theorem A.27 in the last appendix, the integral of any closed form over a boundary is zero. Thus, by linearity of the integral, if C is a cycle, then $\int_C \omega$ depends only on the equivalence class of C in the quotient space of cycles modulo boundaries. Homology theory is the study of this quotient space, which may be thought of simultaneously as a topological invariant and as classifying contours of integration for closed forms. After studying various forms of homology, we dualize our constructions and define cohomology of differential forms. Just as $\int_C \omega$ depends only on the homology class of the chain C, it also depends only on the cohomology class of the form ω.

B.1 Chain complexes and homology theory

Instead of working only with cycles of integration, we develop homology in a more general setting. This approach better illustrates underlying structure, and allows us to reuse results in different contexts. We therefore introduce the following abstract definitions, which generalize some of the properties discussed above.

(i) A ***chain complex*** is a collection $C = \{C_n : n = 0, 1, 2, \ldots\}$ of complex vector spaces, not necessarily finite dimensional, together with a ***boundary operator*** ∂, which for all n is a linear map ∂_n from the space of ***n-chains*** C_n to the space of $(n-1)$-chains C_{n-1} that satisfies $\partial^2 = 0$ (meaning $\partial_n \circ \partial_{n+1} = 0$ for all n, so that "a boundary has no boundary""). By definition, $\partial = 0$ on C_0.

(ii) The vector space of ***n-cycles*** $Z_n \subseteq C_n$ of a chain complex C is the kernel of ∂_n, and the group $B_n \subset C_n$ of ***n-boundaries*** of C is the image of ∂_{n+1}.

(iii) The nth ***homology group*** of C is the vector space quotient

$$H_n(C) := Z_n / B_n \,.$$

The notation $H_*(C)$ is used to refer collectively to $H_n(C)$ for all n. Cycles in the same equivalence class are called ***homologous***.

Remark. Because we work with complex vector spaces, $H_*(C)$ is sometimes called ***homology with coefficients in*** $\mathbb{C}$ to distinguish it from the analogous construction with integer coefficients. While the theory with integer coefficients is richer, taking coefficients in a field better suits the purposes of computing integrals. With integer coefficients, the spaces of chains, cycles, and boundaries are $\mathbb{Z}$-modules, and their quotients are abelian groups, hence "homology group" rather than "homology vector space."

To discuss the homology of a manifold, we must define an appropriate chain complex. One natural candidate is the chain complex defined by smooth chains together with the boundary map discussed in Section A.3 of Appendix A. For many purposes, however, it is convenient to relax our smoothness condition. If $\mathcal{M}$ is any Hausdorff topological space then a ***singular n-simplex*** in $\mathcal{M}$ is a *continuous* (not necessarily smooth) map $\sigma : \Delta^n \to \mathcal{M}$ from the standard n-simplex Δ^n to $\mathcal{M}$, and a ***singular n-chain*** in $\mathcal{M}$ is a complex linear combination of singular n-simplices in $\mathcal{M}$. Just as for smooth chains, we may use the natural ordering of the faces of Δ^n to define a canonical boundary map through (A.3.1), taking a singular n-simplex to a singular $(n-1)$-simplex and extending linearly to singular chains. It is a foundational result in homology theory that the homology of a manifold is unchanged whether one studies smooth or singular chains.

Proposition B.1. *Let $\mathcal{M}$ be a differentiable manifold, let C be the chain complex whose chains are linear combinations of singular simplices on $\mathcal{M}$, and let C' be the chain complex whose chains are linear combinations of smooth* *simplices on $\mathcal{M}$. Then for any $n \in \mathbb{N}$ the homology groups $H_n(C)$ and $H_n(C')$ are isomorphic.*

Proof See [Eil47]. □

We write $C(\mathcal{M})$ for the chain complex defined by the singular n-chains in $\mathcal{M}$. The homology group $H_n(C(\mathcal{M}))$ is written $H_n(\mathcal{M})$ and called the ***nth singular homology group*** of $\mathcal{M}$. One can think of the rank of the homology group $H_n(\mathcal{M})$ – i.e., the minimum size of a generating set of the group – as indicating how many unique cycles in $\mathcal{M}$ don't bound anything.

Example B.2. The rank of $H_1(\mathcal{M})$ represents the number of nonequivalent circles that can be drawn on $\mathcal{M}$ without bounding something in $\mathcal{M}$. The rank of $H_1(\mathcal{M})$ for a connected space $\mathcal{M}$ is thus zero if $\mathcal{M}$ is simply connected, however the converse does not hold. Homology with coefficients in $\mathbb{C}$ cannot detect the presence of a cycle γ such that γ does not bound anything but k times γ does (homology with integer coefficients is more discerning but typically more complicated to compute). ◁

Topological invariance of homology

The first crucial property of homology is that it is a topological invariant. A ***topological map*** (or simply ***map***) between two topological spaces is a continuous function between them, while a ***chain map*** between two chain complexes is a function between them that commutes with their boundary operators. More precisely, if $(\mathcal{A}, \partial^A)$ and $(\mathcal{B}, \partial^B)$ are two chain complexes then a chain map between them can be considered to be a collection of functions $f = (f_0, f_1, \dots)$ with $f_n : \mathcal{A}_n \to \mathcal{B}_n$ mapping from the n-chains of $\mathcal{A}$ to the n-chains of $\mathcal{B}$, such that $\partial_n^B \circ f_n = f_{n-1} \circ \partial_n^A$ for all n.

A topological map from X to Y induces a chain map from the singular chain complex of X to the singular chain complex of Y. A map $f : \mathcal{A} \to \mathcal{B}$ between chain complexes in turn induces a homomorphism f_* on homology groups by applying the map to representatives for the equivalence classes of cycles modulo boundaries. Both of these induced maps are functorial.

Proposition B.3. *If the topological spaces X and Y are homeomorphic then the singular homology groups $H_n(X)$ and $H_n(Y)$ are isomorphic for all n.*

Proof A homeomorphism between topological spaces is a topological map whose inverse is also a topological map. Hence, a homeomorphism between two spaces induces an isomorphism between the homology groups of the spaces. □

Homology groups are preserved under more than just homeomorphism. If X and Y are topological spaces then two maps $f, g : X \to Y$ are said to be ***homotopic maps*** if there is a continuous map $\mathbf{H} : X \times [0, 1] \to Y$, called a ***homotopy***, such that $\boldsymbol{H}(x, 0) = f(x)$ and $\boldsymbol{H}(x, 1) = g(x)$ for all x. A topological map $f : X \to Y$ is called a ***homotopy equivalence*** if there is a topological map $g : Y \to X$ such that $f \circ g$ is homotopic to the identity on Y and $g \circ f$ is homotopic to the identity on X. If $f : X \to Y$ is a homotopy equivalence then we say that X and Y are ***homotopic spaces***.

We claim that homotopic maps induce equal maps on homology, hence homotopy equivalent spaces have naturally identical homology. To see this one proves, on the chain level, that a homotopy equivalence between topological spaces induces a ***chain homotopy equivalence*** between the singular chain complexes, which in turn induces an isomorphism between the homology groups.

Proposition B.4. *If a topological map $f : X \to Y$ is a homotopy equivalence then the singular homology groups $H_n(X)$ and $H_n(Y)$ are isomorphic for all n, with f_* inducing one such isomorphism.*

Proof See Theorem 2.10 and Corollary 2.11 of [Hat02]. □

Exercise B.1. Explain why a homeomorphism is always a homotopy equivalence.

Suppose $H : X \times [0, 1] \to X$ is a homotopy with $H(x, 0) = x$ for all X, and $Y \subseteq X$ is a subspace such that $H(x, 1) \in Y$ for all x. If $H(y, t) \in Y$ for all y and t (so that X ends in Y and Y stays in Y) then we call H a ***weak deformation retract*** and say Y is a weak deformation retract of X. If $H(y, t) = y$ for all y and t then we call H a ***strong deformation retract*** (or simply ***deformation retract***) and say that Y is a strong deformation retract of X.

Exercise B.2. Prove that if Y is a weak deformation retract of X then X and Y are homotopy equivalent.

Remark. Imagine looking at a space X that starts deforming in a continuous manner, with points allowed to collide and pass through each other, ultimately ending up in a different space Y. The space Y can be smaller, even a single point, but Exercise B.2 implies that as long as every point that starts in Y stays in Y then the homology of X is isomorphic to the homology of Y.

Various homologies and their equivalence

Working with the homology of smooth chains is nice because they are what we integrate over, but this approach has some disadvantages. Most prominently, the large amount of freedom involved means the collection of chains on a manifold is huge (not having a countable basis) even though the homology groups are finite dimensional. Because of this we introduce new types of spaces with more structure.

- A ***cell complex*** or ***CW-complex*** X is a Hausdorff space defined using a specific inductive procedure. Let X^0 be a discrete collection of points and for any $n \geq 0$ let X^n be the quotient space, with quotient topology, defined by the disjoint union $X^{n-1} \sqcup_\alpha \Delta^n_\alpha$ of X^{n-1} with a collection of standard n-simplices Δ^n_α, where we identify each $x \in \partial\Delta^n_\alpha$ with some $\phi_\alpha(x) \in X^{n-1}$ using ***gluing maps*** $\phi_\alpha : \partial\Delta^n_\alpha \to X^{n-1}$. The set X^n is the ***n-skeleton*** of X, and contains its ***k-cells*** Δ^k_α for $k \leq n$. We consider *finite dimensional* cell complexes, meaning $X = X^n$ for some natural number n and the smallest such n is called the ***dimension of the cell complex*** X. Each simplex Δ^n_α corresponds to a map $\sigma^n_\alpha : \Delta^n_\alpha \to X$ defined by embedding Δ^n_α into $X^{n-1} \sqcup_\alpha \Delta^n_\alpha$ and then taking the quotient by the gluing maps.
- A ***Δ-complex*** is a cell complex X where the gluing map ϕ_α for any simplex Δ^n_α maps each $(n-1)$-dimensional face F of Δ^n_α homeomorphically to one of the $(n-1)$-simplices in X, preserving the ordering of vertices and agreeing with the previously defined gluing map ϕ_F on ∂F. A Δ-complex X may be viewed as a collection of maps $\sigma_\alpha : \Delta^{n_\alpha} \to X$ such that the restriction of σ_α to the interior of Δ^{n_α} is injective, each point of X lies in exactly one such restriction, and the restriction of σ_α to a face of Δ^{n_α} is another one of the maps $\sigma_\beta : \Delta^{n_\alpha - 1} \to X$ in the collection defining X.
- A ***simplicial complex*** is a Δ-complex where each gluing map is injective (so that distinct faces in the boundary of each simplex are glued to distinct lower-dimensional simplices) and each n-simplex is uniquely determined by its vertices. A simplicial complex X may be viewed as a set of simplices such that every face of a simplex in X is also in X and the nonempty intersection of any two simplices $\Delta_1, \Delta_2 \in X$ is a face of both Δ_1 and Δ_2.

We say that a space S is ***represented*** by a cell complex, Δ-complex, or simplicial complex X if S and X are homeomorphic. The representation of a space by a simplicial complex is called a ***triangulation*** of the space. Figure B.1 shows some examples representing a sphere and a circle.

The ***CW approximation theorem*** [Hat02, Proposition 4.13] states that any Hausdorff space X can be *approximated* by a cell complex $\tilde{X}$ meaning, among other things, that X and $\tilde{X}$ have the same singular homology groups. We are most interested in algebraic varieties, or their complements, which are examples of semi-algebraic sets and can therefore be triangulated [BPR03, Theorem 5.43].

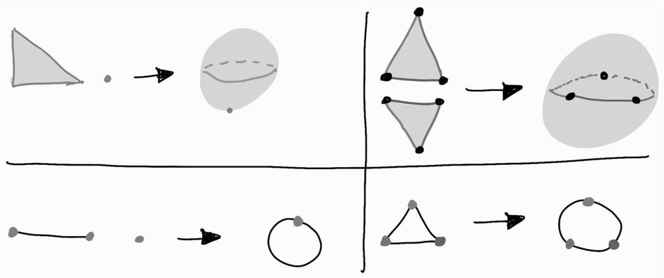

Figure B.1 *Top left:* The sphere S^2 can be represented as a cell complex with one 0-cell and one 2-cell whose boundary is mapped to the 0-cell, but this is not a Δ-complex. *Top right:* A representation of S^2 as a Δ-complex containing three 0-cells, three 1-cells, and two 2-cells. *Bottom left:* The circle S^1 can be represented as a Δ-complex with one 0-cell and one 1-cell whose boundary is identified with the 0-cell, but this is not a triangulation. *Bottom right:* A triangulation of S^1 with three 0-cells and three 1-cells.

The difficulties with singular homology for general topological spaces still arise for Δ and simplicial complexes. However, the additional structure present allows us to define a more rigid notion of homology. If X is a simplicial complex then the ***simplicial homology*** of X is the homology defined by the chain complex C whose n-chains are complex linear combinations of the n-simplices in X, with the boundary map again defined by (A.3.1). Simplicial homology is crucial for calculation, due to the following result.

Proposition B.5. *If X is a simplicial complex then the nth singular homology group of X is isomorphic to the nth simplicial homology group of X. The simplicial homology groups of X are algorithmically computable.*

Proof See [Hat02, Theorem 2.27] for the statement on the equivalence of the homologies, and [Mun84, Theorem 11.5] for an algorithm to determine simplicial homology by representing the linear boundary maps of X as matrices and computing their Smith normal forms. □

Among other corollaries, Proposition B.5 implies that a space which can be represented by a Δ-complex with a finite number of simplices has finitely generated homology groups. Although simplicial complexes have more structure than Δ-complexes, it can often be more efficient to represent a space as a Δ-complex compared to representing it as a simplicial complex.

Example B.6. We compute the homology of the circle S^1 by representing it as the Δ-complex X with one point $X^0 = \{p\}$ and one line segment $X^1 = \{\ell\}$ with both endpoints of ℓ glued at p. The 0-chains in this representation are the complex multiples of p, while the 1-chains are the complex multiples of ℓ. Any 0-chain is trivially a cycle, and the only boundary is $\partial\ell = p - p = 0$, so $H_0(S^1) = \mathbb{C}$. This computation also shows that every 1-chain is a cycle, and there are no non-trivial boundaries because there are no 2-cycles, so $H_1(S^1) = \mathbb{C}$ while $H_k(S^1) = 0$ for $k \geq 2$. ◁

Exercise B.3. Compute the homology of the circle by triangulating it, verifying you get the same result as Example B.6. Compute the homology of the sphere S^2 using similar techniques.

B.2 Tools for homology

Although we can compute the homology of a variety, or the complement of a variety, by triangulating the space and computing simplicial homology, this can be very expensive (such algorithms are generally considered efficient if they run in single-exponential instead of doubly-exponential time; see [Bas08] for a survey of complexity results in this area). Furthermore, instead of computing the entire set of homology groups of a singular variety, we often only need some partial information in our integral manipulations, for instance throwing away topological information that does not affect dominant asymptotics.

Because of such considerations, it is useful to have additional tools to compute homology. One of the most effective approaches is to work recursively, studying a space X using a subspace $A \subset X$ and the quotient space X/A. The relationship between the homology groups of X, X/A, and A is explicit but intricate, with the homologies fitting into a type of nesting structure. We thus require some additional algebraic constructions to describe precisely what is going on.

An ***exact sequence*** of abelian groups (and in particular, complex vector spaces) is a sequence of maps

$$\cdots \to X_{n+1} \xrightarrow{f_{n+1}} X_n \xrightarrow{f_n} X_{n-1} \to \ldots,$$

where the image of each map is equal to the kernel of the next. For instance, an exact sequence of the form $0 \xrightarrow{\epsilon} A \xrightarrow{\alpha} B$ says $0 = \text{Image}(\epsilon) = \text{Kernel}(\alpha)$, so α is injective, while an exact sequence of the form $A \xrightarrow{\alpha} B \xrightarrow{\epsilon} 0$ says $\text{Image}(\alpha) = \text{Kernel}(\epsilon) = B$, so α is surjective. A ***short exact sequence*** is an exact sequence of the form

$$0 \to X \to Y \to Z \to 0,$$

meaning the map from X to Y is injective and the map from Y to Z is surjective.

Remark B.7. When X, Y, and Z are finite dimensional complex vector spaces of dimensions k, ℓ, and m, respectively, then a short exact sequence $0 \to X \to Y \to Z \to 0$ implies $\ell = k + m$ and $Y = X \oplus Z$ as a direct sum. However, this splitting is not natural: X embeds naturally into Y but there is no canonical choice of coset representatives for Y/Z.

A ***short exact sequence of chain complexes*** is a map of chain complexes which is a short exact sequence on the n-chains for all n. One very useful fact about short exact sequences of chain complexes is that they give rise to long exact homology sequences.

Theorem B.8 (Zig-Zag Lemma)**.** *Let* $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ *be a short exact sequence of chain complexes. Then there is an exact sequence*

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial_*} H_n(A) \xrightarrow{\alpha_*} H_n(B) \xrightarrow{\beta_*} H_n(C) \xrightarrow{\partial_*} H_{n-1}(A) \to \dots, \tag{B.2.1}$$

where α_* *and* β_* *are the homology maps induced by the chain maps* α *and* β.

Proof See [Mun84, Lemma 24.1]. □

The exact sequence (B.2.1) in Theorem B.8 is known as the ***long exact sequence*** on homology. The homology map ∂_* has a natural but unwieldy definition; instead of defining it in general we describe it explicitly in the situation most relevant to us in Corollary B.11 below.

Relative homology and excision

Our goal is to apply Theorem B.8 to a short exact sequence of chain complexes related to embedding a subspace A into a space X and then taking the quotient to map into X/A. In fact, we consider a slightly more general setting which is also useful for our asymptotic calculations.

A ***pair of spaces*** (X, A) is any pair of topological spaces with A a subspace of X. A ***pair map*** $f : (X, A) \to (X', A')$ between pairs of spaces is any (topological) map $f : X \to X'$ such that $f(A) \subset A'$. The inclusion $A \hookrightarrow X$ induces an inclusion of chain complexes $C(A) \hookrightarrow C(X)$, and we let $C(X, A)$ denote the ***pair complex*** whose n-chains are the quotient group $C_n(X)/C_n(A)$. The ***relative homology*** of the pair (X, A) is the homology $H_*(X, A) = H_*(C(X, A))$ of the pair complex.

One may think of relative homology roughly as the homology of X if the subspace A were to be shrunk to a point: $H_n(X, A)$ contains ***relative cycles*** $\gamma \in C_n(X)$ with $\partial\gamma \in C_{n-1}(A)$ modulo ***relative boundaries*** $\beta = \partial\zeta + \alpha$, where $\zeta \in C_{n+1}(X)$ and $\alpha \in C_n(A)$. We thus search for cycles that do not bound, but are willing to count a chain as a cycle if its boundary is in A; see Figure B.2.

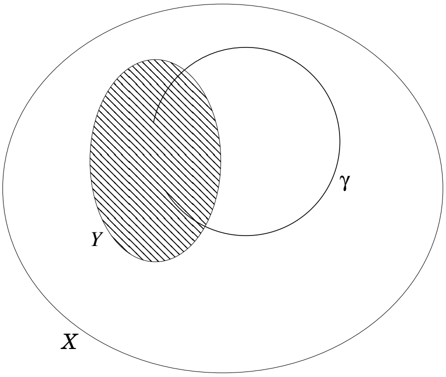

Figure B.2 γ is a relative cycle in $C(X, Y)$.

We often use relative homology to “ignore” regions that do not contribute to dominant asymptotics in our integral manipulations.

The concept of homotopy equivalence is defined for pairs of spaces similarly to the definition for spaces, and homotopy equivalent pairs have the same relative homology.

Example B.9. Let X be the unit ball in $\mathbb{R}^n$ and A be the unit sphere S^{n-1}. To compute the homology of the pair (X, A) we consider the unit n-simplex Δ^n and its boundary: topologically $(\Delta^n, \partial\Delta^n)$ and (B_n, S^{n-1}) are homeomorphic, so their relative homologies are the same. The simplicial chain complex for Δ^n contains all faces of Δ^n, hence it has simplices in dimensions 0 through n. The complex of the pair is non-trivial only in dimension n because every face of dimension less than n is supported on $\partial\Delta^n$. Proposition B.5 (that simplicial and singular homology coincide) extends to pairs composed of a simplicial complex and a subcomplex. Therefore, the singular homology of the pair (B^n, S^{n-1}) is computed by this rather small chain complex, leading to $H_k(B^n, S^{n-1}) \cong \mathbb{C}$ when $k = n$ and 0 otherwise. ◁

Exercise B.4. What is the relative homology of (S^2, S^2_-), where S^2 is the 2-sphere and S^2_- is the closed southern hemisphere?

Relative homology is useful for approximating integrals due to the following result.

Proposition B.10 (asymptotics depend only on relative homology class)**.** *Let X be a manifold of dimension n with submanifold Y also of dimension n, and let ϕ be a smooth complex-valued function on X satisfying $\Re\{\phi\} \leq \beta$ on Y for some $\beta \in \mathbb{R}$. Suppose that $\omega = \omega_\lambda = \exp(\lambda\,\phi(z))\,\eta$ for some closed k-form η on*

X with $k \le n$. If C and C' are k-chains on X with $C \equiv C'$ in $H_k(X,Y)$ then, as $\lambda \to \infty$,

$$\int_C \omega_\lambda = \int_{C'} \omega_\lambda + O\left(e^{\lambda\beta}\right) .$$

Proof By definition, the difference between C and C' is a relative boundary $C - C' = \partial \mathcal{D} + C''$ with C'' supported on Y. Using Stokes's Theorem (Theorem A.24),

$$\left| \int_C \omega - \int_{C'} \omega \right| = \left| \int_{\partial \mathcal{D}} \omega + \int_{C''} \omega \right| = \left| \int_{\mathcal{D}} d\omega + \int_{C''} \omega \right|$$
$$= \left| \int_{C''} \omega \right| \le \int_{C''} e^{\lambda\beta} |\eta| \le K e^{\lambda c'} ,$$

where $K = \int_{C'} |\eta|$ and the appearance of $|\eta|$ in an integral means that when η is pulled back to integrate, we integrate the modulus of the result. □

Let (X, A) be a pair for which A is a strong deformation retract of an open neighborhood in X. It can be shown [Mun84, Ex. 39.2] that for all $n \ge 1$ the relative homology group $H_n(X, A)$ is isomorphic to the singular homology group $H_n(X/A)$ of the quotient X/A, obtained from X by *shrinking A to a point*. This gives a way of computing the homology of a quotient X/A: applying Theorem B.8 to the short exact sequence of chain complexes $0 \to C(A) \to C(X) \to C(X, A) \to 0$ gives a long exact sequence computing $H_*(X, A)$, and hence $H_*(X/A)$, from $H_*(X)$, $H_*(A)$ and some knowledge of the maps in the long exact sequence.

Corollary B.11. *Let A be a subspace of X. Then there is a* ***long exact sequence of the pair*** *(X, Y),*

$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial_*} H_{n-1}(A) \to \cdots ,$$

where i_* and j_* are the maps induced by the inclusions of A into X, and X into (X, A), respectively, and ∂_* is the map induced by taking a relative cycle $\gamma \in C_n(X)$ to its boundary $\partial\gamma \in C_{n-1}(A)$. When A is a deformation retract of an open neighborhood in X then this long exact sequence holds with $H_n(X, A)$ replaced by $\tilde{H}_n(X/A)$, where the ***reduced homology group*** *$\tilde{H}_n$ is the same H_n when $n > 0$, and has dimension one less when $n = 0$.*

Exercise B.5 (computing homology of S^{n-1}). Let $X = \Delta^n$ and let Y be the subcomplex of cells with dimension strictly less than n. Use the long exact sequence for the pair (X, Y) to determine $H_{n-1}(Y)$.

One important feature of relative homology is the ***excision property***.

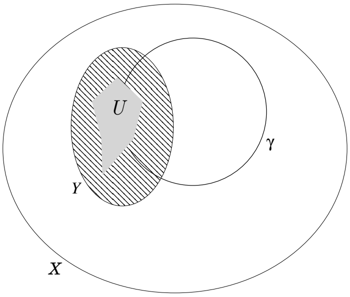

Figure B.3 $(X \setminus U, Y \setminus U) \hookrightarrow (X, Y)$ is a homology isomorphism.

Proposition B.12 (excision theorem)**.** *Given subspaces* $U \subset Y \subset X$ *such that* $\overline{U} \subset \operatorname{int}(Y)$*, the inclusion* $(X \setminus U, Y \setminus U) \hookrightarrow (X, Y)$ *induces an isomorphism* $H_*(X \setminus U, Y \setminus U) \cong H_*(X, Y)$*.*

Proof See [Hat02, Theorem 2.20]. □

Informally, Proposition B.12 says that the relative homology of (X, Y) *cannot see* the interior of U; Figure B.3 gives an illustration. Relative homology can also be used to define another important homology theory for cell complexes. If $X = X^n$ is a cell complex then the ***cellular chain complex*** of X is the complex

$$\cdots \to H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{\partial_n} H_{n-1}(X^{n-1}, X^{n-2}) \to \cdots,$$

where we define $X_{-1} = \varnothing$ and ∂_n is the composition of the boundary operator mapping $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1})$ with the inclusion of $H_{n-1}(X^{n-1})$ into $H_{n-1}(X^{n-1}, X^{n-2})$. The homology groups of this cellular chain complex form the ***cellular homology groups*** of X. Cellular homology is *homology of (relative) homology*, and can be surprisingly useful. Although its definition might seem technical and contrived, $H_n(X^n, X^{n-1})$ can be interpreted easily as linear combinations of the n-cells in X. The following result makes cellular homology a tool for computation.

Proposition B.13. *If X is a cell complex then the nth singular homology group of X is isomorphic to the nth cellular homology group of X.*

Among other things, Proposition B.13 implies that if X can be represented by a cell complex with no n-cells then $H_n(X) = 0$. See [Hat02, Section 2.2] for a full discussion of cellular homology, its implications, and a proof of Proposition B.13.

The homology of a union

Instead of decomposing a space X using a subspace and the corresponding quotient space, we can instead represent X as a union $X = A \cup B$ for subspaces $A, B \subset X$ such that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$ and describe the homology of X in terms of the homologies of A, B, and $A \cap B$. In particular, there is a short exact sequence of chain complexes defined by the short exact sequence in each dimension,

$$0 \to C_n(A \cap B) \xrightarrow{p} C_n(A) \oplus C_n(B) \xrightarrow{s} C_n(A) + C_n(B) \to 0,$$

where $p(a) = (-a, a)$ and $s(a, b) = a + b$, with $C_n(A) + C_n(B) \subseteq C_n(X)$ denoting a sum inside the space of chains on X. The corresponding long exact sequence is known as the ***Mayer–Vietoris sequence***.

Theorem B.14 (Mayer–Vietoris sequence)**.** *Let $A, B \subset X$ be such that X is the union of the interiors of A and B. Then the inclusion of the chain complex $C_*(A) + C_*(B) \hookrightarrow C_*(X)$ induces an isomorphism in homology. It follows that there is a long exact homology sequence*

$$\cdots \to H_k(A \cap B) \to H_k(A) \oplus H_k(B) \to H_k(X) \to H_{k-1}(A \cap B) \to \cdots .$$

Proof See [Hat02, Proposition 2.21] or [Mun84, Theorem 33.1]. □

Exercise B.6. Use the Mayer–Vietoris sequence to re-compute the homology of S^n by decomposing it as the union of two hemispheres, expanded a little so their interiors cover S^n. How does the result of this computation relate to a geometric understanding of two balls glued along their boundary? Try visualizing in dimensions one and two for greatest intuition.

Attachments and the homology of a product

Relative homology is useful for our asymptotic computations because it allows us to “ignore” points that are asymptotically negligible. In practice, to study complex integrals whose domains of integration are allowed to vary in some set $\mathcal{M}$ we describe $\mathcal{M}$ by attaching different spaces together as needed. In essence, we express $\mathcal{M}$ on-the-fly as a cell complex until we have enough information to perform the necessary asymptotic computations (much more information on this approach is given in the next appendix on Morse theory). This section, our final on the basics of homology, describes product complexes, attachments, and products of pairs.

Let C' and C'' be chain complexes with boundary maps ∂' and ∂''. The ***tensor product complex*** (or simply ***product complex***) of C' and C'' is the chain complex $C = C' \otimes C''$ whose n-chains C_n are defined by the direct sum $C_n = \bigoplus_{k=0}^{n} C'_k \otimes C''_{n-k}$, where a basis for the tensor product $C'_k \otimes C''_{n-k}$ is given by elements $\sigma_k \otimes \tau_{n-k}$ as σ_k ranges over a basis for C'_k and τ_{n-k} ranges over a basis for C''_{n-k}. The boundary operator ∂ of C is defined by $\partial(\sigma_k \otimes \tau_{n-k}) = (\partial' \sigma_k) \otimes \tau_{n-k} + (-1)^k \sigma_k \otimes (\partial'' \tau_{n-k})$. The product $Z = X \times Y$ of two simplicial complexes X and Y is naturally a cell complex, and this definition is constructed so that the product chain complex $C(X) \otimes C(Y)$ is isomorphic [Mun84, Theorem 57.1] to the cellular chain complex of Z.

Example B.15. Consider $C = \Delta_1$ as a cell complex with two 0-cells $\{[0], [1]\}$ and one 1-cell $\sigma_1 = [0, 1]$ oriented from 0 to 1, meaning that $\partial\sigma_1 = [1] - [0]$. Then $C \times C$ has four 0-cells $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$, four 1-cells $\{\sigma_1 \times [0], \sigma_1 \times [1], [0] \times \sigma_1, [1] \times \sigma_1\}$, and one 2-cell $\sigma_1 \times \sigma_1$. ◁

Exercise B.7. Write the circle S^1 as a cell complex, then describe the product cell complex $S^1 \times S^1$. Compute the homology of $S^1 \times S^1$ from this cell complex.

The homology of a product is given by the *Künneth formula*. Because we study homology with complex coefficients, the formula is relatively simple.

Theorem B.16 (Künneth product formula). *If C' and C'' are the singular chain complexes for two cell complexes then there is a natural isomorphism*

$$\bigoplus_{p+q=n} H_p(C') \otimes H_q(C'') \to H_n(C' \otimes C'') .$$

Proof See [Mun84, Theorem 58.5]. □

Exercise B.8. Use the Künneth formula to compute the homology of $S^1 \times S^1$ and verify it is the same as you computed in Exercise B.7.

Generalizing the attaching maps for cell complexes, the ***attachment*** of a space Y to a space X along a closed subset $Y_0 \subseteq Y$ by the map $\phi : Y_0 \to X$ is the topological quotient $(X \sqcup Y)/\phi$ obtained from the disjoint union of X and Y by identifying each $y \in Y_0$ with $\phi(y) \in X$. The triple (Y, Y_0, ϕ) is known as ***attachment data***.

Remark. When ϕ is one-to-one, attachments are more or less the same as unions; that is, both X and Y naturally embed in the attachment and their union covers the attachment. In general, an attachment can be thought of as a union of a space X with a quotient Y/ϕ, where points $y_1, y_2 \in Y_0$ are identified if $\phi(y_1) = \phi(y_2)$. While the two spaces in a union play symmetric roles, the attachment of a space Y to a space X is described asymmetrically. This coincides with the framework of *filtered spaces* built up by successive attachments, described in the next appendix. Thus attachments, while not entirely new, provide a useful way to build up a space.

Finally, we define a product on pairs by

$$(X', Y') \times (X'', Y'') = (X' \times X'', X' \times Y'' \cup Y' \times X'') .$$

One nice property of this definition is that the singular chain complex $C(X, Y)$ for a pair $(X, Y) = (X', Y') \times (X'', Y'')$ turns out to be isomorphic to a tensor product of the singular chain complexes for the pairs (X', Y') and (X'', Y''). The Künneth product formula for chain complexes results in a homology formula for products of pairs.

Corollary B.17 (Künneth formula for pairs)**.** *For pairs* (X', X'') *and* (Y', Y'')*,*

$$\begin{aligned} H_n(X' \times X'', X' \times Y'' \cup Y' \times X'') &= H_n((X', Y') \times (X'', Y'')) \\ &\cong \bigoplus_{p+q=n} H_p(X', Y') \otimes H_q(X'', Y'') . \end{aligned}$$

B.3 Cohomology

Given a chain complex

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

we may replace each vector space C_n by its dual C^n consisting of linear maps from C_n to $\mathbb{C}$. As for homology, we consider cohomology with complex coefficients. The elements of C^n are called ***n*-cochains** and the boundary map $\partial_n : C_n \to C_{n-1}$ on n-chains induces a dual map $\delta^n : C^{n-1} \to C^n$ on n-cochains: if $f \in C^{n-1}$ is a linear map from C_{n-1} to $\mathbb{C}$ then $\delta^n(f) \in C^n$ is the linear map $f \circ \partial_n$ from C_n to $\mathbb{C}$. It can be verified that $\delta^n \circ \delta^{n-1} = 0$ for all n, so we have a ***cochain complex***

$$\cdots \leftarrow C^{n+1} \xleftarrow{\delta^n} C^n \xleftarrow{\delta^{n-1}} C^{n-1} \leftarrow \cdots .$$

The quotient of the kernel of δ^n (the ***n*-cocycles**) by the image of δ^{n-1} (the ***n*-coboundaries**) is called the ***n*th cohomology group** of C and is denoted $H^n(C)$. The value of an n-cocycle ν evaluated at an n-cycle σ depends only on the cohomology class $[\nu]$ of ν and the homology class $[\sigma]$ of σ, so this evaluation defines a product $\langle \omega, \eta \rangle$ for $\omega \in H^n(C)$ and $\eta \in H_n(C)$. If $C = C(X)$ is the singular chain complex (or smooth chain complex) of a topological space X, then we use the notation $C^n(X) = C^n$ for the ***singular n*-cochains** of X and $H^n(X) = H^n(C)$ for the ***n*th singular cohomology group** of X. If X is a cell complex then $H^n(X)$ is the dual space of $H_n(X)$.

The functor taking a topological space to its singular or smooth chain complex is covariant, as is the functor from a chain complex to its homology groups. Hence, as noted above, any map $f : X \to Y$ of topological spaces induces maps $f_* : C(X) \to C(Y)$ and $f_* : H_*(X) \to H_*(Y)$. Conversely, the singular or smooth cochain complex of a space is a contravariant functor, so a map $f : X \to Y$ induces a map $f^* : H^*(Y) \to H^*(X)$.

On a manifold we may identify a p-form ω with the smooth p-cochain defined by $\alpha \mapsto \int_\alpha \omega$. Using the definition of the coboundary δ and Stokes's Theorem,

$$\delta\omega(C) = \omega(\partial C) = \int_{\partial C} \omega = \int_C d\omega = d\omega(C)\,.$$

In other words, $\delta\omega = d\omega$, so cocycles correspond to closed forms and we have the following.

Theorem B.18 (integral depends only on homology class)**.** *Let ω be a closed p-form holomorphic on an embedded complex manifold $\mathcal{M} \subseteq \mathbb{C}^n$ (if $p = n$ then ω is always closed). Let C be a singular p-cycle on $\mathcal{M}$. Then $\int_C \omega$ depends on C only via the homology class $[C]$ of C in $H_p(\mathcal{M})$ and on ω only via the cohomology class $[\omega]$ of ω in $H^p(\mathcal{M})$.* □

This theorem is one reason for our detour into topology. Another is the de Rham Theorem. Let ι be the map that takes the smooth p-form ω and maps it to the p-cochain $C \mapsto \int_C \omega$ as C varies over p-chains. This map is in general not a bijection: there may be linear maps on chains that are not represented by integrals of smooth forms. Nevertheless, the induced map ι_* will be an isomorphism from the singular cohomology of $\mathcal{M}$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$ to the cohomology H^*_{DR} of the ***de Rham complex*** of smooth p-forms with coboundary given by the differential operator d.

Theorem B.19 (de Rham Theorem)**.** *Let X be a real manifold. The identification of p-forms with cochains induces an isomorphism $H^*_{\mathrm{DR}}(X) \cong H^*(X)$.*

Proof See [Lee03, Theorem 18.7]. □

Remark. A product called the ***cup product*** may be defined on cochains, satisfying a product rule with respect to the d operator. The cup product endows cohomology with the structure of a graded $\mathbb{C}$-algebra, and the isomorphism in the de Rham Theorem is in fact a ring isomorphism, mapping the wedge product to the cup product.

B.4 Topology of complex manifolds

For us, complex manifolds usually arise as (subsets of) varieties or the complements of varieties in $\mathbb{C}_*^d$, with ACSV requiring the integration of holomorphic forms over chains of real dimension p contained in complex p-manifolds. This final section shows how the complex structure of a complex p-manifold makes it behave in several ways like a real p-manifold, even though it actually has real dimension $2p$.

Proposition B.20 (Andreotti–Frankel Theorem). *If X is a complex p-manifold embedded in $\mathbb{C}^n$ for some $n \geq p$ then X is homotopy equivalent to a CW complex of dimension at most p. It follows that the singular homology groups $H_k(X)$ and singular cohomology groups $H^k(X)$ vanish for all $k > p$.*

Proof Andreotti and Frankel [AF59] proved this for smooth algebraic (and analytic) varieties using Morse theory. A sketch is given at the end of Appendix C. □

Remark B.21. The complex projective space $\mathbb{CP}^k$ is a complex manifold having nonvanishing homology in all even dimensions up to $2k$. Therefore it violates the conclusions of the Andreotti–Frankel Theorem, and cannot be embedded in $\mathbb{C}^n$ for any n. This contrasts to the real case, where the Whitney embedding theorem states that any real k-manifold can be embedded into $\mathbb{R}^{2k}$.

In fact, we can compute homology by considering only holomorphic forms. If $\mathcal{M}$ is a complex p-manifold then the operator $\omega \mapsto d\omega$ preserves holomorphicity, so the holomorphic forms on $\mathcal{M}$ define a sub-cochain complex $C^{n,\mathrm{holo}}$ of the de Rham complex C^n, called the ***holomorphic de Rham complex***. The inclusion $C^{n,\mathrm{holo}} \hookrightarrow C^n$ does not, in general, induce an isomorphism on cohomology, but once again this difficulty can be overcome by restricting to manifolds embedded in complex space.

Proposition B.22 (holomorphic de Rham cohomology). *Let $\mathcal{M}$ be a complex p-manifold embedded in $\mathbb{C}^n$. Then the inclusions $C^{n,holo}(\mathcal{M}) \hookrightarrow C^n(\mathcal{M})$ induce an isomorphism of cohomology rings. In particular, $H^{holo,k}(\mathcal{M}) \cong H^k(\mathcal{M})$ for all $k \geq 0$.* □

Proof See Voisin [Voi02] and the notes at the end of this appendix. □

We finish this appendix by observing a corollary of Proposition B.10 in the complex setting.

Corollary B.23 (asymptotics depend only on relative homology class). *Let X be a complex manifold of dimension n with submanifold Y also of dimension*

n, and define $\omega = \omega_\lambda = \exp(\lambda\,\phi(z))\,\eta$ for some holomorphic n-form η and holomorphic function ϕ on X. When $C \equiv C'$ in $H_n(X, Y)$ with $\mathrm{Re}\{\phi\} \le \beta$ on Y then, as $\lambda \to \infty$,

$$\int_C \omega_\lambda = \int_{C'} \omega_\lambda + O\left(e^{\lambda\beta}\right).$$

Proof This follows immediately from Proposition B.10, because $d\omega = 0$ for a holomorphic n-form on a complex n-manifold. □

Notes

From their origins near the end of the nineteenth century, homology and cohomology have become crucial tools in many areas of mathematics. Much of our presentation of the material in this appendix follows Hatcher [Hat02] and Munkres [Mun84], and further details can be found in those sources.

The Andreotti–Frankel Theorem is true in much greater generality than Proposition B.20: for instance, it holds for all algebraic varieties in complex affine space, regardless of whether they are smooth or singular. This was first proved in [Kar79] via stratified Morse theory. The complement of a variety $\mathcal{V}_Q$ is biholomorphically equivalent to the variety $\mathcal{V}_{1-z_{d+1}Q}$ in one greater dimension, hence complements of d-dimensional affine algebraic varieties are also homotopy equivalent to d-dimensional cell complexes.

Voisin [Voi02] proves Proposition B.22 by showing that holomorphic de Rham *hypercohomology* (cohomology with coefficients in a sheaf resolution) is the same as the ordinary de Rham cohomology, hence the same as smooth cohomology and singular cohomology. For *Stein spaces*, such as embedded complex manifolds, this resolution is flat and holomorphic de Rham hypercohomology boils down to the cohomology of the holomorphic de Rham complex itself. Special cases were known earlier; for example, if $\mathcal{A}$ is a complex hyperplane arrangement then Brieskorn [Bri73] showed that the forms df/f as f varies over annihilators of hyperplanes in $\mathcal{A}$ generate the cohomology ring of the complement $\mathcal{M}$ of $\mathcal{A}$.

Additional exercises

Exercise B.9. Define the ***Möbius strip*** as the quotient of the cell complex representing the unit square as $\Delta_1 \times \Delta_1$ via the three identifications $(0,0) \sim (1,1)$, $(0,1) \sim (1,0)$ and $(0) \times \Delta_1 \sim -(1) \times \Delta_1$.

(1) What is the dimension of this cell complex?

(2) Give a basis for each space Z_0, B_0, Z_1, B_1, Z_2, and B_2.
(3) Compute the homology of the Möbius strip with coefficients in $\mathbb{C}$ from this cell complex.
(4) What changes, if anything, if you use coefficients in $\mathbb{Z}$ instead of coefficients in $\mathbb{C}$?

Exercise B.10. Let $X = \mathbb{C}_* = \mathbb{C} \setminus \{0\}$ be the punctured plane, the simplest case of the complement of a hyperplane arrangement. To establish the complex de Rham Theorem for $H^1(X)$ we need to show that holomorphic 1-forms ω and θ map to the same element of the dual of $H_1(X)$ if and only if they differ by a coboundary df.

(1) Use Theorem B.18 to prove the forward implication.
(2) Compute the homology of X by verifying that the embedding of S^1 into X is a homotopy equivalence.
(3) Let ω be any holomorphic 1-form on X. Use Stokes's Theorem to show that $\int_C \omega = 0$ for any C homologous to zero in $H_1(X)$.
(4) Let γ be the unit circle oriented, say, counterclockwise, and let $\eta = \omega - c\,dz/z$, where $c = (2\pi i)^{-1} \int_\gamma \omega$. Show that $\int_C \eta = 0$ for every $C \in Z_1(X)$. *Hint:* Use part 2.
(5) Show that $\int_\gamma \omega = \int_\gamma \theta$ implies $\omega - \theta = df$ for some holomorphic function f. *Hint:* You can construct f by integrating from an arbitrary fixed basepoint.

Exercise B.11. Let X be the complex curve $\{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$. By the Andreotti–Frankel Theorem, it is homotopy equivalent to a cell complex of (real) dimension 1. Demonstrate this by finding a deformation retract of X onto a one-dimensional manifold.