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RESEARCH ARTICLE

Motivic Steenrod operations in characteristic *p*

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Abstract

For a prime p and a field k of characteristic p, we define Steenrod operations P^n_k on motivic cohomology with $\mathbb{F}_p$-coefficients of smooth varieties defined over the base field k. We show that P^n_k is the pth power on $H^{2n,n}(-,\mathbb{F}_p) \cong CH^n(-)/p$ and prove an instability result for the operations. Restricted to mod p Chow groups, we show that the operations satisfy the expected Adem relations and Cartan formula. Using these new operations, we remove previous restrictions on the characteristic of the base field for Rost's degree formula. Over a base field of characteristic 2, we obtain new results on quadratic forms.

Contents

1. Introduction

Voevodsky constructed motivic reduced power operations P^n_F for $n \geq 0$ where the base field F is a perfect field with char(F) not equal to the characteristic $p > 0$ of the coefficient field [35]. These operations were used in the proof of the Bloch–Kato conjecture. Hoyois, Kelly and Østvær later obtained operations and completeley determined the Steenrod algebra for a general base field F with char$(F) \neq p$ [17]. Brosnan gave an elementary construction of Steenrod operations on mod p Chow groups over a base field of characteristic $\neq p$ [1]. Steenrod operations on Chow groups have been used succesfully in the study of quadratic forms over a base field of characteristic $\neq 2$ and to prove degree formulas in algebraic geometry, as in [6] and [24].

For a prime p, Voevodsky's construction of Steenrod operations for the coefficient field $\mathbb{F}_p$ uses the calculation of the motivic cohomology of BS_p. However, when defined over a base field k of characteristic p, $B\mathbb{Z}/p$ is contractible [25, Proposition 3.3]. Hence, over the base field k, $H^{*,*}(BS_p, \mathbb{F}_p) \cong H^{*,*}(k, \mathbb{F}_p)$, and so one cannot carry out Voevodsky's construction. It has also been an open problem to just define Steenrod operations on the mod p Chow groups of smooth schemes over a field of characteristic p. Haution made progress on this problem by constructing the first $p-1$ homological Steenrod operations on Chow groups mod p and p-primary torsion over any base field [12], defining the first Steenrod square on mod 2 Chow groups over any base field [13] and constructing weak forms of the second and third Steenrod squares over a field of characteristic 2 [15]. Note that in articles where Steenrod squares (or weak forms of Steenrod squares) on mod 2 Chow groups are used, the nth Steenrod square on mod 2 Chow groups corresponds to the $2n$th Steenrod square on mod 2 motivic cohomology, since the Bockstein homomorphism is 0 on mod 2 Chow groups.

For p a prime, we use the results of Frankland and Spitzweck in [8] to define Steenrod operations $P_k^n : H^{i,j}(-, \mathbb{F}_p) \to H^{i+2n(p-1), j+n(p-1)}(-, \mathbb{F}_p)$ for $n \geq 0$ on the mod p motivic cohomology of smooth schemes over a field k of characteristic p. Note that some authors use the notation $H^i(-, \mathbb{Z}(j))$ in place of $H^{i,j}(-, \mathbb{Z})$ to denote motivic cohomology. For $n \geq 1$, we show that P_k^n is the pth power on $H^{2n,n}(-, \mathbb{F}_p) = CH^n(-)/p$, and we prove an instability result for the Steenrod operations. Restricted to mod p Chow groups, I prove that the P_k^i satisfy expected properties such as Adem relations and the Cartan formula. we also show that the operations P_k^n agree with the operations P_K^n, constructed by Voevodsky for $\mathrm{char}(K) = 0$, on the mod p Chow rings of flag varieties in characteristic 0.

To show that the P_k^i satisfy the Adem relations and Cartan formula on mod p Chow groups, we show that the Steenrod operations satisfy the Adem relations and Cartan formula on mod p motivic cohomology up to some error terms. These error terms vanish when we restrict to mod p Chow groups. If the dual Steenrod algebra has the conjectured form (meaning that the map Ψ_k from Theorem 2.3 is an isomorphism), then the error terms encountered in these arguments vanish on mod p motivic cohomology. Our proofs would then simplify to give the Adem relations and Cartan formula for motivic cohomology with mod p coefficients.

In Section 9, I extend Rost's degree formula [24, Theorem 6.4] to a base field of arbitrary characteristic. The degree formula we obtain at odd primes seems to be new.

In Section 11, we use the new operations to study quadratic forms defined over a base field of characteristic 2. Previous results or proofs have avoided the case of quadratic forms in characteristic 2, since Steenrod squares were not available. We recall a conjecture of Hoffmann and Totaro on the possible values of the first Witt index of an anisotropic quadratic form.

Conjecture 1.1. *Let φ be an anisotropic quadratic form over a field F such that* $\dim\varphi \geq 2$. *Then* $\mathfrak{i}_1(\varphi) \leq 2^{v_2(\dim\varphi - \mathfrak{i}_1(\varphi))}$.

As documented in [20], Conjecture 1.1 was first made by Hoffmann in 1998 assuming that the base field is of characteristic $\neq 2$. Using Steenrod squares on mod 2 Chow groups, Karpenko proved this conjecture for anisotropic quadratic forms over base fields fields of characteristic $\neq 2$ [18]. In [31], Totaro extended Conjecture 1.1 to fields of characteristic 2. Using algebraic methods, Scully proved Conjecture 1.1 for totally singular anisotropic quadratic forms over base fields of characteristic 2 [28]. We also remark that Haution previously used a weak form of the first homological Steenrod square to prove a result on the parity of the first Witt index for nonsingular anisotropic quadratic forms over a field of characteristic 2 [14, Theorem 6.2].

In this article, we prove Conjecture 1.1 for nonsingular anisotropic quadratic forms over base fields of characteristic 2. Our My proof copies the arguments of [18] and makes use of the new Steenrod squares defined on mod 2 Chow groups over base fields of characteristic 2. In a recent preprint, Karpenko proved Conjecture 1.1 for the remaining cases of anisotropic quadratic forms over base fields of characteristic 2 [20]. That proof uses the Steenrod squares constructed in this article, along with other new ideas.

Other new results on quadratic forms over base fields of characteristic 2 are also included in Section 11. Using the Steenrod squares defined in this article, it should be possible to extend other results on quadratic forms to the case where the base field has characteristic 2.

2. Prior results on the dual Steenrod algebra and setup

Let k be a field of characteristic $p > 0$. For a base scheme S, let Sm_S denote the category of quasi-projective separated smooth schemes of finite type over S, let $H(S)$ denote the unstable motivic homotopy category of spaces over S defined by Morel and Voevodsky [25], let $H_\bullet(S)$ denote the pointed unstable motivic homotopy category of spaces over S and let $SH(S)$ denote the stable motivic homotopy category of spectra over S [33]. Let

$$\Sigma^\infty_+ : \mathrm{Sm}_S \to SH(S),$$

$$\Sigma^\infty_+ : H(S) \to H_\bullet(S) \to SH(S)$$

denote the infinite $\mathbb{P}^1$-suspension functors.

We recall some results from [8] and [30] in the categories $H(k)$ and $SH(k)$. Let $B\mu_p \in H(k)$ denote the geometric motivic classifying space of the group scheme μ_p over k of the pth roots of unity. Let $H\mathbb{F}^k_p \in SH(k)$ denote the motivic Eilenberg–MacLane spectrum representing mod p motivic cohomology. Let $v \in H^{2,1}(B\mu_p, \mathbb{F}_p)$ denote the pullback of the first Chern class $c_1 \in H^{2,1}(B\mathbb{G}_m, \mathbb{F}_p)$. From the computation of the motivic cohomology of $B\mu_p$ in [35, Theorem 6.10], there exists a unique $u \in H^{1,1}(B\mu_p, \mathbb{F}_p)$ such that $\beta(u) = v$, where β denotes the Bockstein homomorphism on mod p motivic cohomology. The class of $\rho = -1$ in $H^{1,1}(k, \mathbb{F}_p) = k^*/k^{*p}$ is 0, and the class $\tau \in H^{0,1}(k, \mathbb{F}_p) = \mu_p(k) = 0$ described in [35, Theorem 6.10] is also 0. We need the following computation, which can be deduced from [35, Theorem 6.10] by setting $\rho = 0$ and $\tau = 0$.

Theorem 2.1. *There is an isomorphism*

$$H^{*,*}(B\mu_p, \mathbb{F}_p) \cong H^{*,*}(k, \mathbb{F}_p)[\![v, u]\!]/(u^2).$$

Note that $H^{*,*}(B\mu_p, \mathbb{F}_p)$ is defined in [35] as a limit of motivic cohomology rings of smooth schemes over the base field. This explains why power series appear in this theorem.

Let $\mathcal{A}^k_{*,*} := \pi_{*,*}(H\mathbb{F}^k_p \wedge H\mathbb{F}^k_p)$. As described in [30, Chapter 10.2], there is a coaction map

$$H^{*,*}(B\mu_p, \mathbb{F}_p) \to \mathcal{A}^k_{-*,-*} \widehat{\otimes}_{\pi_{-*,-*}H\mathbb{F}^k_p} H^{*,*}(B\mu_p, \mathbb{F}_p). \tag{1}$$

We use the left $H\mathbb{F}^k_p$-module structure on $H\mathbb{F}^k_p \wedge H\mathbb{F}^k_p$ for this coaction map. For $i \geq 0$ and $j \geq 1$, classes $\tau_i \in \mathcal{A}^k_{2p^i-1,p^i-1}$ and $\xi_j \in \mathcal{A}^k_{2p^j-2,p^j-1}$ are defined by the coaction map:

$$u \mapsto u + \Sigma_{i\geq 0}\tau_i \otimes v^{p^i},$$

$$v \mapsto v + \Sigma_{j\geq 1}\xi_j \otimes v^{p^j}.$$

Proposition 2.2. $\tau_i^2 = 0$ *for all* $i \geq 0$.

Proof. We use the argument of [35, Theorem 12.6]. First, we assume that $\mathrm{char}(k) = 2$. Under the coaction map 1,

$$u^2 = 0 \mapsto u^2 + \Sigma_{i\geq 0}\tau_i^2 \otimes v^{2^{i+1}} = 0.$$

For $i \geq 0$, the coefficient of $v^{2^{i+1}}$ equals $0 = \tau_i^2$.

Now we assume that $p = \mathrm{char}(k)$ is odd. Let $i \geq 0$. As $\mathcal{A}^k_{*,*}$ is graded-commutative under the first grading, we have $\tau_i^2 = (-1)^{(2p^i-1)(2p^i-1)}\tau_i^2 = -\tau_i^2$, which implies that $\tau_i^2 = 0$. □

In this article, we shall consider finite sequences $\alpha = (\epsilon_0, r_1, \epsilon_1, r_2, \ldots)$ of integers such that $\epsilon_i \in \{0, 1\}$ and $r_j \geq 0$ for all $i \geq 0$ and $j \geq 1$. From now on, it will be assumed that any sequence α in this article satisfies these conditions. To a sequence α, associate a monomial $\omega(\alpha) = \tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\cdots \in \mathcal{A}^k_{*,*}$ of bidegree (p_α, q_α). The sequences α induce a morphism

$$\Psi_k : \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \to H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$$

of left $H\mathbb{F}_p^k$-modules. Frankland and Spitzweck proved the following theorem [8, Theorem 1.1], which allows us to define Steenrod operations on mod p motivic cohomology over the base field k.

Theorem 2.3. *The morphism*

$$\Psi_k : \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \to H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k$$

is a split monomorphism of left $H\mathbb{F}_p^k$-modules.

It is conjectured that Ψ_k is an isomorphism. Frankland and Spitzweck proved this theorem by comparing Ψ_k to the corresponding isomorphism

$$\Psi_K : \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K \to H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \tag{2}$$

of left $H\mathbb{F}_p^K$-modules for $\mathrm{char}(K) = 0$. From now on, $\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K$ will be identified with $H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K$ as left $H\mathbb{F}_p^K$-modules through Ψ_K whenever K is a field of characteristic 0. Let D be a complete unramified discrete valuation ring with closed point $i : \mathrm{Spec}(k) \to \mathrm{Spec}(D)$ and generic point $j : \mathrm{Spec}(K) \to \mathrm{Spec}(D)$, where $K = \mathrm{Frac}(D)$. For example, when $k = \mathbb{F}_p$, $D = \mathbb{Z}_p$ and $K = \mathbb{Q}_p$.

For a morphism $f : S_1 \to S_2$ of base schemes, let $f_* := Rf_* : SH(S_1) \to SH(S_2)$ and $f^* := Lf^* : SH(S_2) \to SH(S_1)$ denote the right derived push-forward and left derived pullback functors, respectively. Pullback f^* is strongly monoidal, while f_* is lax monoidal. Furthermore, f_* commutes with all suspensions $\Sigma^{i,j}$ [8, Lemma 7.5]. Note also that f_* preserves coproducts [8, Lemma 7.4].

For a separated Noetherian scheme S of finite dimension, let $\widehat{H}\mathbb{Z}^S \in SH(S)$ denote the motivic E_∞ ring spectrum constructed by Spitzweck in [30] and let $\widehat{H}\mathbb{F}_p^S := \widehat{H}\mathbb{Z}^S/p$. Let $D(\widehat{H}\mathbb{Z}^S)$ denote the homotopy category of left $\widehat{H}\mathbb{Z}^S$-modules. See [3, Section 7.2] and [8, Sections 2 and 3] for a discussion on the homotopy category of left R-modules $D(R)$ for a highly structured ring spectrum R. There is a forgetful functor $U_S : D(\widehat{H}\mathbb{Z}^S) \to SH(S)$.

The spectrum $\widehat{H}\mathbb{Z}^S$ enjoys a number of desirable properties. It is Cartesian, which means that for a morphism $f : S_1 \to S_2$ of base schemes, the induced morphism $f^*\widehat{H}\mathbb{Z}^{S_2} \to \widehat{H}\mathbb{Z}^{S_1}$ is an isomorphism in $SH(S_1)$ of E_∞ ring spectra [30, Chapter 9]. Throughout this article, we will frequently identify $f^*\widehat{H}\mathbb{Z}^{S_2}$ with $\widehat{H}\mathbb{Z}^{S_1}$ whenever we are given a morphism $f : S_1 \to S_2$ of base schemes (see also [8, Section 2]). Hence, the square

$$\begin{tikzcd} D(\widehat{H}\mathbb{Z}^{S_2}) \arrow[r, "f^*"] \arrow[d, "U_{S_2}"] & D(\widehat{H}\mathbb{Z}^{S_1}) \arrow[d, "U_{S_1}"] \\ SH(S_2) \arrow[r, "f^*"] & SH(S_1) \end{tikzcd}$$

commutes.

For $S = \mathrm{Spec}(F)$ with F a field, $\widehat{H}\mathbb{Z}^S$ is isomorphic as an E_∞ ring spectrum to the usual Eilenberg–MacLane spectrum $H\mathbb{Z}^S$ constructed by Voevodsky [30, Theorem 6.7]. For the discrete valuation ring D, $\widehat{H}\mathbb{Z}^D$ represents Bloch–Levine motivic cohomology as defined in [23].

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We briefly describe the definition of Bloch–Levine motivic cohomology in [23] for a discrete valuation ring D. Let $X \to \mathrm{Spec}(D)$ be a morphism of finite type with X irreducible. If the image of the generic point η_X of X is $\mathrm{Spec}(k)$, then define $\dim(X) := \dim(X_{\mathrm{Spec}(k)})$. Otherwise, define $\dim(X) := \dim(X_{\mathrm{Spec}(K)}) + 1$. For $n \geq 0$, let $\Delta^n := \mathrm{Spec}(D[t_0, \dots, t_n]/\Sigma_i t_i - 1)$ denote the algebraic n-simplex over D. Let $z_q(X, r)$ denote the free abelian group generated by all irreducible closed subschemes $C \subset \Delta^r \times_{\mathrm{Spec}(D)} X$ of dimension $r+q$ such that C meets each face of $\Delta^r \times_{\mathrm{Spec}(D)} X$ properly. Then set $z^q(X, r) = z_{\dim(X)-q}(X, r)$ to get a pullback homomorphism $z^q(X, r) \to z^q(X, r-1)$ for each face of Δ^r. Then the Zariski hypercohomology of the complex $z^q(X, *)$ with alternating face maps is Bloch–Levine motivic cohomology (with the appropriate shift).

Theorem 2.4. *The morphism $H\mathbb{F}_p^k \cong i^*(\widehat{H}\mathbb{F}_p^D) \to i^* j_* H\mathbb{F}_p^K \cong i^* j_* j^* \widehat{H}\mathbb{F}_p^D$ in $D(H\mathbb{F}_p^k)$ induced by adjunction induces a splitting $i^* j_* H\mathbb{F}_p^K \cong H\mathbb{F}_p^k \oplus \Sigma^{-1,-1} H\mathbb{F}_p^k$ in $D(H\mathbb{F}_p^k)$. There is also a splitting $i^* j_* H\mathbb{Z}^K \cong H\mathbb{Z}^k \oplus \Sigma^{-1,-,1} H\mathbb{Z}^k$ in $D(H\mathbb{Z}^k)$ [8, Lemma 4.10].*

Definition 2.1. Let $\pi : i^* j_* H\mathbb{F}_p^K \to H\mathbb{F}_p^k$ and $\pi_0 : i^* j_* H\mathbb{F}_p^K \to \Sigma^{-1,-1} H\mathbb{F}_p^k$ denote the projections induced by the splitting from Theorem 2.4.

Let $\eta : id. \to j_* j^*$ denote the unit map. From now on, we shall denote all adjunction morphisms $i^* E \to i^* j_* j^* E$ for $E \in SH(D)$ by $i^*\eta$. We will also denote all $\Sigma^{s,t}\pi, \Sigma^{s,t}\pi_0$ by π and π_0, respectively, to make the text easier to read. The morphisms Ψ_k and Ψ_K lift to a morphism

$$\Psi_D : \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} \widehat{H}\mathbb{F}_p^D \to \widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D$$

in $D(\widehat{H}\mathbb{F}_p^D)$ [8, Lemma 3.10]. Applying $i^*\eta$ to Ψ_D gives a commuting square

$$\begin{tikzcd}
\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \arrow[r, "\Psi_k"] \arrow[d, "i^*\eta"] & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} i^* j_* H\mathbb{F}_p^K \arrow[r, "i^* j_* \Psi_K"] & i^* j_* (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K)
\end{tikzcd} \tag{3}$$

in $D(H\mathbb{F}_p^k)$. Let $r : H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \to \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k$ be the retraction of Ψ_k defined by the following composite [8, Theorem 5.1]:

$$\begin{tikzcd}
H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[r, "i^*\eta"] & i^* j_* (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^* j_* \Psi_K^{-1}"] & \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} i^* j_* H\mathbb{F}_p^K \arrow[d, "\oplus\pi"] \\
 & & \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k.
\end{tikzcd}$$

For $S = k, K$, or D (use $\widehat{H}\mathbb{F}_p^D$), let $\mu_1^S : H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S \to H\mathbb{F}_p^S$ denote the multiplication morphism. There is also a multiplication morphism

$$\mu_2^S : (H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S) \wedge (H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S) \to H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S$$

defined in the standard way by interchanging the two middle $H\mathbb{F}_p^S$ terms and then applying $\mu_1^S \wedge \mu_1^S$.

For a sequence α_0, define $i^*\eta_{\alpha_0} : H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \to \Sigma^{p_{\alpha_0},q_{\alpha_0}} H\mathbb{F}_p^k$ in $D(H\mathbb{F}_p^k)$ to be the composite

$$\begin{tikzcd} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[r, "i^*\eta"] & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^*j_*\Psi_K^{-1}"] & \bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} i^*j_*H\mathbb{F}_p^K \arrow[d, "proj."] \\ & & \Sigma^{p_{\alpha_0},q_{\alpha_0}} i^*j_*H\mathbb{F}_p^K \arrow[d, "\pi"] \\ & & \Sigma^{p_{\alpha_0},q_{\alpha_0}} H\mathbb{F}_p^k. \end{tikzcd} \tag{4}$$

The morphism $i^*\eta_{\alpha_0}$ is a retraction of the morphism

$$H\mathbb{F}_p^k \wedge \omega(\alpha_0) : \Sigma^{p_{\alpha_0},q_{\alpha_0}} H\mathbb{F}_p^k \to H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k.$$

From the work of Friedlander and Suslin [9, Corollary 12.2] and Voevodsky [34], Bloch's higher Chow groups are isomorphic to motivic cohomology as defined by Voevodsky. The isomorphism between motivic cohomology and Bloch's higher Chow groups is compatible with pullback maps and product structures [30, Theorem 6.7]. See also [22].

Theorem 2.5. *Let F be a field and let $X \in \mathrm{Sm}_F$. Then*

$$H^{n,i}(X, \mathbb{Z}) \cong CH^i(X, 2i - n)$$

for all n and $i \geq 0$.

Let $n, i \geq 0$ such that $n > 2i$. From Theorem 2.5, $H^{n,i}(X, A) = 0$ for any coefficient ring A and $X \in \mathrm{Sm}_F$.

3. Definition of operations

In this section, we use the results of Frankland and Spitzweck in [8] to define new Steenrod operations P_k^n for $n \geq 0$. Let

$$i_L, i_R : H\mathbb{F}_p^S \to H\mathbb{F}_p^S \wedge H\mathbb{F}_p^S$$

denote the left and right $H\mathbb{F}_p^S$-module maps, respectively, for $S = D$ (use $\widehat{H\mathbb{F}}_p^D$), k, or K. Motivated by the corresponding duality in characteristic 0, we want to define operations $P_k^n \in H\mathbb{F}_p^{k\,*,*} H\mathbb{F}_p^k$ for $n \geq 0$ by taking operations dual to the ξ_1^n.

Definition 3.1. Let α be a sequence. Define $P_k^\alpha \in H\mathbb{F}_p^{k\,*,*} H\mathbb{F}_p^k$ by $P_k^\alpha := i^*\eta_\alpha \circ i_R$. For $n \geq 0$, let $P_k^n = P_k^{(0,n,0,\ldots)}$. Let $\beta_k = P_k^{(1,0,\ldots)}$.

There are corresponding operations P_K^α in characteristic 0 defined from 2 by

$$H\mathbb{F}_p^K \xrightarrow{i_R} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \xrightarrow{proj.} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^K.$$

Definition 3.2. To define a homomorphism

$$\Phi : H\mathbb{F}_p^{K\,*,*} H\mathbb{F}_p^K \to H\mathbb{F}_p^{k\,*,*} H\mathbb{F}_p^k$$

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of graded additive groups, let $f : H\mathbb{F}_p^K \to \Sigma^{k,l} H\mathbb{F}_p^K$ be given. Define $\Phi(f) : H\mathbb{F}_p^k \to \Sigma^{k,l} H\mathbb{F}_p^k$ by $\Phi(f) = \pi \circ i^* j_*(f) \circ i^*\eta$.

$$H\mathbb{F}_p^k \xrightarrow{i^*\eta} i^* j_* H\mathbb{F}_p^K \xrightarrow{i^* j_*(f)} \Sigma^{k,l} i^* j_* H\mathbb{F}_p^K \xrightarrow{\pi} \Sigma^{k,l} H\mathbb{F}_p^k. \tag{5}$$

From the definition of Φ, it is clear that $\Phi(id.) = id$. The following lemma will be important for proving that the operations P_k^n restricted to mod p Chow groups satisfy the Adem relations and Cartan formula:

Lemma 3.1. *Let $X \in \mathrm{Sm}_k$ and let $f : \Sigma_+^\infty X \to \Sigma^{2m,m} H\mathbb{F}_p^k$ be given.*

1. *Let α_0 be a sequence. Consider the morphism*

$$g_{\alpha_0} : H\mathbb{F}_p^k \to \Sigma^{p_{\alpha_0}-1, q_{\alpha_0}-1} H\mathbb{F}_p^k$$

given by the composite

$$H\mathbb{F}_p^k \xrightarrow{i^*\eta} i^* j_* H\mathbb{F}_p^K \xrightarrow{i^* j_*(P_K^{\alpha_0})} i^* j_* \Sigma^{p_{\alpha_0}, q_{\alpha_0}} H\mathbb{F}_p^K \xrightarrow{\pi_0} \Sigma^{p_{\alpha_0}-1, q_{\alpha_0}-1} H\mathbb{F}_p^k.$$

Then $\Sigma^{2m,m} g_{\alpha_0} \circ f = 0$.

2. *The composite*

$$\begin{array}{ccccc}
\Sigma_+^\infty X & \xrightarrow{f} & \Sigma^{2m,m} H\mathbb{F}_p^k & \xrightarrow{i_R} & \Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \\
&&&& \downarrow{\scriptstyle i^*\eta} \\
&&&& i^* j_*(\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \\
&&&& \downarrow{\scriptstyle i^* j_* \Psi_K^{-1}} \\
&&&& \bigoplus_\alpha \Sigma^{p_\alpha+2m, q_\alpha+m} i^* j_* H\mathbb{F}_p^K \\
&&&& \downarrow{\scriptstyle \oplus\pi_0} \\
&&&& \bigoplus_\alpha \Sigma^{2m+p_\alpha-1, m+q_\alpha-1} H\mathbb{F}_p^k
\end{array}$$

is equal to 0.

3. *Let $g : \Sigma_+^\infty X \to i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K$ for some $m \in \mathbb{N}$. Then $g = i^*\eta \circ g_0$ for some $g_0 : \Sigma_+^\infty X \to \Sigma^{2m,m} H\mathbb{F}_p^k$. Here we can take $g_0 = g \circ r_0$, where r_0 is any retraction of $i^*\eta$.*

Proof. Note that for any sequence α of bidegree (p_α, q_α), $p_\alpha \geq 2q_\alpha$, which implies that $p_\alpha - 1 > 2(q_\alpha - 1)$. For (1) and (2), Theorem 2.5 implies that

$$\mathrm{Hom}_{SH(k)}(\Sigma_+^\infty X, \Sigma^{2m+p_\alpha-1, m+q_\alpha-1} H\mathbb{F}_p^k) = H^{2m+p_\alpha-1, m+q_\alpha-1}(X, \mathbb{F}_p) = 0$$

for any sequence α. □

Theorem 3.2.

1. *We have $\Phi(H^{*,*}(K, \mathbb{F}_p)) \subset H^{*,*}(k, \mathbb{F}_p)$.*
2. *Let α be a sequence. Then $\Phi(P_K^\alpha) = P_k^\alpha$. In particular, for the Bockstein β_K and reduced power operations P_K^n constructed by Voevodsky in characteristic 0, $\Phi(P_K^n) = P_k^n$ for $n \geq 0$ and $\Phi(\beta_K) = \beta_k$. Also, P_k^0 is the identity, since P_K^0 is the identity.*

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3. *Let* $X \in \mathrm{Sm}_k$ *and let* $f : \Sigma_+^\infty X \to \Sigma^{2m,m} H\mathbb{F}_p^k$ *be given. Let* α *be a sequence and let* $h : H\mathbb{F}_p^K \to \Sigma^{i,j} H\mathbb{F}_p^K$ *be given. Then*

$$\Phi(h \circ P_K^\alpha)(f) = \Phi(h)(P_k^\alpha(f)).$$

Proof. We first prove (1). Let $a \in H^{*,*}(K, \mathbb{F}_p)$. The element a corresponds to a morphism $f_a : H\mathbb{F}_p^K \to \Sigma^{m,n} H\mathbb{F}_p^K$ in $D(H\mathbb{F}_p^K)$. The functors i^*, j_* restrict to functors $i^* : D(\widehat{H\mathbb{F}}_p^D) \to D(H\mathbb{F}_p^k)$ and $j_* : D(H\mathbb{F}_p^K) \to D(\widehat{H\mathbb{F}}_p^D)$. Hence, $i^* j_*(f_a)$ is a morphism in $D(H\mathbb{F}_p^k)$. From the definition of Φ, it follows that $\Phi(f_a)$ is a morphism in $D(H\mathbb{F}_p^k)$. Thus, $\Phi(a) := \Phi(f_a) \in H^{*,*}(k, \mathbb{F}_p)$.

We now prove (2). Let α be a sequence. Applying the natural transformation $i^* \to i^* j_* j^*$ to $i_R : \widehat{H\mathbb{F}}_p^D \to \widehat{H\mathbb{F}}_p^D \wedge \widehat{H\mathbb{F}}_p^D$, we obtain the following commuting square in $SH(k)$:

$$\begin{tikzcd}
H\mathbb{F}_p^k \arrow[r, "i_R"] \arrow[d, "i^*\eta"] & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^* j_* H\mathbb{F}_p^K \arrow[r, "i^* j_*(i_R)"] & i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K).
\end{tikzcd}$$

From the definition of $i^*\eta_\alpha$ 4, the following diagram commutes:

$$\begin{tikzcd}
H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[r, "i^*\eta_\alpha"] \arrow[d, "i^*\eta"] & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \\
i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "proj"] & i^* j_* \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K. \arrow[u, "\pi"']
\end{tikzcd}$$

Putting these two diagrams together yields the following commuting diagram:

$$\begin{tikzcd}
H\mathbb{F}_p^k \arrow[r, "i_R"] \arrow[d, "i^*\eta"] & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[r, "i^*\eta_\alpha"] \arrow[d, "i^*\eta"] & \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k \\
i^* j_* H\mathbb{F}_p^K \arrow[r, "i^* j_*(i_R)"] & i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "proj"] & i^* j_* \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^K. \arrow[u, "\pi"']
\end{tikzcd} \tag{6}$$

The top row of this diagram gives P_k^α, while the composite starting at $H\mathbb{F}_p^k$ in the top left and continuing along the bottom, row ending with π, gives $\Phi(P_K^\alpha)$. Hence, $\Phi(P_K^\alpha) = P_k^\alpha$.

Now we prove (3). Consider the following diagram:

$$\begin{tikzcd}
\Sigma_+^\infty X \arrow[d, "f"] & & \\
\Sigma^{2m,m} H\mathbb{F}_p^k \arrow[r, "P_k^\alpha"] \arrow[d, "i^*\eta"] & \Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^k \arrow[r, "\Phi(h)"] \arrow[d, "i^*\eta"] & \Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K \arrow[r, "i^* j_* P_K^\alpha"] & i^* j_* \Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^K \arrow[r, "i^* j_* h"] & i^* j_* \Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^K \arrow[d, "\pi"] \\
& & \Sigma^{i+2m+p_\alpha, j+m+q_\alpha} H\mathbb{F}_p^k.
\end{tikzcd} \tag{7}$$

As $\Phi(P_K^\alpha) = P_k^\alpha$, Lemma 3.1 implies that the composite

$$i^*\eta \circ P_k^\alpha \circ f : \Sigma_+^\infty X \to i^* j_* \Sigma^{2m+p_\alpha, m+q_\alpha} H\mathbb{F}_p^K$$

in diagram (7) is equal to

$$i^* j_* P_K^\alpha \circ i^*\eta \circ f.$$

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Thus, from diagram (7),

$$\Phi(h)(P^\alpha_k(f)) = \pi \circ i^*\eta \circ \Phi(h) \circ P^\alpha_k \circ f = \pi \circ i^* j_*(h) \circ i^* j_*(P^\alpha_K) \circ i^*\eta \circ f = \Phi(h \circ P^\alpha_K)(f)$$

as desired. □

Remark 1. Theorem 3.2 says that Φ commutes with compositions, up to some error terms. These error terms vanish on mod p Chow groups. In the next section, we will use the third part of this theorem to get Adem relations for the Steenrod operations P^n_k restricted to mod p Chow groups. Essentially, we just apply Φ to the Adem relations in characteristic 0.

We next prove that the operations P^n_k commute with base change of the field k on mod p Chow groups. For a morphism of fields $f : \mathrm{Spec}(F_1) \to \mathrm{Spec}(F_2)$, the pullback functor $f^* : SH(F_2) \to SH(F_1)$ induces a homomorphism $H\mathbb{F}_p^{F_2\,*,*} H\mathbb{F}_p^{F_2} \to H\mathbb{F}_p^{F_1\,*,*} H\mathbb{F}_p^{F_1}$. For $\mathrm{char}(F_2) \neq p$, $f^*(P^n_{F_2}) = P^n_{F_1}$, since the dual Steenrod algebra has the expected form in this case [17, Theorem 1.1]. However, for our situation where the base field is of characteristic p, we do not yet know the full structure of the dual Steenrod algebra.

Let $f_1 : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$ be the structure map. In the following commuting diagram, f_2, f_3, i_0, and j_0 are maps compatible with f_1:

$$\begin{tikzcd}
\mathrm{Spec}(k) \arrow[r, "f_1"] \arrow[d, "i"] & \mathrm{Spec}(\mathbb{F}_p) \arrow[d, "i_0"] \\
\mathrm{Spec}(D) \arrow[r, "f_2"] & \mathrm{Spec}(\mathbb{Z}_p) \\
\mathrm{Spec}(K) \arrow[r, "f_3"] \arrow[u, "j"] & \mathrm{Spec}(\mathbb{Q}_p). \arrow[u, "j_0"']
\end{tikzcd}$$

Proposition 3.3. *Let $X \in \mathrm{Sm}_k$ and let $g : \Sigma^\infty_+ X \to \Sigma^{2m,m} H\mathbb{F}_p^k$ be given. Then $P^n_k(g) = f_1^*(P^n_{\mathbb{F}_p})(g)$ for all $n \geq 0$.*

Proof. Let $\eta_0 : 1 \to j_{0*}j_0^*$ denote the unit map. Let $f_2^*\widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \to f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p}$ be the map $f_2^*\eta_0$ induced by the isomorphism $j_0^*\widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \to H\mathbb{F}_p^{\mathbb{Q}_p}$. The exchange transformation $f_2^* j_{0*} \to j_* f_3^*$ induces a morphism $f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \to j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}$. Let $f_2^*\widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \to j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}$ be the map ηf_2^* induced by the isomorphism

$$j^* f_2^* \widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \cong f_3^* j_0^* \widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \to f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}.$$

Putting these maps together, we get the following square, which commutes by adjunction:

$$\begin{tikzcd}
f_2^*\widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \arrow[r, "f_2^*\eta_0"] \arrow[d, "id."] & f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[d] \\
f_2^*\widehat{H\mathbb{F}}_p^{\mathbb{Z}_p} \arrow[r, "\eta f_2^*"] & j_* f_3^* H\mathbb{F}_p^{\mathbb{Q}_p}.
\end{tikzcd} \tag{8}$$

Applying the exchange transformation $f_2^* j_{0*} \to j_* f_3^*$ to $P^n_{\mathbb{Q}_p}$, we get the following commuting square:

$$\begin{tikzcd}
f_2^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[r, "f_2^* j_{0*} P^n_{\mathbb{Q}_p}"] \arrow[d] & f_2^* j_{0*} \Sigma^{2n(p-1),n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[d] \\
j_* H\mathbb{F}_p^K \arrow[r, "j_* P^n_K"] & j_* \Sigma^{2n(p-1),n(p-1)} H\mathbb{F}_p^K.
\end{tikzcd}$$

Applying i^* (and the connection isomorphism $i^* f_2^* \cong f_1^* i_0^*$) to these two squares and combining with $g : \Sigma^\infty_+ X \to \Sigma^{2m,m} H\mathbb{F}_p^k$, we obtain the following commuting diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ X \arrow[r, "g"] \arrow[d, "id."] & \Sigma^{2m,m} H\mathbb{F}_p^k \arrow[r, "f_1^* i_0^* \eta_0"] \arrow[d, "id."] & f_1^* i_0^* j_{0*} \Sigma^{2m,m} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[r, "f_1^* i_0^* j_{0*} P^n_{\mathbb{Q}_p}"] \arrow[d] & f_1^* i_0^* j_{0*} \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[d] \\
\Sigma^\infty_+ X \arrow[r, "g"] & \Sigma^{2m,m} H\mathbb{F}_p^k \arrow[r, "i^* \eta"] & i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K \arrow[r, "i^* j_* P^n_K"] & i^* j_* \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^K .
\end{tikzcd} \tag{9}$$

Let $\pi' : i_0^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \to H\mathbb{F}_p^{\mathbb{F}_p}$ and $\pi_0' : i_0^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \to \Sigma^{-1,-1} H\mathbb{F}_p^{\mathbb{F}_p}$ be projection morphisms induced by the isomorphism $i_0^* j_{0*} H\mathbb{F}_p^{\mathbb{Q}_p} \cong H\mathbb{F}_p^{\mathbb{F}_p} \oplus \Sigma^{-1,-1} H\mathbb{F}_p^{\mathbb{F}_p}$ of Theorem 2.4. Consider the following diagram:

$$\begin{tikzcd}
& & & \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^k \\
\Sigma^\infty_+ X \arrow[r, "g"] \arrow[d, "id."] & \Sigma^{2m,m} H\mathbb{F}_p^k \arrow[r, "f_1^* i_0^* \eta_0"] \arrow[d, "id."] & f_1^* i_0^* j_{0*} \Sigma^{2m,m} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[r, "f_1^* i_0^* j_{0*} P^n_{\mathbb{Q}_p}"] \arrow[d] & f_1^* i_0^* j_{0*} \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^{\mathbb{Q}_p} \arrow[d] \arrow[u, "f_1^* \pi'"'] \\
\Sigma^\infty_+ X \arrow[r, "g"] & \Sigma^{2m,m} H\mathbb{F}_p^k \arrow[r, "i^* \eta"] & i^* j_* \Sigma^{2m,m} H\mathbb{F}_p^K \arrow[r, "i^* j_* P^n_K"] & i^* j_* \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^K \arrow[d, "\pi"] \\
& & & \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^k .
\end{tikzcd} \tag{10}$$

From Theorem 3.2, the composite $\Sigma^\infty_+ X \to \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^k$ given by the upper half of diagram (10) is equal to $f_1^*(P^n_{\mathbb{F}_p})(g)$, and the composite $\Sigma^\infty_+ X \to \Sigma^{2(m+n(p-1)),m+n(p-1)} H\mathbb{F}_p^k$ given by the lower half is equal to $P^n_k(g)$. As diagram (9) commutes, Lemma 3.1 then implies that $f_1^*(P^n_{\mathbb{F}_p})(g) = P^n_k(g)$. □

We can now prove that the Steenrod operations P^n_k commute with base change on mod p Chow groups. Let $f : \mathrm{Spec}(k_1) \to \mathrm{Spec}(k_2)$ be given, where k_1, k_2 are fields of characteristic p. Let $h : \mathrm{Spec}(k_2) \to \mathrm{Spec}(\mathbb{F}_p)$ be the structure map.

Corollary 3.4. *Let $X \in \mathrm{Sm}_{k_2}$. Let $n \geq 0$. The following square commutes:*

$$\begin{tikzcd}
CH^*(X)/p \arrow[r, "P^n_{k_2}"] \arrow[d, "f^*"] & CH^*(X)/p \arrow[d, "f^*"] \\
CH^*(X_{k_1})/p \arrow[r, "P^n_{k_1}"] & CH^*(X_{k_1})/p .
\end{tikzcd}$$

Proof. From Proposition 3.3, $h^* P^n_{\mathbb{F}_p}$ agrees with $P^n_{k_2}$ on $CH^*(X)/p$ and $f^* h^* P^n_{\mathbb{F}_p}$ agrees with $P^n_{k_1}$ on $CH^*(X_{k_1})/p$. Let $g : \Sigma^\infty_+ X \to \Sigma^{2m,m} H\mathbb{F}_p^{k_2}$ be given. Then

$$f^*(P^n_{k_2}(g)) = f^*(h^* P^n_{\mathbb{F}_p}(g)) = f^* h^*(P^n_{\mathbb{F}_p})(f^* g) = P^n_{k_1}(f^* g),$$

as required. □

Proposition 3.5. *The morphism $\beta_k = P_k^{(1,0,\ldots)}$ defined in Definition 3.1 is equal to the Bockstein homomorphism β on mod p motivic cohomology.*

Proof. We let β denote the Bockstein homomorphism on mod p motivic cohomology over any base scheme. The Bockstein homomorphism β in characteristic 0 is known to be dual to τ_0. Hence,

$\beta = P_K^{(1,0,\ldots)} = \beta_K$. Applying the natural transformation $i^* \to i^* j_* j^*$ to the diagram

$$\begin{tikzcd}
\widehat{H\mathbb{Z}}^D \arrow[r, "\cdot p"] & \widehat{H\mathbb{Z}}^D \arrow[r] & (\widehat{H\mathbb{Z}}^D)/p \arrow[r] \arrow[rr, bend left, "\beta"] & \Sigma^{1,0}\widehat{H\mathbb{Z}}^D \arrow[r, "proj."] & \Sigma^{1,0}\widehat{H\mathbb{Z}}^D/p
\end{tikzcd}$$

in $SH(D)$ yields the following commuting diagram in $SH(k)$:

$$\begin{tikzcd}
H\mathbb{Z}^k \arrow[r, "\cdot p"] \arrow[d, "i^*\eta"] & H\mathbb{Z}^k \arrow[r, "proj."] \arrow[d, "i^*\eta"] & H\mathbb{F}_p^k \arrow[r] \arrow[d, "i^*\eta"] \arrow[rr, bend left, "\beta"] & \Sigma^{1,0}H\mathbb{Z}^k \arrow[r, "proj."] \arrow[d, "i^*\eta"] & \Sigma^{1,0}H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^*j_*H\mathbb{Z}^K \arrow[r, "\cdot p"] & i^*j_*H\mathbb{Z}^K \arrow[r, "proj."] & i^*j_*H\mathbb{F}_p^K \arrow[r] \arrow[rr, bend right, "i^*j_*\beta_K"'] & \Sigma^{1,0}i^*j_*H\mathbb{Z}^K \arrow[r, "proj."] & \Sigma^{1,0}i^*j_*H\mathbb{F}_p^K \arrow[d, "\pi"] \\
 & & & & \Sigma^{1,0}H\mathbb{F}_p^k .
\end{tikzcd} \tag{11}$$

From Theorem 3.2, $\Phi(\beta_K) = \beta_k$. The composite in diagram (11) that starts at $H\mathbb{F}_p^k$ in the top row and goes immediately down to $\Sigma^{1,0}H\mathbb{F}_p^k$ is equal to $\Phi(\beta_K)$. As the diagram commutes and $\pi \circ i^*\eta = id.$, it follows that $\Phi(\beta_K) = \beta = \beta_k$. □

4. Adem relations

In this section we use the map $\Phi : H\mathbb{F}_p^{K\,*,*}H\mathbb{F}_p^K \to H\mathbb{F}_p^{k\,*,*}H\mathbb{F}_p^k$ 5 and Theorem 3.2 to show that the operations P_k^n for $n \geq 0$ satisfy the expected Adem relations when restricted to mod p Chow groups. The proof uses the corresponding Adem relations in characteristic 0, which can be found in [27, Théorème 4.5.1] for $p = 2$ and [27, Théorème 4.5.2] for odd p. First we state the Adem relations for $p = 2$ over the base K of characteristic 0. Let $\tau \in H^{0,1}(K, \mathbb{F}_2)$ denote the class of $-1 \in \mu_2(K)$ and let $\rho \in H^{1,1}(K, \mathbb{F}_2)$ denote the class of $-1 \in K^*/K^{*2}$. Set $\mathrm{Sq}_k^{2n} := P_k^n$ and $\mathrm{Sq}_k^{2n+1} = \beta_k \mathrm{Sq}_k^{2n}$ for $n \geq 0$.

Theorem 4.1. *Let $a, b \in \mathbb{N}$ with $a < 2b$.*

1.
$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j + \sum_{\substack{j=1 \\ j\, odd}}^{\lfloor \frac{a}{2} \rfloor} \rho \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j-1} \mathrm{Sq}_K^j$$

   if a is even and b is odd.

2.
$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{\substack{j=0 \\ j\, odd}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j$$

   if a and b are odd.

3.
$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{j=0}^{\lfloor \frac{a}{2} \rfloor} \tau^{j \bmod 2} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j$$

   if a and b are even.

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4.
$$\mathrm{Sq}_K^a \mathrm{Sq}_K^b = \sum_{\substack{j=0 \\ j\,even}}^{\lfloor \frac{a}{2} \rfloor} \binom{b-1-j}{a-2j} \mathrm{Sq}_K^{a+b-j} \mathrm{Sq}_K^j + \sum_{\substack{j=1 \\ j\,odd}}^{\lfloor \frac{a}{2} \rfloor} \rho \binom{b-1-j}{a-1-2j} \mathrm{Sq}_K^{a+b-j-1} \mathrm{Sq}_K^j$$

if a is odd and b is even.

Next we state the characteristic 0 Adem relations for p odd.

Theorem 4.2. 1. *Let $a, b \in \mathbb{N}$ with $a < pb$. Then*

$$P_K^a P_K^b = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_K^{a+b-j} P_K^j.$$

2. *Let $a, b \in \mathbb{N}$ with $a \le pb$. Then*

$$P_K^a \beta_K P_K^b = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} \beta_K P_K^{a+b-j} P_K^j +$$
$$\sum_{j=0}^{\lfloor \frac{a-1}{p} \rfloor} (-1)^{a+j+1} \binom{(p-1)(b-j)-1}{a-pj-1} P_K^{a+b-j} \beta_K P_K^j.$$

The Adem relations can now be proven for the operations P_k^n restricted to mod p Chow groups.

Theorem 4.3. *Let $X \in \mathrm{Sm}_k$ and let $x \in H^{2m,m}(X, \mathbb{F}_p) = CH^m(X)/p$ for some $m \ge 0$. Let $a, b \in \mathbb{N}$ such that $a < pb$. Then*

$$P_k^a(P_k^b(x)) = \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_k^{a+b-j}(P_k^j(x)).$$

Proof. From Theorem 3.2, $P_k^a(P_k^b(x)) = \Phi(P_K^a P_K^b)(x)$. Then use the Adem relations in characteristic 0 to rewrite $P_K^a P_K^b \in H\mathbb{F}_p^{K\,*,*} H\mathbb{F}_p^K$. Note that the Bockstein β_k is the 0 homomorphism on mod p Chow groups. If $p = 2$, $\Phi(\mathrm{Sq}_K^n)(x) = \mathrm{Sq}_k^n(x) = 0$ whenever n is odd. Thus, applying Theorem 3.2 yields

$$P_k^a(P_k^b(x)) = \Phi(P_K^a P_K^b)(x) = \Phi\Big(\sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_K^{a+b-j} P_K^j\Big)(x)$$
$$= \sum_{j=0}^{\lfloor \frac{a}{p} \rfloor} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P_k^{a+b-j}(P_k^j(x)). \qquad \square$$

5. Coaction map for smooth X

In this section, for $X \in \mathrm{Sm}_k$, we describe a coaction map

$$\lambda_X : H^{*,*}(X, \mathbb{F}_p) \to \pi_{-*,-*}\Big(\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k\Big) \otimes_{\pi_{-*,-*} H\mathbb{F}_p^k} H^{*,*}(X, \mathbb{F}_p)$$

such that the actions of the cohomology operations P_k^n defined in Section 3 on $H^{*,*}(X, \mathbb{F}_p)$ are determined by λ_X. We show that λ_X is a ring homomorphism when restricted to mod p Chow groups. This will allow us to prove the Cartan formula in the next section.

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There is a multiplication morphism

$$m : \Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \wedge \Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \to \bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k \tag{12}$$

defined as $m = r \circ \mu_2^k \circ (\Psi_k \wedge \Psi_k)$. The morphism m defines multiplication on

$$\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big)^{*,*}(\Sigma_+^\infty X)$$

and

$$\pi_{*,*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big).$$

For sequences α_1, α_2, Proposition 2.2 allows us to calculate the product

$$r_*(\omega(\alpha_1))r_*(\omega(\alpha_2)) \in \pi_{*,*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big)$$

in terms of another sequence $\alpha_1 + \alpha_2$ by using the relations $\tau_i^2 = 0$ for $i \geq 0$.

Proposition 5.1. *The natural ring homomorphism*

$$\pi_{-*,-*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \otimes_{\pi_{-*,-*}H\mathbb{F}_p^k} H^{*,*}(X,\mathbb{F}_p) \to \Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big)^{*,*}(\Sigma_+^\infty X)$$

is an isomorphism.

Proof. The suspension spectrum $\Sigma_+^\infty X \in SH(k)$ is compact. Hence,

$$\mathrm{Hom}_{SH(k)}(\Sigma^{s,t}\Sigma_+^\infty X, \bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k) \cong \bigoplus_{\alpha} \mathrm{Hom}_{SH(k)}(\Sigma^{s,t}\Sigma_+^\infty X, \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k)$$

for all $s, t \in \mathbb{Z}$. □

Definition 5.1. Using the isomorphism

$$\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big)^{*,*}(\Sigma_+^\infty X) \cong \pi_{-*,-*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \otimes_{\pi_{-*,-*}H\mathbb{F}_p^k} H^{*,*}(X,\mathbb{F}_p)$$

from Proposition 5.1, define an additive homomorphism of graded abelian groups

$$\lambda_X : H^{*,*}(X,\mathbb{F}_p) \to \pi_{-*,-*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \otimes_{\pi_{-*,-*}H\mathbb{F}_p^k} H^{*,*}(X,\mathbb{F}_p)$$

by the composite

$$\begin{array}{ccc} H\mathbb{F}_p^{k\,*,*}(\Sigma_+^\infty X) & \xrightarrow{\quad i_{R*}\quad} & (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k)^{*,*}(\Sigma_+^\infty X) \\ & & \big\downarrow{\scriptstyle r_*} \\ & & \pi_{-*,-*}\Big(\bigoplus_{\alpha} \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \otimes_{\pi_{-*,-*}H\mathbb{F}_p^k} H^{*,*}(X,\mathbb{F}_p). \end{array} \tag{13}$$

Proposition 5.2. *Restricted to mod p Chow groups, λ_X preserves multiplication.*

Proof. Let $f : \Sigma_+^\infty X \to \Sigma^{2m,m} H\mathbb{F}_p^k$ and $g : \Sigma_+^\infty X \to \Sigma^{2n,n} H\mathbb{F}_p^k$ be given. We need to show that $\lambda_X(fg) = \lambda_X(f)\lambda_X(g)$. The right $H\mathbb{F}_p^k$ map i_R is a morphism of commutative ring spectra. Hence, i_{R*} is a homomorphism of rings. Hence, we need to prove that $r_*(i_{R*}(f)i_{R*}(g)) = r_*(i_{R*}(f))r_*(i_{R*}(g))$.

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Applying the natural transformation $i^* \to i^* j_* j^*$ to μ_2^D, we get a commuting diagram:

$$\begin{tikzcd}
(H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \arrow[r, "\mu_2^k"] \arrow[d, "i^*\eta"] & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^* j_*((H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K)) \arrow[r, "i^* j_* \mu_2^K"] & i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d, "\oplus\pi"] \\
& \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k.
\end{tikzcd} \tag{14}$$

We will factor the left vertical morphism in this diagram. Consider the triangle

$$\begin{tikzcd}
(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge (\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \arrow[r, "\eta\wedge\eta"] \arrow[d, "\eta"] & j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[dl] \\
j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K), &
\end{tikzcd} \tag{15}$$

where the morphism on the hypotenuse is defined by the lax monoidal property of j_*. Note that the counit morphism $\epsilon : j^* j_* \to id$. is an isomorphism, since j is open. By adjunction, the morphism on the hypotenuse of diagram (15) is induced by the isomorphism

$$\epsilon \wedge \epsilon : j^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \to (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K).$$

The morphism η on the left leg of triangle (15) is induced by the isomorphism

$$j^*\eta : j^*((\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge (\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D)) \to (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K).$$

Using the property that pullback is strongly monoidal, we then have the following commuting triangle:

$$\begin{tikzcd}
j^*(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \wedge j^*(\widehat{H}\mathbb{F}_p^D \wedge \widehat{H}\mathbb{F}_p^D) \arrow[r, "j^*\eta\wedge j^*\eta"] \arrow[d, "j^*\eta"] & j^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge j^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[dl, "\epsilon\wedge\epsilon"] \\
(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge (H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K). &
\end{tikzcd}$$

Thus, by adjunction, triangle (15) commutes.

Applying i^* to triangle (15) shows that the commuting diagram (14) is a subdiagram of the commuting diagram

$$\begin{tikzcd}
(H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \arrow[r, "\mu_2^k"] \arrow[d, "i^*\eta\wedge i^*\eta"] & H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k \arrow[dd, "i^*\eta"] \\
i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d] & \\
i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^* j_* \mu_2^K"] & i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d, "\oplus\pi"] \\
& \bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k.
\end{tikzcd} \tag{16}$$

From diagram (3),

$$(i^*\eta \wedge i^*\eta) \circ (\Psi_k \wedge \Psi_k) : \left(\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k\right) \wedge \left(\bigoplus_\alpha \Sigma^{p_\alpha, q_\alpha} H\mathbb{F}_p^k\right) \to i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^* j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K)$$

is equal to the composite $(i^* j_* \Psi_K \wedge i^* j_* \Psi_K) \circ (i^*\eta \wedge i^*\eta)$. Hence, diagram (16) implies that the multiplication morphism $m = r \circ \mu_2^k \circ (\Psi_k \wedge \Psi_k)$ on

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$$\Big(\bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \wedge \Big(\bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big)$$

is equal to the following composite:

$$\begin{tikzcd}
\Big(\bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \wedge \Big(\bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k\Big) \arrow[d, "((i^*j_*\Psi_K)\circ i^*\eta)\wedge((i^*j_*\Psi_K)\circ i^*\eta)"] & \\
i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d] & \\
i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^*j_*\mu_2^K"] & i^*j_*(H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d, "\oplus\pi"] \\
& \bigoplus_\alpha \Sigma^{p_\alpha,q_\alpha} H\mathbb{F}_p^k.
\end{tikzcd} \tag{17}$$

To show that $r_*(i_{R*}(f)i_{R*}(g)) = r_*(i_{R*}(f))r_*(i_{R*}(g))$, consider the following commuting diagram, where Δ is the diagonal morphism:

$$\begin{tikzcd}
\Sigma_+^\infty X \arrow[d, "\Delta"] & \\
\Sigma_+^\infty X \wedge \Sigma_+^\infty X \arrow[d, "f\wedge g"] & \\
\Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k \arrow[d, "i_R\wedge i_R"] & \\
(\Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (\Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \arrow[r, "\mu_2^k"] \arrow[d, "i^*\eta\wedge i^*\eta"] & \Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k \arrow[dd, "i^*\eta"] \\
i^*j_*(\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(\Sigma^{2n,n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d] & \\
i^*j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^*j_*\mu_2^K"] & i^*j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d, "\oplus\pi"] \\
& \bigoplus_\alpha \Sigma^{p_\alpha+2m+2n,q_\alpha+m+n} H\mathbb{F}_p^k.
\end{tikzcd} \tag{18}$$

The composite $\oplus\pi \circ i^*\eta \circ \mu_2^k \circ (i_R \wedge i_R) \circ (f \wedge g) \circ \Delta$ in this diagram is equal to $r_*(i_{R*}(f)i_{R*}(g))$. From Lemma 3.1, we can replace $i^*\eta \wedge i^*\eta$ in this diagram with $i^*\eta \wedge i^*\eta \circ r \wedge r$ to obtain an equivalent map:

$$\begin{tikzcd}
\Sigma_+^\infty X \arrow[d, "\Delta"] & \\
\Sigma_+^\infty X \wedge \Sigma_+^\infty X \arrow[d, "f\wedge g"] & \\
\Sigma^{2m,m} H\mathbb{F}_p^k \wedge \Sigma^{2n,n} H\mathbb{F}_p^k \arrow[d, "i_R\wedge i_R"] & \\
(\Sigma^{2m,m} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \wedge (\Sigma^{2n,n} H\mathbb{F}_p^k \wedge H\mathbb{F}_p^k) \arrow[d, "r\wedge r"] & \\
(\bigoplus_\alpha \Sigma^{p_\alpha+2m,q_\alpha+m} H\mathbb{F}_p^k) \wedge (\bigoplus_\alpha \Sigma^{p_\alpha+2n,q_\alpha+n} H\mathbb{F}_p^k) \arrow[d, "i^*\eta\wedge i^*\eta"] & \\
i^*j_*(\Sigma^{2m,m} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \wedge i^*j_*(\Sigma^{2n,n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d] & \\
i^*j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[r, "i^*j_*\mu_2^K"] & i^*j_*(\Sigma^{2(m+n),m+n} H\mathbb{F}_p^K \wedge H\mathbb{F}_p^K) \arrow[d, "\oplus\pi"] \\
& \bigoplus_\alpha \Sigma^{p_\alpha+2m+2n,q_\alpha+m+n} H\mathbb{F}_p^k.
\end{tikzcd} \tag{19}$$

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From diagram (17), the composite given by diagram (19) is equal to $\Sigma^{2(m+n),m+n} m \circ (r \wedge r) \circ (i_R \wedge i_R) \circ (f \wedge g) \circ \Delta = r_*(i_{R*}(f)) r_*(i_{R*}(g))$. Thus, $r_*(i_{R*}(f) i_{R*}(g)) = r_*(i_{R*}(f)) r_*(i_{R*}(g))$, as desired. □

6. Cartan formula

In this section, we use the coaction map constructed in the previous section to prove a Cartan formula for the operations P^n_k restricted to mod p Chow groups. Let $X \in \mathrm{Sm}_k$. Let $\langle \cdot, \cdot \rangle$ denote the pairing between $\mathcal{A}^k_{*,*}$ and $H\mathbb{F}_p^{k\,*,*} H\mathbb{F}_p^k$. Let $n \geq 0$. For $x \in H^{*,*}(X, \mathbb{F}_p)$ with $\lambda_X(x) = \Sigma y_i \otimes x_i$, we have $P^n_k(x) = \Sigma \langle y_i, P^n_k \rangle x_i$.

Proposition 6.1. *Let $x, y \in CH^*(X)/p$ and $i \geq 0$. Then*

$$P^i_k(xy) = \sum_{j=0}^{i} P^j_k(x) P^{i-j}_k(y).$$

Proof. From the definition of P^i_k, $\langle \xi^i_1, P^i_k \rangle = 1$ and $\langle \omega(\alpha), P^i_k \rangle = 0$ for all sequences $\alpha \neq (0, i, 0, 0, \ldots)$. Using coaction map (13), write

$$\lambda_X(x) = \sum_q \omega(\alpha^1_q) \otimes x_q$$

and

$$\lambda_X(y) = \sum_r \omega(\alpha^2_r) \otimes y_r$$

for some sequences α^1_q, α^2_r. Then

$$\lambda_X(xy) = \sum_{q,r} ((\omega(\alpha^1_q)\omega(\alpha^2_r) \otimes x_q y_r).$$

For any 2 sequences α^1_q, α^2_r appearing in these sums, we have $\omega(\alpha^1_q)\omega(\alpha^2_r) = 0$ if the relation $\tau^2_m = 0$ from Proposition 2.2 applies for some $m \geq 0$, or else $\omega(\alpha^1_q)\omega(\alpha^2_r) = \pm\omega(\alpha^1_q + \alpha^2_r)$.

From the definition of λ_X,

$$P^i_k(xy) = \sum_{q,r} \langle (\omega(\alpha^1_q)\omega(\alpha^2_r), P^i_k \rangle x_q y_r.$$

Proposition 2.2 implies that if $\omega(\alpha_1)\omega(\alpha_2) = a\xi^i_1$ for two sequences α_1, α_2 and $a \neq 0 \in H^{*,*}(k, \mathbb{F}_p)$, then $a = 1$ and $\omega(\alpha_1) = \xi^j_1$, $\omega(\alpha_2) = \xi^{i-j}_1$ for some $0 \leq j \leq i$. As P^i_k is dual to ξ^i_1, the only terms for which $\langle \omega(\alpha^1_q + \alpha^2_r), P^i_k \rangle \neq 0$ are of the form $\omega(\alpha^1_{q_j}) = \xi^j_1, \omega(\alpha^2_{r_j}) = \xi^{i-j}_1$ for $0 \leq j \leq i$. Hence,

$$P^i_k(xy) = \sum_{j=0}^{i} \langle \omega(\alpha^1_{q_j} + \alpha^2_{r_j}), P^i_k \rangle x_{q_j} y_{r_j} = \sum_{j=0}^{i} P^j_k(x) P^{i-j}_k(y), \tag{20}$$

as required. □

7. *p*th power and instability

In this section, for $n \in \mathbb{N}$, we prove that P^n_k is the pth power on $CH^n(-)/p$. Letting $f : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$ denote the structure map, it suffices to prove that $f^*(P^n_{\mathbb{F}_p})(\iota_n) = \iota^p_n$ for the canonical element $\iota_n \in H^{2n,n}(K_{n,k}, \mathbb{F}_p)$, where $K_{n,k} \in H(k)$ is the motivic Eilenberg–MacLane space representing $H^{2n,n}(-, \mathbb{F}_p)$. This proof makes use of Morel's S^1-recognition principle.

We refer to [7, Section 3] as a reference for the S^1-recognition principle. For a base scheme S, let $\mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$ denote the category of Nisnevich local presheaves of spaces on Sm_S. The unstable motivic homotopy category $H(S)$ can be described as the full subcategory of $\mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$ of presheaves that are $\mathbb{A}^1$-invariant. Let $L_{\mathrm{mot}} : \mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S) \to H(S)$ denote the $\mathbb{A}^1$-localization functor. Let $SH^{S^1}(S)$ denote the stable motivic homotopy category of S^1-spectra. For a morphism $f : S_1 \to S_2$ of base schemes, we have the adjoint functors of pullback $f^* := Lf^*$ and push-forward $f_* := Rf_*$:

$$f^* : H(S_2) \rightleftarrows H(S_2) : f_*.$$

For $f : S_1 \to S_2$ smooth, f^* admits a left adjoint $f_\#$ such that $f_\#(X) = X \in H(S_2)$ for any $X \in \mathrm{Sm}_{S_1}$.

For $C = \mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_S)$, consider the n-fold bar constructions $\mathrm{B}^n_{\mathrm{nis}}$ that are adjoint to the nth S^1-deloopings Ω^n:

$$\mathrm{B}^n_{\mathrm{nis}} : \mathrm{Mon}_{\mathcal{E}_n}(C) \rightleftarrows C : \Omega^n.$$

We also consider the infinite bar construction

$$\mathrm{B}^\infty_{\mathrm{nis}} : \mathrm{CMon}(C) = \mathrm{Mon}_{\mathcal{E}_\infty}(C) \rightleftarrows \mathrm{Stab}(C) : \Omega^\infty,$$

where $\mathrm{Stab}(C) := C \otimes \mathrm{Spt}$ denotes the S^1-stabilization of C. Similarly, for $C = H(S)$ we have the nth S^1-deloopings Ω^n:

$$\mathrm{B}^n_{\mathrm{mot}} : \mathrm{Mon}_{\mathcal{E}_n}(C) \rightleftarrows C : \Omega^n$$

and infinite bar construction

$$\mathrm{B}^\infty_{\mathrm{mot}} : \mathrm{CMon}(C) = \mathrm{Mon}_{\mathcal{E}_\infty}(C) \rightleftarrows \mathrm{Stab}(C) : \Omega^\infty.$$

For later use, note that $\mathrm{B}^n_{\mathrm{nis}}$ and $\mathrm{B}^\infty_{\mathrm{nis}}$ commute with pullbacks.

Definition 7.1. Define $X \in \mathrm{Mon}(H(S))$ to be strongly $\mathbb{A}^1$-invariant if $\mathrm{B}_{\mathrm{nis}}X \simeq \mathrm{B}_{\mathrm{mot}}X$. Define $X \in \mathrm{CMon}(H(S))$ to be strictly $\mathbb{A}^1$-invariant if $\mathrm{B}^n_{\mathrm{nis}}X \simeq \mathrm{B}^n_{\mathrm{mot}}X$ for all $n \geq 0$.

Most of the proof of the following proposition was suggested by Marc Hoyois.

Proposition 7.1. *Let k be a perfect field of characteristic p and let $i : Spec(k) \to Spec(D)$ be a closed embedding where D is a complete unramified discrete valuation ring with generic point $j : Spec(K) \to Spec(D)$. Fix $n > 0$. Let $K_{n,D} := \Omega^\infty_{\mathbb{P}^1}\Sigma^{2n,n}\widehat{H}\mathbb{F}^D_p$. Then the morphism $i^*K_{n,D} \to K_{n,k}$ induced by $i^*\Sigma^{2n,n}\widehat{H}\mathbb{F}^D_p \cong \Sigma^{2n,n}H\mathbb{F}^k_p$ is an isomorphism in $H(k)$.*

Proof. We first prove that $K_{n,D}$ is connected. Let R be a Henselian local ring that is essentially smooth over D. From [11, Corollary 4.2], the Bloch–Levine Chow groups $CH^m(R)$ of R vanish for $m \geq 1$. Thus, $\pi_0^{\mathrm{nis}}(K_{n,D}(\mathrm{Spec}(R))) \simeq *$, since $K_{n,D} \in H(D)$ represents the codimension n mod p Bloch–Levine Chow group.

Now we prove that $i^*K_{n,D}$ is connected. As $j : \mathrm{Spec}(K) \to \mathrm{Spec}(D)$ is smooth, $j^*K_{n,D} \simeq K_{n,K}$. Consider the homotopy pushout P in $\mathrm{PSh}_{\mathrm{nis}}(\mathrm{Sm}_D)$ of the following diagram:

$$\begin{tikzcd} j_\# K_{n,K} \arrow[r] \arrow[d] & K_{n,D} \\ \mathrm{Spec}(K). & \end{tikzcd}$$

The morphism $j_\# K_{n,K} \to \mathrm{Spec}(K)$ induces a bijection on π_0^{nis}. Hence, $\pi_0^{\mathrm{nis}}(K_{n,D}) \simeq \pi_0^{\mathrm{nis}}(P)$. From the gluing square [25, Theorem 2.21],

$$L_{\mathrm{mot}}(P) \simeq i_* i^*(K_{n,D}).$$

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From [25, Corollary 3.22], it follows that $i_*i^*(K_{n,D})$ is connected, since $K_{n,D}$ is connected. Let $k \to S_k$ be an essentially smooth homomorphism of rings, where S_k is Henselian local. The ring S_k admits a lift S_D where $D \to S_D$ is essentially smooth and S_D is Henselian local. Hence, $i^*(K_{n,D})(S_k) \simeq i_*i^*(K_{n,D})(S_D)$ is connected. Thus, $i^*K_{n,D} \in H(k)$ is connected. In particular, $\pi_0^{\mathrm{nis}}(i^*(K_{n,D}))$ is strongly $\mathbb{A}^1$-invariant. The S^1-recognition principle [7, Theorem 3.1.12] then implies that $i^*K_{n,D}$ is strictly $\mathbb{A}^1$-invariant. Note that $K_{n,k}$ is also strictly $\mathbb{A}^1$-invariant, since $\pi_0^{\mathrm{nis}}(K_{n,k})$ is strongly $\mathbb{A}^1$-invariant.

From [30, Theorem 8.18], we have

$$\mathrm{B}^\infty_{\mathrm{mot}} i^*(K_{n,D}) \cong i^*(\mathrm{B}^\infty_{\mathrm{mot}} K_{n,D}) \cong i^*(\Omega^\infty_{\mathbb{G}_m}\Sigma^{2n,n}\widehat{H}\mathbb{F}_p^D) \cong \Omega^\infty_{\mathbb{G}_m}\Sigma^{2n,n}H\mathbb{F}_p^k \cong \mathrm{B}^\infty_{\mathrm{mot}} K_{n,k}$$

in $SH^{S^1}(k)$. Then [7, Corollary 3.1.15] implies that $i^*K_{n,D} \cong K_{n,k}$ in $H(k)$. □

Proposition 7.2. *Let k be a field of characteristic p with structure map $f : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$ and let $\iota_n \in H^{2n,n}(K_{n,k}, \mathbb{F}_p)$ be the canonical element. Then $f^*P^n_{\mathbb{F}_p}(\iota_n) = \iota_n^p$.*

Proof. First, assume that k is perfect. Let D be a discrete valuation ring having k as a residue field with inclusion morphism $i : \mathrm{Spec}(k) \to \mathrm{Spec}(D)$ and generic point $j : \mathrm{Spec}(K) \to \mathrm{Spec}(D)$. From Proposition 7.1, $i^*K_{n,D} \cong K_{n,k}$. Over all base schemes S, let ι_n denote the canonical element in $H^{2n,n}(K_{n,S}, \mathbb{F}_p)$. Apply $i^* \to i^*j_*j^*$ to the natural morphism $\iota_n : \Sigma^\infty_+ K_{n,D} \to \Sigma^{2n,n}\widehat{H}\mathbb{F}_p^D$ to get the following commuting square:

$$\begin{tikzcd}
\Sigma^\infty_+ K_{n,k} \arrow[r, "\iota_n"] \arrow[d, "i^*\eta"] & \Sigma^{2n,n} H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^*j_*\Sigma^\infty_+ K_{n,K} \arrow[r, "i^*j_*\iota_n"] & \Sigma^{2n,n} i^*j_* H\mathbb{F}_p^K .
\end{tikzcd}$$

Apply $i^*\eta : i^* \to i^*j_*j^*$ to the morphism $\Sigma^\infty_+ K_{n,D} \to \Sigma^{2pn,pn}\widehat{H}\mathbb{F}_p^D$ in $SH(D)$ corresponding to ι_n^p to get the commutative diagram

$$\begin{tikzcd}
\Sigma^\infty_+ K_{n,k} \arrow[r, "\iota_n^p"] \arrow[d, "i^*\eta"] & \Sigma^{2pn,pn} H\mathbb{F}_p^k \arrow[d, "i^*\eta"] & \\
i^*j_*\Sigma^\infty_+ K_{n,K} \arrow[r, "i^*j_*\iota_n^p"] & i^*j_*\Sigma^{2pn,pn} H\mathbb{F}_p^K \arrow[r, "\pi"] & \Sigma^{2pn,pn} H\mathbb{F}_p^k .
\end{tikzcd} \tag{21}$$

From [35, Lemma 9.8], $i^*j_*\iota_n^p = i^*j_*P^n_K(\iota_n)$. Hence, the bottom row of diagram (21) can be rewritten as

$$i^*j_*\Sigma^\infty_+ K_{n,K} \xrightarrow{i^*j_*\iota_n} i^*j_*\Sigma^{2n,n} H\mathbb{F}_p^K \xrightarrow{i^*j_*P^n_K} i^*j_*\Sigma^{2pn,pn} H\mathbb{F}_p^K \xrightarrow{\pi} \Sigma^{2pn,pn} H\mathbb{F}_p^k .$$

From Theorem 3.2 and the foregoing commuting diagrams, $P^n_k(\iota_n) = \pi \circ (i^*j_*P^n_K) \circ (i^*j_*\iota_n) \circ i^*\eta$. Hence, from diagram (21) we have $P^n_k(\iota_n) = \pi \circ (i^*j_*\iota_n^p) \circ i^*\eta = \iota_n^p$.

For k not perfect, we have an essentially smooth morphism $f : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$, and $f^*(K_{n,\mathbb{F}_p}) \cong K_{n,k}$ [17, Theorem 2.11]. As $\mathbb{F}_p$ is perfect, we then have $f^*(P^n_{\mathbb{F}_p}(\iota_n)) = f^*(P^n_{\mathbb{F}_p})(\iota_n) = f^*(\iota_n^p) = \iota_n^p$. □

From Proposition 3.3, we have the following corollary:

Corollary 7.3. *Let $X \in \mathrm{Sm}_k$. Then P^n_k is the pth power on $CH^n(X)/p$.*

Now that we know that $f^*(P^n_{\mathbb{F}_p})$ is the pth power on $H^{2n,n}(-, \mathbb{F}_p)$ for all $n \geq 1$, we can prove an instability result. Let $f : \mathrm{Spec}(k) \to \mathrm{Spec}(\mathbb{F}_p)$ be the structure morphism.

Proposition 7.4. *Let $m, q, n \geq 0$ be integers such that $n > m - q$ and $n \geq q$. Let $X \in H(k)$ and let $x \in H^{m,q}(X, \mathbb{F}_p)$. Then $f^*(P^n_{\mathbb{F}_p})(x) = 0$.*

Proof. Voevodsky's proof in [35, Lemma 9.9] works here, since $f^*(P^n_{\mathbb{F}_p})$ is the pth power on $H^{2n,n}(-,\mathbb{F}_p)$ by Proposition 7.2. □

Corollary 7.5. *Let $X \in \mathrm{Sm}_k$. Then P^n_k is the 0 map on $CH^m(X)/p$ for $m < n$.*

8. Proper push-forward

In this section, we restrict our attention to mod p Chow groups on Sm_k. The ring of mod p Chow groups is an oriented cohomology pretheory in the sense of [26, Section 1], with perfect integration given by proper push-forward on Chow groups. Consider the total cohomological Steenrod operation $P_k := P^0_k + P^1_k + P^2_k + \cdots : CH^*(-)/p \to CH^*(-)/p$. From the Cartan formula (Section 6), P_k is a ring morphism of oriented cohomology pretheories in the sense of [26, Definition 1.1.7].

Let $\mathbb{Z}[[c_1, c_2, \ldots]]$ denote the power series ring on Chern classes c_i for $i \geq 1$, and let $w \in \mathbb{Z}[[c_1, c_2, \ldots]]$ denote the total characteristic class corresponding to the polynomial $f(x) = 1 + x^{p-1}$. For $p = 2$, w is the total Chern class. Let $X \in \mathrm{Sm}_k$. For a line bundle L on X, $w(L) = 1 + c_1^{p-1}(L) \in CH^*(X)$. For a vector bundle V on X that has a filtration by subbundles with quotients given by line bundles $L_1, \ldots, L_m$, $w(V) = w(L_1) \cdots w(L_m)$. Let w_i denote the ith homogeneous component of w for $i \geq 0$. We have $w_i = 0$ if $p - 1$ does not divide i. Define the total homological Steenrod operation $P^X := w(-T_X) \circ P_k : CH^*(X)/p \to CH^*(X)/p$, where T_X is the tangent bundle on X. For $i \geq 0$, let P^X_i denote the $(p-1)i$th homogeneous component of P^X. The following proposition is a consequence of the general Riemann–Roch formulas proved by Panin in [26]:

Proposition 8.1. *Let $f : X \to Y$ be a projective morphism with $X, Y \in \mathrm{Sm}_k$. Then*

$$\begin{array}{ccc} CH^*(X)/p & \xrightarrow{P^X} & CH^*(X)/p \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle f_*} \\ CH^*(Y)/p & \xrightarrow{P^Y} & CH^*(Y)/p \end{array}$$

commutes.

Proof. This is [26, Theorem 2.5.4]. See [26, Section 2.6] for a discussion relevant to our situation. The main ingredients are that the operations P^n_k satisfy the Cartan formula and that P^n_k is the pth power on $CH^n(-)/p$. □

Restricting to the case $p = \mathrm{char}(k) = 2$, we obtain a Wu formula from the work of Panin [26, Theorem 2.5.3]. Here, $w = c$ is the total Chern class and Sq denotes the total Steenrod square P_k on $CH^*(-)/2$.

Proposition 8.2. *Let $X, Y \in \mathrm{Sm}_k$ and let $i : X \hookrightarrow Y$ be a closed embedding with normal bundle N. Then*

$$i_*(c(N)) = \mathrm{Sq}([X])$$

in $CH^*(Y)/2$, where $[X] \in CH^*(Y)/2$ denotes the mod 2 cycle class of X.

9. Rost's degree formula

Now that we have Steenrod operations on mod p Chow groups of Sm_k, we can prove Rost's degree formula [24, Theorem 6.4] without any restrictions on the characteristic of the base field. We closely follow the presentation of Merkurjev [24], where Steenrod operations (assuming restrictions on the characteristic of the base field) are used to prove degree formulas. In [16], Haution extended the Rost degree formulas to base fields of characteristic 2.

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For a variety X over k, let n_X denote the greatest common divisor of $\deg(x)$ over all closed points $x \hookrightarrow X$. Let $X \in \mathrm{Sm}_k$ be projective of dimension $d > 0$. Applying Proposition 8.1 to the structure morphism $X \to \mathrm{Spec}(k)$ and $[X] \in CH_d(X)/p$, we see that $p \mid \deg(w_d(-T_X))$.

Proposition 9.1. *Let $f : X \to Y$ be a morphism of projective varieties $X, Y \in \mathrm{Sm}_k$ of dimension $d > 0$. Then $n_Y \mid n_X$ and*

$$\frac{\deg(w_d(-T_X))}{p} \equiv \deg(f) \cdot \frac{\deg(w_d(-T_Y))}{p} \mod n_Y.$$

Proof. The proof in [24, Theorem 6.4] works here. From Proposition 8.1, $f_*(w_d(-T_X)) \equiv \deg(f) w_d(-T_Y) \in CH_0(Y)/p$. We then take the degree homomorphism to finish the proof. □

10. Specialization map

Fix a complete unramified discrete valuation ring D with residue field $i : \mathrm{Spec}(k) \to \mathrm{Spec}(D)$ and fraction field $j : \mathrm{Spec}(K) \to \mathrm{Spec}(D)$ as before. Let $X \in \mathrm{Sm}_D$ with special fiber X_k and generic fiber X_K. As described in [10, Chapter 20.3], there are specialization maps $\sigma_n : CH^n(X_K) \to CH^n(X_k)$ defined for all $n \geq 0$. The specialization maps can be defined at the level of cycles. Namely, for an irreducible closed subvariety $Z_K \subset X_K$ of codimension n, let Z_k denote the special fiber of the reduced closed subscheme $\overline{Z_K} \subset X$ associated to $Z_K \subset X$. Then $\sigma_n(\langle Z_K \rangle) = \langle Z_k \rangle \in CH^n(X_k)$. Also let σ_n denote the specialization map induced on mod p Chow groups.

We now show that the Steenrod operations P_k^n defined on $CH^*(X_k)/p$ are compatible with the operations P_K^n defined on $CH^*(X_K)/p$.

Proposition 10.1. *Let $m \geq 0$ and let $Z_K \subset X_K$ be a closed subvariety of codimension n. Let $\langle Z_K \rangle \in CH^n(X_K)/p$ denote the mod p cycle class of Z_K. Then*

$$P_k^m(\sigma_n(\langle Z_K \rangle)) = \sigma_{n+m(p-1)}(P_K^m(\langle Z_K \rangle)) \in CH^{n+m(p-1)}(X_k)/p.$$

Proof. The mod p cycle class of $\overline{Z_K} \subset X$ induces a map

$$f_D : \Sigma_+^\infty X \to \Sigma^{2n,n} \widehat{H}\mathbb{F}_p^D$$

in $SH(D)$. The map $i^* f_D$ gives the mod p cycle class of Z_k (the special fiber of $\overline{Z_K} \subset X$), and $j^* f_D$ gives the mod p cycle class of Z_K. Applying the natural transformation $i^*\eta : i^* \to i^* j_* j^*$ to f_D gives a commuting square:

$$\begin{array}{ccc} \Sigma_+^\infty X_k & \xrightarrow{i^* f_D} & \Sigma^{2n,n} H\mathbb{F}_p^k \\ \downarrow{\scriptstyle i^*\eta} & & \downarrow{\scriptstyle i^*\eta} \\ i^* j_* \Sigma_+^\infty X_K & \xrightarrow{i^* j_* j^* f_D} & i^* j_* \Sigma^{2n,n} H\mathbb{F}_p^K. \end{array} \tag{22}$$

From Theorem 3.2, $P_k^m = \Phi(P_K^m) = \pi \circ i^* j_* P_K^m \circ i^*\eta$. Hence, from diagram (22),

$$\pi \circ i^*\eta \circ P_k^m \circ i^* f_D = \pi \circ i^* j_* P_K^m \circ i^* j_* j^* f_D \circ i^*\eta$$

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in the following commuting diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ X_k \arrow[r, "i^* f_D"] \arrow[d, "i^*\eta"] & \Sigma^{2n,n} H\mathbb{F}_p^k \arrow[r, "P^m_k"] \arrow[d, "i^*\eta"] & \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^* j_* \Sigma^\infty_+ X_K \arrow[r, "i^* j_* j^* f_D"] & i^* j_* \Sigma^{2n,n} H\mathbb{F}_p^K \arrow[r, "i^* j_* P^m_K"] & i^* j_* \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^k \arrow[d, "\pi"] \\
& & \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^k .
\end{tikzcd} \tag{23}$$

Write $P^m_K(\langle Z_K \rangle) = \sum_{l=1}^q a_l \left\langle Z^l_K \right\rangle$ for some $q, a_l \in \mathbb{Z}$ and closed subvarieties $Z^l_K \subset X_K$ of codimension $n + m(p-1)$. Taking the associated reduced closed subschemes in X gives an element $\sum_{l=1}^q a_l \left\langle \overline{Z}^l_K \right\rangle \in H^{2(n+m(p-1)),n+m(p-1)}(X, \mathbb{F}_p)$ which corresponds to a morphism

$$g : \Sigma^\infty_+ X \longrightarrow \Sigma^{2(n+m(p-1)),n+m(p-1)} \widehat{H}\mathbb{F}_p^D .$$

For $1 \leq l \leq q$, let Z^l_k denote the special fiber of $\overline{Z}^l_K$. Taking pullbacks, $i^* g$ gives

$$\sum_{l=1}^q a_l \left\langle Z^l_k \right\rangle \in H^{2(n+m(p-1)),n+m(p-1)}(X_k, \mathbb{F}_p)$$

and $j^* g = \sum_{l=1}^q a_l \left\langle Z^l_K \right\rangle = P^m_K(\langle Z_K \rangle)$. Applying $i^*\eta$ to g yields a commuting diagram:

$$\begin{tikzcd}
\Sigma^\infty_+ X_k \arrow[r, "i^* g"] \arrow[d, "i^*\eta"] & \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^k \arrow[d, "i^*\eta"] \\
i^* j_* \Sigma^\infty_+ X_K \arrow[r, "i^* j_* j^* g"] & i^* j_* \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^K \arrow[d, "\pi"] \\
& \Sigma^{2(n+m(p-1)),n+m(p-1)} H\mathbb{F}_p^k .
\end{tikzcd} \tag{24}$$

From diagrams (23) and (24), we get

$$i^* g = \sum_{l=1}^q a_l \left\langle Z^l_k \right\rangle = \pi \circ i^* j_* j^* g \circ i^*\eta = \pi \circ i^* j_* (P^m_K(\langle Z_K \rangle)) \circ i^*\eta$$

$$= \pi \circ i^* j_* P^m_K \circ i^* j_* j^* f_D \circ i^*\eta = P^m_k(\langle Z_k \rangle),$$

as required. □

We recall some facts about flag varieties, using [21] as a reference. Let G_k be a split reductive group over k with Borel subgroup B_k and Weyl group W. From the Bruhat decomposition,

$$G_k / B_k = \coprod_{w \in W} B_k w B_k / B_k .$$

For $w \in W$, the closure X^w_k of $B_k w B_k / B_k$ in G_k / B_k is called a Schubert variety and

$$B_k w B_k / B_k \cong \mathbb{A}_k^{l(w)},$$

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where $l(w)$ is the length of w in W. Let $P_k \supseteq B_k$ be a parabolic subgroup of G_k. We have $P_k = BW_PB$ for some subgroup $W_P \leq W$. There is a related $W^P \subset W$, such that for each $w \in W^P$, B_kwB_k/B_k is isomorphic to B_kwB_k/P_k under the quotient morphism $G_k/B_k \to G_k/P_k$ [21, Lemma 1.2]. We also have a cell decomposition

$$G_k/P_k = \coprod_{w \in W^P} B_kwB_k/P_k.$$

This cell decomposition is independent of the field k. It follows that the total Chow group $CH^*(G_k/P_k)$ is freely generated as an additive group by the cycle classes $\langle Y_k^w \rangle$ of the images Y_k^w of the Schubert varieties X_k^w for $w \in W^P$.

Chevalley [2] and Demazure [5] showed that the Chow rings

$$CH^*(G_{F_1}/P_{F_1}) \text{ and } CH^*(G_{F_2}/P_{F_2})$$

are isomorphic for any two fields F_1, F_2. The isomorphism is given by mapping the class of a Schubert subscheme $Y_{F_1}^w$ to $Y_{F_2}^w$ for $w \in W^P$. We now prove that the Steenrod operations P_k^n and P_K^n give the same action on $H^{2*,*}(G_k/P_k, \mathbb{F}_p) \cong CH^*(G_k/P_k)/p \cong CH^*(G_K/P_K)/p \cong H^{2*,*}(G_K/P_K, \mathbb{F}_p)$.

Corollary 10.2. *Let $n \geq 0$ and let $w_0 \in W^P$. Then*

$$P_K^n(\langle Y_{w_0}^K \rangle) = \sum_{w \in W^P} a_w \langle Y_w^K \rangle$$

in $CH^*(G_K/P_K)/p$ for some $a_w \in \mathbb{Z}$, and

$$P_k^n(\langle Y_{w_0}^k \rangle) = \sum_{w \in W^P} a_w \langle Y_w^k \rangle .$$

Proof. We refer to [4] for facts about integral models of split reductive groups. Let $w \in W$ and let X_D^w be the reduced closed subscheme of G_D/B_D associated to B_DwB_D/B_D. Note that X_D^w is flat over $\mathrm{Spec}(D)$. For any field F and morphism $\mathrm{Spec}(F) \to \mathrm{Spec}(D)$, the fiber $X_D^w \times_{\mathrm{Spec}(D)} \mathrm{Spec}(F)$ in G_F/B_F is isomorphic to X_F^w [29, Theorem 2]. The main point to check is that the fibers of X_D^w over $\mathrm{Spec}(D)$ are reduced.

Now assume that $w \in W^P$. Let Y_D^w denote the image of X_D^w in G_D/P_D. Then $Y_D^w \times_{\mathrm{Spec}(D)} \mathrm{Spec}(F) \cong Y_F^w$ for any field F and morphism $\mathrm{Spec}(F) \to \mathrm{Spec}(D)$. Proposition 10.1 then applies to finish the proof. □

11. Applications to quadratic forms

In this section, we use the Steenrod squares Sq_k^{2n} to prove new results about nonsingular quadratic forms over a field k of characteristic 2. The results we prove have analogues in characteristic $\neq 2$ conveniently found in [6, Sections 79–82] where the only missing ingredient for extending to characteristic 2 was the existence of Steenrod squares satisfying expected properties.

Recall that a quadratic form (q, V) over k is nonsingular if the associated radical $V^\perp$ is of dimension at most 1 and q is nonzero on $V^\perp \setminus 0$. Equivalently, (q, V) is nonsingular if the associated projective quadric is smooth. Note that nonsingular quadratic forms are called nondegenerate in [6]. In characteristic 2, anisotropic quadratic forms are not necessarily nonsingular. Let (q, V) be a nonsingular anisotropic quadratic form defined over k and let X be the associated projective quadric of dimension D. Over some field extension F of k, the quadric X_F becomes split. A computation of $CH^*(X_F)$ can be found in [6, Chapter XIII]. Let $h \in CH^1(X_F)$ denote the pullback of the hyperplane class in $\mathbb{P}(V)$ and let $l_d \in CH_d(X_F)$ denote the class of a d-dimensional subspace in X_F, where $d = \lfloor D/2 \rfloor$. Let $l_{d-i} = h^i \cdot l_d$ for $0 \leq i \leq d$.

Proposition 11.1. *As an additive group, $CH^*(X_F)$ is freely generated by h^i, l_i for $0 \le i \le d$. For the ring structure, $h^{d+1} = 2l_{D-d-1}$, $l_d^2 = 0$ if 4 does not divide D, and $l_d^2 = l_0$ if 4 divides D.*

From Corollary 10.2, the action of the Steenrod squares Sq_F^{2n} on $CH^*(X_F)/2$ agrees with the action of Steenrod squares on the mod 2 Chow ring of a split quadric in characteristic 0. We refer to [6, Corollary 78.5] for the calculation of the action of Steenrod squares on the mod 2 Chow ring of a split quadric in characteristic 0.

Proposition 11.2. *For any $0 \le i \le d$ and $j \ge 0$,*

$$\mathrm{Sq}_F^{2j}(h^i) = \binom{i}{j} h^{i+j} \text{ and } \mathrm{Sq}_F^{2j}(l_i) = \binom{D+1-i}{j} l_{i-j}.$$

To state our results, we recall the definition of relative higher Witt indices. Let φ be a nonsingular quadratic form over a field F and let $F(\varphi)$ denote the function field of the associated quadric. Let φ_{an} denote the anisotropic part of φ and let $\mathfrak{i}_0(\varphi)$, the Witt index of φ, denote the dimension of a maximal isotropic subspace for φ. Start with $\varphi_0 := \varphi_{an}$ and $F_0 := F$. Inductively define $F_i := F_{i-1}(\varphi_{i-1})$ and $\varphi_i := (\varphi_{F_i})_{an}$ for $i > 0$. There exists an integer $\mathfrak{h}(\varphi)$, called the height of φ, such that $\dim\varphi_{\mathfrak{h}(\varphi)} \le 1$. For $1 \le j \le \mathfrak{h}(\varphi)$, we then define the jth relative higher Witt index $\mathfrak{i}_j(\varphi)$ to be $\mathfrak{i}_0(\varphi_{F_j}) - \mathfrak{i}_0(\varphi_{F_{j-1}})$.

Proposition 11.3. *Let φ be a nonsingular anisotropic quadratic form over k such that $\dim\varphi \ge 2$. Then $\mathfrak{i}_1(\varphi) \le 2^{v_2(\dim\varphi - \mathfrak{i}_1(\varphi))}$.*

Proof. The proof of [6, Proposition 79.4] works in this case and uses the computation of the Steenrod squares on the mod 2 Chow ring of a split quadratic given by Proposition 11.2 along with Corollary 3.4 on base change of the Steenrod squares. From the Cartan formula (Section 6) and results on shell triangles in [6, Sections 72,73] that were proved in arbitrary characteristic, we see that the conclusion of [6, Lemma 79.3] holds for nonsingular anisotropic quadratic forms in characteristic 2. □

To finish, we extend 3 more results of Karpenko on quadratic forms in characteristic $\neq 2$ to the case of nonsingular anisotropic quadratic forms in characteristic 2. Let φ be a nonsingular anisotropic quadratic form defined over a field k of characteristic 2 with relative higher Witt indices $\mathfrak{i}_j := \mathfrak{i}_j(\varphi)$ as defined previously for $j = 1, \ldots, \mathfrak{h} := \mathfrak{h}(\varphi)$.

Proposition 11.4. *Assume that $\mathfrak{h} > 1$. Then*

$$v_2(\mathfrak{i}_1) \ge \min(v_2(\mathfrak{i}_2), \ldots, v_2(\mathfrak{i}_{\mathfrak{h}})) - 1.$$

Proof. The analogue of this proposition in characteristic $\neq 2$ can be found in [6, Corollary 81.19]. The proof there works over a base field of characteristic 2 using the properties we have established for the Steenrod squares Sq_k^{2n}. The conclusions of [6, Lemma 80.1] and [6, Theorem 80.2] hold in our situation, since Sq_k^{2n} acts by squaring on $CH^n(-)/2$ by Proposition 7.2 and the total homological Steenrod square commutes with proper push-forward by Proposition 8.1. □

We next discuss the characteristic 2 analogue of the “holes in I^n" result [6, Corollary 82.2]. For a field F, the quadratic Witt group $I_q(F)$ is defined as the quotient of the Grothendieck group of the monoid of isometry classes of even-dimensional nonsingular quadratic forms by the subgroup generated by the hyperbolic plane [6, Section 8]. There is an action of the Witt ring $W(F)$ of nondegenerate symmetric bilinear forms on $I_q(F)$. Let $I(F) \subset W(F)$ denote the fundamental ideal of $W(F)$ and set $I_q^n(F) := I^{n-1}(F) \cdot I_q(F)$ for $n \ge 1$. Let k be a field of characteristic 2. Mimicking the proof of [6, Corollary 82.2], with $I_q^n(k)$ used in place of $I^n(k)$, gives the following result (let $n \ge 1$):

Proposition 11.5. *Let $\varphi \in I_q^n(k)$ be a nonsingular anisotropic quadratic form such that $\dim\varphi < 2^{n+1}$. Then there exists $0 \le i \le n$ such that $\dim\varphi = 2^{n+1} - 2^{i+1}$.*

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Our last result concerns u-invariants of fields. The u-invariant $u(F)$ of a field F is defined to be the smallest non-negative integer (or ∞ if there is no such integer) $u(F)$ such that every nonsingular quadratic form φ over F with $\dim\varphi > u(F)$ is isotropic.

In [32], Vishik constructed characteristic 0 fields of u-invariant $2^r + 1$ for all $r \geq 3$. Karpenko used Steenrod squares on mod 2 Chow groups to show that for any $r \geq 3$ and any field F of characteristic $\neq 2$, F is contained in a field of u-invariant $2^r + 1$ [19]. Karpenko's constructions now extend to fields of characteristic 2 through the use of the Steenrod squares Sq_k^{2n} defined in this article for k of characteristic 2.

Proposition 11.6. *Let k be a field of characteristic 2 and let $r \geq 3$. Then k is a subfield of a field of u-invariant $2^r + 1$.*

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