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TABLEAUX REALIZATION OF GENERALIZED VERMA MODULES

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ABSTRACT. We construct the tableaux realization of generalized Verma modules over the Lie algebra $\mathrm{sl}(3,\mathbb{C})$. By the same procedure we construct and investigate the structure of a new family of generalized Verma modules over $\mathrm{sl}(n,\mathbb{C})$.

1**. Introduction.** The structure theory of Verma modules over semisimple complex finite-dimensional Lie algebras ([3]) is based on the theory of finite-dimensional modules over the Lie algebra $\mathrm{sl}(2,\mathbb{C})$. The classical methods of investigation of finite-dimensional $\mathrm{sl}(2,\mathbb{C})$-modules provide a clear geometrical realization for these modules by choosing an eigenbasis with respect to a Cartan subalgebra. In this basis it is possible to write down explicit formulae defining an action of the generating elements.

During the last decade there appeared many papers (see [2, 6] and references therein) where a class of the so-called stratified generalized Verma modules (different from those studied in [13]) was investigated. For the simplest case of such modules the structure theorem generalizing the well-known BGG theorem ([3, Theorem 7.6.23]) was proved in [6]. The proof is based on the technical results from [7] where the structure of a generalized Verma $\mathrm{sl}(3,\mathbb{C})$-module induced from a simple weight $\mathrm{sl}(2,\mathbb{C})$-module without lowest and highest weights was studied. The main result in [7] was obtained by a hard direct calculation. The same method was used in [6] to obtain an analogues result in the case of the Lie algebra of type B_2. In [12] an analogue of the BGG theorem was obtained by a geometrical realization of generalized Verma modules induced from a "well-embedded" $\mathrm{sl}(k,\mathbb{C})$ subalgebra of the algebra $\mathrm{sl}(n,\mathbb{C})$.

In the present paper we propose a geometrical realization (which we call the tableaux realization with respect to the Gelfand-Zetlin subalgebra) for a larger family of generalized Verma modules. In this way we easily reobtain without any calculation all the results from [7] for modules having the tableaux realization. This enables us to obtain a structure theorem for a large class of generalized Verma modules over $\mathrm{sl}(n,\mathbb{C})$ induced from an arbitrary semisimple "well-embedded" subalgebra. Moreover, in some special cases we construct a composition series for a generalized Verma module. In $\mathrm{sl}(3,\mathbb{C})$ case this was done in [7]. But those methods could not be applied to a non-simply-laced case or to the case investigated in [12]. Moreover, even the structure of the maximal submodule in a generalized Verma module is unknown. We describe the structure of the maximal submodule and construct a composition series of some generalized Verma modules over $\mathrm{sl}(n,\mathbb{C})$. We also formulate a conjecture for all cases.

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Let us briefly describe the structure of the paper. In Section 2 we give all necessary basic preliminaries. In Section 3 we collect all information on Gelfand-Zetlin modules. In Section 4 we define a set of modules with the tableaux realization and present some examples of such modules including finite-dimensional modules, generic GZ-modules and a subclass of Verma modules. In Section 6 we construct the tableaux realization for a huge class of α-stratified generalized Verma modules over $\mathrm{sl}(3,\mathbb{C})$. Using the same procedure in Section 7 we construct a new family of generalized Verma modules and investigate their structure (Proposition 3, Corollary 3). Finally, in Section 8 we obtain a structure theorem for a class of generalized Verma $\mathrm{sl}(n,\mathbb{C})$-modules induced form an arbitrary "well-embedded" semi-simple subalgebra (Theorem 1). As a corollary we obtain a criterion of irreducibility for such modules (Corollary 4).

2. **Preliminaries.** Let $\mathbb{C}$ denote the field of complex numbers, $\mathbb{Z}$ denote the ring of integers and $\mathbb{N}$ denote the set of positive integers. We will also denote by $\mathbb{Z}_+$ the set of all non-negative integers.

For a Lie algebra $\mathfrak{A}$ by $U(\mathfrak{A})$ and $Z(\mathfrak{A})$ we will denote the universal enveloping algebra of $\mathfrak{A}$ and the center of $U(\mathfrak{A})$ correspondingly.

Let $\mathfrak{G}$ be a simple finite-dimensional complex Lie algebra with a fixed Cartan subalgebra $\mathfrak{H}$ and the root system Δ. Let π be a basis of Δ and $\Delta = \Delta_- \cup \Delta_+$ be the decomposition of Δ into positive and negative roots with respect to π. For $\alpha \in \Delta$ let $\mathfrak{G}_\alpha$ be the root subspace of $\mathfrak{G}$ corresponding to the root α and X_α be the corresponding element from a fixed Weyl-Chevalley basis. For $\alpha \in \Delta_+$ we also set $H_\alpha = [X_\alpha, X_{-\alpha}]$.

Let $\mathfrak{N}_\pm$ denote Lie subalgebra of $\mathfrak{G}$ generated by $X_{\pm\alpha}$, where α runs through Δ_+. Then

$$\mathfrak{G} = \mathfrak{N}_- \oplus \mathfrak{H} \oplus \mathfrak{N}_+$$

is a triangular decomposition of $\mathfrak{G}$.

For a $\mathfrak{G}$-module V and $\lambda \in \mathfrak{H}^*$ set

$$V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{H}\}.$$

If V_λ is non-trivial we will say that V_λ is a weight subspace of V and in this case we will call λ a weight of V. A module V will be called *weight module* provided

$$V = \bigoplus_{\lambda \in \mathfrak{H}^*} V_\lambda.$$

Each non-zero element from V_λ will be called *weight element*. Note, that here we do not assume that V_λ is finite-dimensional. For a weight $\mathfrak{G}$-module V we denote by $\mathrm{Supp}\, V$ the set of all weights of V. Clearly, each submodule and each quotient of a weight module is a weight module.

From now on all the modules are assumed to be weight modules.

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3. **Gelfand-Zetlin modules.** In this section we collect all necessary information about the so-called Gelfand-Zetlin modules. We will follow closely [4, 14] (see also [1]).

Consider the Lie algebra $\mathfrak{G}_m = \mathrm{gl}(m, \mathbb{C})$ for $m \geq 1$. We fix the following notations: $U_m = U(\mathfrak{G}_m)$ and $Z_m = Z(\mathfrak{G}_m)$, $m \geq 1$. For a fixed $n \geq 1$ set $\mathfrak{G} = \mathfrak{G}_n$, $U = U_n$ and for $m < n$ identify $\mathfrak{G}_m$ with the Lie subalgebra of $\mathfrak{G}$ generated by the matrix units $\{e_{ij} \mid i, j = 1, \ldots, m\}$. In this way we obtain the following inclusions:

$$\mathfrak{G}_1 \subset \mathfrak{G}_2 \subset \cdots \subset \mathfrak{G}_n = \mathfrak{G} \quad \text{and} \quad U_1 \subset U_2 \subset \cdots \subset U_n = U.$$

Denote by Γ the subalgebra of $\mathfrak{G}$, generated by $\{Z_m \mid m = 1, \ldots, n\}$. Following [4] we will call it Gelfand-Zetlin subalgebra (GZ-subalgebra) of U.

It is well-known (see for example [14]), that Γ is a polynomial algebra in $n(n+1)/2$ variables c_{lk}, where

$$c_{lk} = \sum_{i_1, \ldots, i_k = 1, \ldots, l} e_{i_1 i_2} e_{i_2 i_3} \cdots e_{i_k i_1}, \quad 1 \leq k \leq l \leq n.$$

Following [4] and [14] we choose a new set of generators for Γ. Set $L = L(n) = \mathbb{C}^{n(n+1)/2}$. The elements from L will be called *tableaux* and will be viewed as double indexed families

$$[l] = \{l_{ij} \mid i = 1, \ldots, n; j = 1, \ldots, i\}.$$

For $[l] \in L$ denote $[l]_i = \{l_{ij} \mid j = 1, \ldots, i\}$ the i-th row of $[l]$. We will also need a subset $L_0 \subset L$ consisting of all $[l]$ such that $l_{nj} = 0$ for all possible j and $l_{ij} \in \mathbb{Z}$ for all $1 \leq j \leq i < n$.

Consider the polynomial algebra Λ in $n(n+1)/2$ variables λ_{ij}, $1 \leq j \leq i \leq n$. We can identify Λ with the algebra of polynomial functions on L by setting $\lambda_{ij}([l]) = l_{ij}$. The product of symmetrical groups $G = S_1 \times S_2 \times \cdots \times S_n$ acts on L in a natural way: S_i permutes the elements of $[l]_i$. This induces a natural action of G on Λ. Define a homomorphism $\imath \colon \Gamma \longrightarrow \Lambda$ in the following way:

$$c_{ij} \longmapsto \sum_{k=1}^{i} (\lambda_{ik} + i)^j \prod_{l \neq k} \left(1 - \frac{1}{\lambda_{ik} - \lambda_{il}}\right).$$

Then $\imath(\Gamma)$ coincides with the set of G-invariants in Λ ([14]). From now on we will identify Γ with its image in Λ. Let γ_{ij} denote the j-th elementary symmetric function in $\lambda_{i1}, \ldots, \lambda_{ii}$. Clearly, $\{\gamma_{ij} \mid 1 \leq j < i \leq n\}$ generates the polynomial ring Γ.

For a $\mathfrak{G}$-module V and $\chi \in \Gamma^*$ set

$$V^\chi = \left\{ v \in V \mid \text{there exists } t \in \mathbb{N} \text{ such that } \big(c - \chi(c)\big)^t v = 0 \text{ for all } c \in \Gamma \right\}.$$

If V^χ is non-trivial we will say that V^χ is a GZ-*root subspace* of V and in this case we will call χ a GZ-*weight* of V. Any non-zero element $v \in V^\chi$ will be called GZ-*element*. A $\mathfrak{G}$-module V will be called *Gelfand-Zetlin module* (GZ-module) provided

$$V = \bigoplus_{\chi \in \Gamma^*} V^\chi \text{ and } \dim V^\chi < \infty \quad \text{for all } \chi \in \Gamma^*.$$

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For a GZ-module V we will denote by GZsupp V the set of all GZ-weights of V. If the action of Γ is diagonalizable on V^χ we will say that V^χ is a GZ-weight space and will denote it by V_χ. Each non-zero element from V_χ will be called GZ-*weight element*.

Since we identify Γ with $\imath(\Gamma)$ we can parametrize an element $\chi \in \Gamma^*$ by the tableau $[l] \in \mathcal{L}$. In this case we will say that the GZ-root (weight) space V^χ (V_χ) has the tableau $[l]$ and will denote it by $V^{[l]}$ ($V_{[l]}$).

Let V be an indecomposable GZ-module and $\chi \in$ GZsupp V. Suppose that χ has a tableau $[l] \in \mathcal{L}$. Then there exists a subset $P([l], V)$ in $[l]+\mathcal{L}_0$ whose elements parametrize all GZ-weights of V ([4]).

Clearly, any weight (with respect to the Cartan subalgebra) $\mathfrak{G}$-module with finite-dimensional weight spaces is a GZ-module. Thus, each Verma module ([3]) is a GZ-module.

Since $\mathrm{gl}(n,\mathbb{C})$ is a splitting central extension of $\mathrm{sl}(n,\mathbb{C})$ one can consider each $\mathrm{gl}(n,\mathbb{C})$-module as an $\mathrm{sl}(n,\mathbb{C})$-module by restriction. Moreover, each $\mathrm{sl}(n,\mathbb{C})$-module defines a $\mathbb{C}$-family ([12]) of $\mathrm{gl}(n,\mathbb{C})$-modules with different "central charges". We will formulate all our results for the reductive algebra $\mathrm{gl}(n,\mathbb{C})$ as it done in [4]. One can easily reformulate them for $\mathrm{sl}(n,\mathbb{C})$ case.

4. **Modules with the tableaux realization.** A tableau $[l] \in \mathcal{L}$ will be called *good* provided $l_{ij} \neq l_{ik}$ for all $1 \leq j < k \leq i < n$ and bad in the opposite case.

A $\mathfrak{G}$-module V is said to have the tableaux realization if it is GZ-weight module, all GZ-weight spaces are one-dimensional and have good tableaux.

Let δ^{ij} be the Kronecker tableau *i.e.* $\delta^{ij}_{ij} = 1$ and $\delta^{ij}_{i'j'} = 0$ for $i \neq i'$ or $j \neq j'$.

LEMMA 1. *Let V be a $\mathfrak{G}$-module which has the tableaux realization. Suppose that $V_{[l]} \neq 0$ for some $[l] \in \mathcal{L}$. Then $V \simeq W$, where $\mathfrak{G}$-module W is defined as follows: it has the $\mathbb{C}$-basis $v_{[t]}$ for $[t] \in P([l], V)$ and the action of elements from $\mathfrak{G}$ is defined by the following formulae:*

$$c_{ij} v_{[t]} = c_{ij}([t]) v_{[t]}, \quad E_i^{\pm} v_{[t]} = \sum_{j=1}^{i} a_{ij}^{\pm}([t]) v_{[t]\pm\delta^{ij}},$$

where $E_i^+ = e_{i\,i+1}$, $E_i^- = e_{i\,i-1}$, $i = 1, \ldots, n-1$, $j = 1, \ldots, i$ and

$$c_{ij}([t]) = \sum_{k=1}^{i} (t_{ik} + i)^j \prod_{s \neq k} \Big(1 - \frac{1}{t_{ik} - t_{is}}\Big);$$

$$a_{ij}^{\pm}([t]) = \mp \frac{\prod_k (t_{i\pm 1\,k} - t_{ij})}{\prod_{k \neq j} (t_{ik} - t_{ij})}.$$

PROOF. Follows from [4, Theorem 24, Proposition 20, Proposition 22]. ∎

The last lemma implies that a $\mathfrak{G}$-module with the tableaux realization has clear geometrical structure. Thus the question to give the tableaux realization for a given module seems to be rather interesting. We will call the formulae from Lemma 1 Gelfand-Zetlin formulae (GZ-formulae). The basis $\{v_{[l]}\}$ will be called GZ-*basis*. Note, that these formulae appear first time in the original papers by Gelfand and Zetlin ([9, 10]).

The following lemma is obvious:

LEMMA 2. *If a $\mathfrak{G}$-module V has the tableaux realization then each submodule and each quotient of V also has the tableaux realization.*

Now we can give some examples of modules having the tableaux realization.

4.1. *Finite-dimensional modules.* The first result from which the theory of GZ-modules arose were the result from [9] where there was proved that each simple finite-dimensional $\mathfrak{G}$-module has a basis $\{v_{[l]}\}$ with $[l]$ such that $l_{ij} \in \mathbb{N}$ for all possible i, j and $l_{ij} \geq l_{i-1\,j} > l_{i\,j+1}$. Moreover, the action of elements of the algebra $\mathfrak{G}$ in this basis is defined by the GZ-formulae. Thus, directly from the definition we obtain that each simple finite-dimensional $\mathfrak{G}$-module has the tableaux realization.

4.2. *Generic* GZ-*modules.* The second class of modules having clear tableaux realization is a huge family of GZ-modules constructed in [4].

Consider a tableau $[l] \in L$ such that $l_{ij} - l_{ik} \notin \mathbb{Z}$ for all possible i and $j \neq k$. Let V be a $\mathbb{C}$-space with the basis $v_{[t]}$, $[t] \in [l] + L_0$. Then GZ-formulae define on V a structure of a $\mathfrak{G}$-module ([4]). Clearly, such modules arise together with their tableaux realization.

An interesting subclass of such modules compose the so-called generic GZ-modules ([12]). These modules can be obtained from the original tableau $[l]$ satisfying the following condition: $l_{i+1\,j} - l_{ik} \notin \mathbb{Z}$ for all possible j, k and $i < n$. One can easily show that each simple generic module does not have non-trivial extensions with a non-isomorphic simple GZ-module in the category of GZ-modules ([4]).

4.3. *Generic Verma modules.* For $\lambda \in \mathfrak{H}^*$ let $M(\lambda)$ be the Verma module corresponding to $\mathfrak{G}$, $\mathfrak{H}$, λ and π ([3]). Since $M(\lambda)$ is a weight module with finite-dimensional weight spaces it is also a GZ-module. But it may happened that some GZ-weights of $M(\lambda)$ have bad tableaux. For example, it follows immediately from the construction of the GZ-basis for finite-dimensional $\mathfrak{G}$-modules that for any dominant integral λ the corresponding module $M(\lambda)$ necessarily has a GZ-weight parametrized by a bad tableau.

Now we construct the tableaux realization for a huge family of Verma modules those are "opposite" in some sense to the modules with dominant integral highest weight. Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ be such that $a_k - a_j \notin \mathbb{Z}$ for all $1 \leq j < k \leq n$. Consider a tableau $[l] = [l](a)$ defined as follows: $l_{ij} = a_j$ for all $1 \leq j \leq i \leq n$. Let $P([l])$ denote the set of all tableaux $[t]$ satisfying the following conditions:

1. $l_{nj} = t_{nj}$, $j = 1, \ldots, n$;
2. $l_{ij} - t_{ij} \in \mathbb{Z}_+$ for all $1 \leq j \leq i < n$;
3. $t_{ij} - t_{i-1\,j} \geq 0$ for all $1 < i \leq n$, $1 \leq j \leq i$.

LEMMA 3. *Let $[t] \in P([l])$ and $a_{ij}^{\pm}([t]) \neq 0$ (here $a_{ij}^{\pm}$ is an expression from the* GZ-*formulae). Then $[t] \pm \delta^{ij} \in P([l])$.*

PROOF. Let $a^+_{ij}([t]) \neq 0$. Hence $t_{ij} \neq t_{i+1\,k}$ for all $1 \leq k \leq i+1$ and thus $t_{ij} < t_{i+1j}$ (here $a < \beta$ means $0 < \beta - a \in \mathbb{R}$) since $[t] \in P([l])$. Clearly, this implies $[t] + \delta^{ij} \in P([l])$. For a^-_{ij} the proof is analogous. ∎

Applying [4, Section 2.3] to the set $P([t])$ we obtain:

COROLLARY 1. *Let $V = V([l])$ be a $\mathbb{C}$-space with the basis $\{v_{[t]}, [t] \in P([l])\}$.*
1. *The* GZ*-formulae define on V the structure of a $\mathfrak{G}$-module;*
2. $\mathfrak{N}_+ v_{[l]} = 0$*;*
3. *If $\mathfrak{N}_+ v_{[t]} = 0$ then $[t] = [l]$.*

PROOF. The first statement is a straightforward corollary from [4]. The second and the third ones follow from the GZ-formulae. ∎

PROPOSITION 1. *Let $\lambda - \rho \in \mathfrak{H}^*$ be a weight of $v_{[l]}$. Then*
1. *V is generated by $v_{[l]}$;*
2. *$M(\lambda) \simeq V$;*
3. *$M(\lambda)$ is irreducible.*

PROOF. The first statement follows from the GZ-formulae by direct calculation. To prove the rest consider $M(\lambda)$. It is a GZ-module with $P\big([l], M(\lambda)\big) \subset P([l])$. Since all tableaux from $P([l])$ are good it follows by direct calculation that V and $M(\lambda)$ have equal dimensions of the weight spaces with respect to $\mathfrak{H}$. Thus $M(\lambda) \simeq V$. But V is simple since $v_{[l]}$ is the unique (up to a scalar) primitive element in V by Corollary 1. ∎

By Proposition 1 we construct the tableaux realization for a family of Verma modules defined by n (or $n-1$ in the case of $\mathrm{sl}(n, \mathbb{C})$) parameters $(a_1, \ldots, a_n)$ lying in "general position" (this means that $a_i - a_j \notin \mathbb{Z}$), *i.e. generic* Verma modules. The following properties of generic Verma modules are easy:

LEMMA 4. *Let $M(\lambda)$ be a generic Verma module then*
1. $\mathrm{Ext}^1\big(M(\lambda), M(\mu)\big) = \mathrm{Ext}^1\big(M(\mu), M(\lambda)\big) = 0$ *for any $\mu \in \mathfrak{H}^*$;*
2. *$M(\lambda) \otimes F$ is completely reducible for any finite-dimensional module F.*

PROOF. The first statement follows from the fact that the GZ-weight lattices of $M(\lambda)$ and $M(\mu)$ have empty intersection. The second one follows from the Kostant theorem ([11]). ∎

And finally we have:

COROLLARY 2. *Let $M(\lambda)$ be a generic Verma module with primitive generator v having the tableau $[l] \in L$. Then $[l] = [l](a)$ for some $a \in \mathbb{C}^n$ (see the definition of the set $P([l])$).*

PROOF. Follows from Proposition 1 and [4, Theorem 17, Theorem 24]. ∎

Regretfully, we can not construct the tableaux realization for an arbitrary Verma module since till now there does not exist any generalization of the GZ-formulae in the case of modules with non-trivial GZ-root subspaces.

5**. Generalized Verma modules.** Consider a subset $S \subset \pi$ and let Δ_S be the root subsystem of Δ generated by S. Denote by $\mathfrak{G}_S$ a semisimple Lie subalgebra of $\mathfrak{G}$ generated by X_α, $\alpha \in \Delta_S$ and by $\mathfrak{N}_S$ the subalgebra of $\mathfrak{N}_+$ generated by all X_α, $\alpha \in \Delta_+ \setminus \Delta_S$. Let $\mathfrak{H}_S = \mathfrak{H} \cap \mathfrak{G}_S$ be a Cartan subalgebra of $\mathfrak{G}_S$ and $\mathfrak{H}^S$ be the complement to $\mathfrak{H}_S$ in $\mathfrak{H}$ with respect to the standard form on $\mathfrak{H}$.

Set $\rho = \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$, $\rho_S = \frac{1}{2}\sum_{\alpha\in\Delta_+\cap\Delta_S}\alpha$ and $\rho^S = \rho - \rho_S$.

Consider a parabolic subalgebra $\mathfrak{B}_S = \mathfrak{G}_S + \mathfrak{H} + \mathfrak{N}_+$. An important role in the representation theory of $\mathfrak{G}$ is played by the so-called generalized Verma modules (GVM), those are the universal modules in a category of $\mathfrak{G}$-modules induced form $\mathfrak{G}_S$ ([2]).

Let V be an irreducible weight $\mathfrak{G}_S$-module and $\lambda \in (\mathfrak{H}^S)^*$. Putting $\mathfrak{N}_S v = 0$ and $hv = (\lambda(h) - \rho^S)v$ for all $v \in V$ and $h \in \mathfrak{H}^S$ we define a $\mathfrak{B}_S$-module structure on V. The module

$$M(\lambda, V) = U(\mathfrak{G}) \bigotimes_{U(\mathfrak{B}_S)} V$$

is called *generalized Verma module corresponding* to π, S, V and λ.

A non-zero element w of a $\mathfrak{G}$-module W will be called *S-primitive* provided $\mathfrak{N}_S v = 0$. Clearly, each element of the form $1 \otimes v \in M(\lambda, V)$ for $0 \neq v \in V$ is S-primitive. The following statement describes the universal property of $M(\lambda, V)$ ([2]):

LEMMA 5. *Let W be a weight $\mathfrak{G}$-module generated by an S-primitive element v such that the module $U(\mathfrak{G}_S)v$ is simple. Then W is a quotient of a generalized Verma module.*

A $\mathfrak{G}$-module V is called *S-stratified* ([2]) provided the action of X_α, $\alpha \in \Delta_S$ is injective on V.

Note, that recently there appeared a number of papers devoted to the study of GVM, see for example [2, 5, 6, 7, 12] and references therein.

6**. Tableaux realization for α-stratified generalized Verma modules over** $\mathrm{sl}(3, \mathbb{C})$**.** Let $\mathfrak{G} = \mathrm{sl}(3, \mathbb{C})$, $\mathfrak{H}$ consist of all diagonal matrices with zero trace, Δ be the standard root system with a basis $\pi = \{\alpha, \beta\}$ and $S = \{\alpha\}$. We fix the standard Weyl-Chevalley basis in $\mathfrak{G}$.

In the case when $S = \{\alpha\}$, S-stratified modules are also called *α-stratified* ([2]). The structure of α-stratified generalized Verma modules over $\mathfrak{G}$ was investigated in [7]. The aim of this section is to construct the tableaux realization for α-stratified GVM $M(\lambda, V)$, $\lambda \in (\mathfrak{H}^\alpha)^*$.

It is well-known ([2]) that $M(\lambda, V)$ is α-stratified if and only if V is an irreducible weight $\mathrm{sl}(2, \mathbb{C})$-module without highest and lowest weights. Unfortunately, there exists an α-stratified GVM such that the action of the "small Casimir operator" $C = (H_\alpha + 1)^2 + 4X_{-\alpha}X_\alpha$ is not diagonalizable on it. And this is the case as soon as our GVM has the biggest possible number of non-trivial submodules ([7]). Thus, we are not able to construct the tableaux realization for such GVMs in our terms. To overcome this difficulty we need a notion of the backward GZ-modules.

The GZ-modules defined in Section 3 correspond to the inclusions $\mathrm{gl}(k,\mathbb{C}) \subset \mathrm{gl}(k+1,\mathbb{C})$ with respect to the left upper corner of the matrix. We will call such modules forward GZ-modules. One can consider another kind of inclusions with respect to the right lower corner. More precisely, one can identify $\mathrm{gl}(k,\mathbb{C})$ with the subalgebra of $\mathrm{gl}(k+1,\mathbb{C})$ generated by matrix units $\{e_{ij} \mid i,j = 2,\dots,k+1\}$. Following the same procedure as in Section 3 one obtains a class of backward GZ-modules. We will call the corresponding GZ-subalgebra backward GZ-subalgebra and will denote it by Γ_b. For backward GZ-modules we will use the same notations as for forward GZ-modules, adding "backward" to them.

Clearly, each weight module with finite-dimensional weight spaces necessarily is a forward GZ-module as well as a backward GZ-module.

LEMMA 6. *There exists a forward* GZ-*module V over* $\mathrm{gl}(n,\mathbb{C})$ *which is not a backward* GZ-*module and vice versa.*

PROOF. Let V be a generic forward GZ-module defined in Section 4. Consider the element $a = e_{n-1\,n}e_{nn-1} \in \Gamma_b$. The GZ-formulae imply immediately that a has no eigenvectors on V an thus V can not be a backward GZ-module. ∎

Since every α-stratified GVM is a weight module with finite-dimensional weight spaces it is both forward and backward GZ-module.

It was mentioned that any Verma module $M(\lambda)$ with a dominant integral λ has no tableaux realization. In this section we will show that almost all analogous α-stratified generalized Verma modules (over $\mathrm{sl}(3,\mathbb{C})$) has the tableaux realization as backward GZ-modules.

Set $\mathfrak{G}^{\beta} = \langle X_\beta, X_{-\beta}, H_\beta\rangle$.

As a first step we construct a family of generic α-stratified GVM together with their tableaux realization.

Consider a map $\Phi\colon \mathbb{C}^4 \longrightarrow L$ defined as follows: $\Phi(a,b,c,x) = [l]$, where

$$l_{i1} = a, \quad i = 1,2,3;\ l_{22} = x;\ l_{32} = b;\ l_{33} = c.$$

Let $[l] = \Phi(a,b,c,x)$. Consider a set $P_\alpha([l])$ consisting of all tableaux $[t]$ such that

1. $t_{3j} = l_{3j}$, $j = 1,2,3$;
2. $t_{22} - l_{22} \in \mathbb{Z}$;
3. $l_{i1} - t_{i1} \in \mathbb{Z}_+$, $i = 1,2$;
4. $t_{11} \leq t_{21}$.

Let $V = V([l])$ be a $\mathbb{C}$-space with the basis $v_{[t]}$, $[t] \in P_\alpha([l])$.

LEMMA 7. *1. If $x - a \notin \mathbb{Z}$ then the backward* GZ-*formulae define on V the structure of a $\mathfrak{G}$-module.*

2. V is α-stratified if and only if $x - c, x - b \notin \mathbb{Z}$.

PROOF. The first statement follows from [4]. The second one follows easily from the backward GZ-formulae. ■

The following statement covers all results from [7] for modules defined above.

PROPOSITION 2. *Suppose that $x - a, x - b, x - c \notin \mathbb{Z}$. Then*
1. V is an α-stratified GVM.
2. All α-primitive generators of V have the following form:

$$\sum_{k=-n}^{n} s_i v_{[l]+k\delta^{22}}, \quad s_i \in \mathbb{C}.$$

3. If $v_{[t]}$ is an α-primitive element then either $b = t_{21} = t_{11}$ or $c = t_{21} = t_{11}$.
4. V is irreducible if and only if $a - b \notin \mathbb{N}$ and $a - c \notin \mathbb{N}$.
5. If $a - b \in \mathbb{N}$ and $a - c \notin \mathbb{N}$ then V has the unique submodule W generated by an α-primitive element which has the tableau $[t] = \Phi(b, a, c, x)$. Both W and V/W are irreducible.
6. If $a - c \in \mathbb{N}$ and $a - b \notin \mathbb{N}$ then V has the unique submodule W generated by an α-primitive element which has the tableau $[t] = \Phi(c, a, b, x)$. Both W and V/W are irreducible.
7. If $a - b \in \mathbb{N}$, $b - c \in \mathbb{N}$ then $W_1 \subset W_2 \subset V$, where W_1 is generated by an α-primitive element which has the tableau $[t] = \Phi(c, a, b, x)$ and W_2 is generated by an α-primitive element which has the tableau $[s] = \Phi(b, a, c, x)$. Modules W_1, W_2/W_1, V/W_2 are irreducible.

PROOF. Let $\mu \in \mathfrak{H}^*$ be the weight of the element $v_{[l]}$. Consider an $U(\mathfrak{G}^\beta)$-submodule $K = \oplus_{k\geq 0} V_{\mu - k\beta}$ of V. One has $\dim K_{\mu-k\beta} = k + 1$ by direct calculation. Thus, comparing the dimensions of the weight spaces of V with the dimensions of the weight spaces in the corresponding GVM $M(\nu', U(\mathfrak{G}_S)v_{[l]})$ ($\nu' \in (\mathfrak{H}^\alpha)^*$) we obtain that $V \simeq M(\nu', U(\mathfrak{G}_S)v_{[l]})$. From the backward GZ-formulae it follows that an action of $X_{\pm\alpha}$ on $U(\mathfrak{G}_S)v_{[l]}$ is injective, hence V is α-stratified. This proves the first statement. The rest follows easily from the backward GZ-formulae. ■

7. **Tableaux realization of generalized Verma modules over** $\mathrm{sl}(n, \mathbb{C})$**.** A trick described in Section 6 allows one to consider a new family of GVM over the Lie algebra $\mathrm{gl}(n, \mathbb{C})$ ($\mathrm{sl}(n, \mathbb{C})$).

Let $\mathfrak{G} = \mathrm{gl}(n, \mathbb{C})$, $\mathfrak{H}$ be the subalgebra of all diagonal matrices, $\pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ be the standard basis of the root system Δ. We fix the standard Weyl-Chevalley basis in $\mathfrak{G}$. Let $S = \{\alpha_2, \ldots, \alpha_{n-1}\}$. The aim of this section is to construct a family of forward GZ-modules which are GVMs with respect to S.

Consider a map $\Phi: L(n-1) \oplus \mathbb{C} \longrightarrow L(n)$ defined as follows: for $[t] \in L_{n-1}, a \in \mathbb{C}$ we set $\Phi\big(([t], a)\big) = [l]$, where

$$l_{i1} = a, \quad i = 1, \ldots, n; \; l_{ij} = t_{i-1\, j-1}, \quad 2 \leq j \leq i \leq n.$$

For $[l] = \Phi\big(([t], a)\big)$ consider a set $P_S([l])$ consisting of all tableaux $[s]$ such that

1. $l_{nj} = s_{nj}$, $j = 1, \ldots, n$.
2. $l_{i1} - s_{i1} \in \mathbb{Z}_+$, $i = 1, \ldots, n-1$.
3. $s_{i1} \geq s_{i-1\,1}$, $i = 2, \ldots, n-1$.
4. $l_{ij} - s_{ij} \in \mathbb{Z}$, $2 \leq j \leq i \leq n-1$.

Let $V = V([l])$ be a $\mathbb{C}$-space with the basis $v_{[s]}$, $[s] \in P_S([l])$. The following lemma is obvious:

LEMMA 8. *If $l_{ij} - l_{ik} \notin \mathbb{Z}$, $1 \leq k < j \leq i < n$; $l_{ij} - l_{i-1\,k} \notin \mathbb{Z}$, $i = 2, \ldots, n$, $j = 2, \ldots, i$, $k = 2, \ldots, i-1$ then* GZ*-formulae define on V the structure of a $\mathfrak{G}$-module.*

For the rest of the section we will assume that $[l]$ satisfies all conditions of Lemma 8. The following lemma describes the basic properties of V:

LEMMA 9. *1. V is a* GVM *with respect to S.*

2. If $v_{[s]}$ is an S-primitive generator of V then $s_{i1} = a$, $i = 1, \ldots, n$.

3. If $v_{[s]}$ is an S-primitive element in V then $[s] = \Phi\big(([f], l_{nj})\big)$ for some $[f] \in L(n-1)$ and $j \in \{1, \ldots, n\}$.

4. V is irreducible if and only if $l_{n1} - l_{nj} \notin \mathbb{N}$ for all $1 \leq j \leq n$.

PROOF. Clearly, V is a quotient of some GVM $M(\lambda, W)$ generated by an S-primitive element of weight μ (see Lemma 5). One can see that the length of an $U(\mathfrak{G}_S)$-module $U(\mathfrak{G}_S)X^k_{-\alpha_1}v_{[l]}$ equals to the length of an $U(\mathfrak{G}_S)$-module $U(\mathfrak{G}_S)M(\lambda, W)_{\mu-k\alpha_1}$ (see [12, Section 6]). Thus $V \simeq M(\lambda, W)$. The rest follows from the GZ-formulae. ∎

The following lemma describes the type of the module V:

LEMMA 10. *Let $W = U(\mathfrak{G}_S)v_{[l]}$. Then W is a forward* GZ*-module over $\mathfrak{G}_S$. Moreover, W is a generic* GZ*-module.*

PROOF. Clearly, for the first part it is sufficient to prove that $v_{[l]}$ is a GZ-weight vector with respect to the forward GZ-subalgebra of $U(\mathfrak{G}_S)$.

Let $\mathfrak{A}_k$, $k = 2, \ldots, n$ be a subalgebra of $\mathfrak{G}$ generated by the matrix units $\{e_{ij} \mid i, j = 1, \ldots, k\}$ and set $(\mathfrak{A}_k)_S = \mathfrak{A}_k \cap \mathfrak{G}_S$. We denote by $\mathfrak{H}_i$ the standard Cartan subalgebra of $\mathfrak{A}_i$ and by $\mathfrak{H}_i^S$ the complement of $\mathfrak{H}_i \cap \mathfrak{G}_S$ in $\mathfrak{H}_i$ with respect to the standard form on $\mathfrak{H}_i$.

Consider the S-homomorphism of Harish-Chandra (see [5] for more details):

$$\varphi_S \colon Z(\mathfrak{A}_i) \longrightarrow Z\big((\mathfrak{A}_i)_S\big) \otimes S(\mathfrak{H}_i^S).$$

For any S-primitive element v and for any $z \in Z(\mathfrak{A}_i)$ we have $zv = \varphi_S(z)v$. By the S-Harish-Chandra theorem ([12, Theorem 3]) there exists a homomorphism $\psi \colon Z\big((\mathfrak{A}_i)_S\big) \otimes S(\mathfrak{H}_i^S) \longrightarrow S(\mathfrak{H}_i)^W$ such that its restriction on $Z\big((\mathfrak{A}_i)_S\big)$ coincides with the Harish-Chandra homomorphism ([3]) and $\psi \circ \varphi_S$ is an isomorphism. Thus for any $u \in Z\big((\mathfrak{A}_i)_S\big)$ there exists $z \in Z(\mathfrak{A}_i)$ and $h \in S(\mathfrak{H}_i^S)$ such that $\varphi_S(z) = u \otimes h$. This proves the first statement.

The second part now follows from the conditions of Lemma 8 and the GZ-formulae. ∎

Now we are able to describe the structure of V.

PROPOSITION 3. *Let* $M = \{l_{nj} \mid 2 \leq j \leq n, l_{n1} - l_{nj} \in \mathbb{N}\}$ *and* $m = |M|$. *Set* $M = \{p_1, \ldots, p_m\}$; $p_j > p_{j+1}$, $j = 1, \ldots, m-1$. *The length of* V *equals* $m+1$. *A composition series for* V *has the following form:*

$$0 = V_{m+1} \subset V_m \subset \cdots \subset V_1 \subset V_0 = V,$$

where V_j *is generated by an* S*-primitive* GZ*-weight element* $v_{[l_j]}$ *and* $[l_j] = \Phi\big(([t_j], p_j)\big)$ *for a suitable* $[t_j]$.

PROOF. One can obtain all these results directly from the construction of $P_S([l])$ and the GZ-formulae. ∎

Let $L(V)$ denote a unique simple quotient of V.

COROLLARY 3. *The sequence*

$$0 \longrightarrow V_1 \xrightarrow{\varepsilon'} V \xrightarrow{\varepsilon} L(V) \longrightarrow 0,$$

where ε' *is the canonical inclusion and* ε *is the canonical projection, is exact.*

It is natural to call this sequence a BGG-resolution of V.

We note that in the case $\pi \setminus S = \{\alpha_{n-1}\}$ the exactness of the analogous sequence is an open problem (see [12, Section 4]). And the complete structure of the corresponding GVMs in that case is still unknown. In that case the only known result is the tableaux realization for a class of generic GVMs. But generic GVMs considered in [12] may have only one proper submodule. In the most interesting case when a GVM has the maximal possible number of proper submodules, it has no tableaux realization. This shows us a brief analogue with Section 6. It happens that the structure of the modules considered in this section is more simple than those considered in [12]. We have already mentioned that our family contains GVMs having more than one proper submodule. Nevertheless, we manage to construct the tableaux realization in all cases.

The following conjecture seems to be reasonable:

CONJECTURE 1. *Let* $\mathfrak{G} = \mathrm{sl}(n, \mathbb{C})$, $\pi \setminus S = \{\alpha_1\}$, V *be a generic* GZ*-module over* $\mathfrak{G}_S$ *(forward or backward). Suppose that the module* $M(\lambda, V)$ *is reducible. Then the maximal submodule of* $M(\lambda, V)$ *has the form* $M(\mu, W)$ *for suitable* μ *and* W.

By Corollary 3 this is the case if $M(\lambda, V)$ has the tableaux realization.

8**. Structure of induced modules.** Proposition 3 allows us to formulate an analogue of the BGG-criterion for the existence of a submodule in a GVM over $\mathrm{gl}(n, \mathbb{C})$ ($\mathrm{sl}(n, \mathbb{C})$) induced from the generic GZ-module over an arbitrary subalgebra $\mathfrak{G}_S$, $S \subset \pi$. More precisely we describe the submodule structure for all GVMs having special form (we are unable to overcome the difficulties which are described in Section 7).

In this section we give all necessary constructions, definitions and formulations. One can prove all the results following the routine procedure of [12].

Consider $\mathfrak{G} = \mathrm{gl}(n, \mathbb{C})$ and $S \subset \pi$, $S \neq \pi$. For $1 \leq j \leq i \leq n$ we will write $j \bowtie i$ if $\{\alpha_j, \alpha_{j+1}, \ldots, \alpha_i\} \subset S$ and $j \not\bowtie i$ in the opposite case.

Let $[l] \in L$ be a tableau satisfying the following *generating conditions*:

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1. $l_{ij} = l_{nj}$ for $1 \leq j \leq i < n$ such that $j \not\bowtie i$;
2. $l_{ik} - l_{ij} \notin \mathbb{Z}$ for $1 \leq k < j \leq i < n$ such that $j \bowtie i$;
3. $l_{ik} - l_{i-1\,j} \notin \mathbb{Z}$ for $1 \leq i < n$, $1 \leq j \leq i-1$, $1 \leq k \leq i$ such that $j \bowtie i$.

Define an action of the Weyl group $W \simeq S_n$ on the set consisting of all tableaux $[l]$ satisfying the generating conditions, as follows: for $\sigma \in W$ set $\sigma([l]) = [t]$, where

1. $t_{nj} = l_{n\sigma(j)}$, $j = 1, \ldots, n$;
2. $t_{ij} = l_{n\sigma(j)}$ for $1 \leq j \leq i < n$ such that $j \not\bowtie i$;
3. $t_{ij} = l_{ij}$ in the remaining cases.

For $\beta \in \Delta_+$ we will denote by $s_\beta \in W$ the reflection corresponding to the root β.

We will write $[l] \simeq [t]$ if the tableau $[s] = [l] - [t]$ satisfies the following conditions:

1. $s_{ij} = 0$ for all $1 \leq j \leq i \leq n$ such that $j \not\bowtie i$;
2. $s_{ij} \in \mathbb{Z}$ for all $1 \leq j \leq i \leq n$ such that $j \bowtie i$.

We assume that $\alpha_n \notin S$.

For $a = (a_1, \ldots, a_k) \in \mathbb{C}^k$ and $\sigma \in S_n$ set $\sigma(a) = (a_{\sigma(1)}, \ldots, a_{\sigma(k)})$. For $a = (a_1, \ldots, a_k)$, $b = (b_1, \ldots, b_k) \in \mathbb{C}^k$ we will write $a \leq b$ if there exists $\sigma \in S_k$ such that $b - \sigma(a) \in \mathbb{Z}_+^k$. Let $[l]$ be a tableau satisfying the generating conditions. For $i \in \{1, \ldots, n\}$ we will denote by $[l]_i^r$ the vector which can be obtained from $[l]_i$ by erasing of those l_{ij}, $1 \leq j \leq i$ for which $j \bowtie i$.

We define a partial order $\leq$ on L as follows: set $[l] \leq [t]$ if and only if the following two conditions are satisfied:

1. $l_{ij} - t_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i < n$ such that $j \not\bowtie i$;
2. $[l]_i^r \leq [t]_i^r$ for $1 \leq i \leq n-1$.

Obviously, there exists a GVM $M(\lambda, V) = M([l])$ generated by an S-primitive element $v_{[l]}$ which has the tableau $[l]$. It follows from Lemma 10 that the $\mathfrak{G}_S$-module $K = U(\mathfrak{G}_S)v_{[l]}$ is an irreducible generic GZ-module over $\mathfrak{G}_S$ (*i.e.* K is irreducible and is a direct sum of generic GZ-submodules with respect to connected components of $\mathfrak{G}_S$). We will denote by $L([l])$ the unique irreducible quotient of $M([l])$. The following result is an obvious property of the defined action:

LEMMA 11. *Let $W_S \subset W$ be the Weyl group of the root system Δ_S and $\sigma \in W_S$. Let $[l]$ be a tableau satisfying the generating conditions. Then $M([l]) \simeq M(\sigma[l])$.*

The following theorem can be obtained in the same way as [12, Theorem 8].

THEOREM 1. *Let $[l]$, $[t]$ be a tableaux which satisfies the generating conditions. The following statements are equivalent:*

1. *$M([t]) \subset M([l])$.*
2. *$L([t])$ is a subquotient of $M([l])$.*
3. *There exists a sequence $\beta_1, \ldots, \beta_k$ of elements from Δ_+ such that*

$$[t] \leq s_{\beta_1}[t] \leq \cdots \leq s_{\beta_k} \cdots s_{\beta_1}[t] \simeq [l].$$

COROLLARY 4. *$M([l])$ is irreducible if and only if $l_{nj} - l_{nk} \notin \mathbb{N}$ for any $1 \leq j < k \leq n$ such that $j \not\bowtie k$.*

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