BULL. AUSTRAL. MATH. SOC. 32A35
VOL. 34 (1986) 461-472

A NOTE ON N-HARMONIC MAJORANTS

HONG OH KIM AND CHANG OCK LEE

Suppose $D(v)$ is the Dirichlet integral of a function v defined on the unit disc U in the complex plane. It is well known that if v is a harmonic function in U with $D(v) < \infty$, then for each p, $0 < p < \infty$, $|v|^p$ has a harmonic majorant in U.

We define the "iterated" Dirichlet integral $D_n(v)$ for a function v on the polydisc U^n of C^n and prove the polydisc version of the well known fact above:

If v is an n-harmonic function in U^n with $D_n(v) < \infty$, then for each p, $0 < p < \infty$, $|v|^p$ has an n-harmonic majorant in U^n.

1. Introduction

For a differentiable real function v defined in the polydisc,

$$U^n = \{(z_1, \ldots z_n) : |z_i| < 1 \text{ for } 1 \leq j \leq n\}$$

in C^n, we define the "iterated" Dirichlet integral $D_n(v)$ as

$$\int \cdots \int_{U^n} |\nabla_1 \otimes \ldots \otimes \nabla_n v|^2 \, dxdy.$$

where $z_j = x_j + iy_j$, $dx = dx_1 \ldots dx_n$, $dy = dy_1 \ldots dy_n$ and

$$\nabla_1 \otimes \ldots \otimes \nabla_n = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right) \otimes \ldots \otimes \left(\frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}\right)$$

Received 6 February 1986.

For example, we have

$$|\nabla_1 \otimes \nabla_2 v|^2 = \left|\frac{\partial^2 v}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 y}{\partial x_1 \partial y_2}\right|^2 + \left|\frac{\partial^2 v}{\partial y_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 v}{\partial y_1 \partial y_2}\right|^2 .$$

If $n = 1$, $D_1(v)$ is the usual Dirichlet integral of v .

A continuous function v is an open set in C^n is n-harmonic if v is harmonic in each complex variable separately, that is,

$$\frac{\partial^2 v}{\partial x_j^2} + \frac{\partial^2 v}{\partial y_j^2} = 0 , \ (1 \le j \le n).$$

The function v on U^n has an n-harmonic majorant if there is an n-harmonic function V such that $v(z) \le V(z)$ throughout U^n .

Let $h^p(U^n)$, $0 < p < \infty$, be the class of all n-harmonic functions v in U^n for which

$$||v||_p = \sup_{0 \le r_1, \dots, r_n < 1} M_p(r_1, \dots ,r_n \ ; v) < \infty$$

where

$$M_p(r_1,\dots,r_n;v) = \left(\int_0^{2\pi} \dots \int_0^{2\pi} |v(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n}|^p \frac{d\theta_1}{2\pi} \dots \frac{d\theta_n}{2\pi}\right)^{1/p} .$$

For $p \ge 1$, M_p is an increasing function of $r_1,\dots,r_n$, separately. If $v \in h^p(U^n)$, $p \ge 1$, it is known that the radial limit

$$v^*(e^{i\theta_1},\dots,e^{i\theta_n}) = \lim_{r_1,\dots,r_n \to 1} v(r_1e^{i\theta_1},\dots,r_ne^{i\theta_n})$$

of v exists almost everywhere and it is an L^p-function on the distinguished boundary.

$$T^n = \{(z_1,\dots,z_n) : |z_j| = 1 \ \text{ for } \ 1 \le j \le n\} .$$

Moreover, $||v||_p = ||v^*||_{L^p(T^n)}$. Also, if $v \in h^p(U^n)$, $p > 1$, then v is equal to the iterated Poisson integral of its radial limit function v^*. That is,

$$v(z_1,\dots,z_n) = \int \dots \int_{T^n} P_{r_1}(\theta_1-t_1) \dots P_{r_n}(\theta_n-t_n) v^*(e^{it_1},\dots,e^{it_n}) \frac{dt_1}{2\pi} \dots \frac{dt_n}{2\pi}$$

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

where $P_r(\theta-t)$ is the Poisson kernel for U. See [1,3,8] for more about h^p.

The prototype of our main theme is the following well-known theorem (see [9] for example):

Let v be a harmonic function in U with $D_1(v) < \infty$. Then for each p, $0 < p < \infty$, the function $|v|^p$ has a harmonic majorant in U.

We prove the polydisc version of the theorem above:

MAIN THEOREM. *Let v be an n-harmonic function in U^n with $D_n(v) < \infty$. Then for each p, $0 < p < \infty$, the function $|v|^p$ has an n-harmonic majorant in U^n.*

We consider only the case $n = 2$, but the procedure can be repeated for an arbitrary n.

2. The iterated Dirichlet integral

PROPOSITION 2.1. *If v is a 2-harmonic function in U^2, then*

$$D_2(v) = 4 \int\cdots\int_{U^2} \left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 dxdy .$$

Proof. By an elementary calculation, we see that $|\nabla_1 \otimes \nabla_2 v|^2$

$$= \left|\frac{\partial^2 v}{\partial x_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 v}{\partial x_1 \partial y_2}\right|^2 + \left|\frac{\partial^2 v}{\partial y_1 \partial x_2}\right|^2 + \left|\frac{\partial^2 v}{\partial y_1 \partial y_2}\right|^2 .$$

$$= \left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 + \frac{1}{r_1^2}\left|\frac{\partial^2 v}{\partial \theta_1 \partial r_2}\right|^2 + \frac{1}{r_2^2}\left|\frac{\partial^2 v}{\partial r_1 \partial \theta_2}\right|^2 + \frac{1}{r_1^2 r_2^2}\left|\frac{\partial^2 v}{\partial \theta_1 \partial \theta_2}\right|^2 .$$

Since v can be expanded as

$$v(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \sum_{k\in Z^2} \hat{v}(k) r_1^{|k_1|} r_2^{|k_2|} e^{ik_1\theta_1 + ik_2\theta_2}$$

where $k = (k_1,k_2)$, a pair of integers, and $\hat{v}(k)$ is the multiple Fourier coefficient of v, we have, by Parseval's identity,

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 d\theta_1 d\theta_2 = (2\pi)^2 \sum_{k\in Z^2}|k_1|^2|k_2|^2|\hat{v}(k)|^2 r_1^{2|k_1|-2} r_2^{2|k_2|-2},$$

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial^2 v}{\partial \theta_1 \partial r_2}\right|^2 d\theta_1 d\theta_2 = (2\pi)^2 \sum_{k\in Z^2}|k_1|^2|k_2|^2|\hat{v}(k)|^2 r_1^{2|k_1|} r_2^{2|k_2|-2},$$

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial^2 v}{\partial r_1 \partial \theta_2}\right|^2 d\theta_1 d\theta_2 = (2\pi)^2 \sum_{k\in Z^2}|k_1|^2|k_2|^2|\hat{v}(k)|^2 r_1^{2|k_1|-2} r_2^{2|k_2|},$$

and

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial^2 v}{\partial \theta_1 \partial \theta_2}\right|^2 d\theta_1 d\theta_2 = (2\pi)^2 \sum_{k\in Z^2}|k_1|^2|k_2|^2|\hat{v}(k)|^2 r_1^{2|k_1|} r_2^{2|k_2|}.$$

So

$$\int\cdot\cdot\int_{U^2}\left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 dxdy = \int\cdot\cdot\int_{U^2} \frac{1}{r_1^2}\left|\frac{\partial^2 v}{\partial \theta_1 \partial r_2}\right|^2 dxdy$$

$$= \int\cdot\cdot\int_{U^2} \frac{1}{r_2^2}\left|\frac{\partial^2 v}{\partial r_1 \partial \theta_2}\right|^2 dxdy$$

$$= \int\cdot\cdot\int_{U^2} \frac{1}{r_1^2 r_2^2}\left|\frac{\partial^2 v}{\partial \theta_1 \partial \theta_2}\right|^2 dxdy$$

$$(2.1) \qquad = \pi^2 \sum_{k\in Z^2} |k_1||k_2||\hat{v}(k)|^2 .$$

Hence we have

$$D_2(v) = 4\int\cdot\cdot\int_{U^2}\left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 dxdy . \qquad \square$$

In general, for an n-harmonic function v in U^n, we have

$$D_n(v) = 2^n \int\cdots\int_{U^n}\left|\frac{\partial^n v}{\partial r_1\cdots\partial r_n}\right|^2 dxdy .$$

Let α be a real number and let

$$v(z_1,z_2) = \sum_{k\in Z^2} \hat{v}(k) r_1^{|k_1|} r_2^{|k_2|} e^{ik_1\theta_1 + ik_2\theta_2}$$

be a 2-harmonic function on U^2. The fractional derivative of v of

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

order α is defined as

$$D^{\alpha,\alpha}v(z_1,z_2) = \sum_{k\in Z^2}(1+|k_1|)^{\alpha}(1+|k_2|)^{\alpha}\hat{v}(k)r_1^{|k_1|}r_2^{|k_2|}e^{ik_1\theta_1+ik_2\theta_2} .$$

The fractional integral of v of order α is defined as $I^{\alpha,\alpha}v = D^{-\alpha,-\alpha}v$. If $\alpha > 0$, the following integral representation can easily be verified:

$$I^{\alpha,\alpha}v(z_1,z_2) = \frac{1}{\Gamma(\alpha)^2}\int_0^1\int_0^1 (\log\frac{1}{\rho})^{\alpha-1}(\log\frac{1}{\sigma})^{\alpha-1}v(\rho z_1,\sigma z_2)\, d\rho d\sigma .$$

PROPOSITION 2.2. *Let v be a 2-harmonic function in U^2. Then $D_2(v) < \infty$ if and only if $D^{\frac{1}{2},\frac{1}{2}}v \in h^2(U^2)$.*

Proof. By Parseval's identity we have

$$||D^{\frac{1}{2},\frac{1}{2}}v||_2^2 = \sum_{k\in Z^2}(1+|k_1|)(1+|k_2|)|\hat{v}(k)|^2 .$$

By (2.1), we obtain that $D^{\frac{1}{2},\frac{1}{2}}v \in h^2(U^2)$ if and only if

$$\int\cdots\int_{U^2} \left|\frac{\partial^2 v}{\partial r_1 \partial r_2}\right|^2 dxdy < \infty ,$$

or, equivalently, if and only if $D_2(v) < \infty$. □

Generally, for an n-harmonic function v in U^n , $D_n(v) < \infty$ if and only if $D^{\frac{1}{2},\ldots,\frac{1}{2}}v \in h^2(U^n)$.

The following proposition is known, but we include a proof.

PROPOSITION 2.3. *For $p > 1$, $v \in h^p(U^2)$ if and only if $|v|^p$ has a 2-harmonic majorant.*

Proof. Let V be a 2-harmonic majorant of $|v|^p$. Then we have, by the mean value property,

$$\int_0^{2\pi}\int_0^{2\pi} |v(r_1e^{i\theta_1},r_2e^{i\theta_2})|^p \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi} \le \int_0^{2\pi}\int_0^{2\pi} V(r_1e^{i\theta_1},r_2e^{i\theta_2})\frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}$$

$$= V(0,0) < \infty ;$$

so $v \in h^p(U^2)$.

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

If $v \in h^p(U^2)$ for $p > 1$, we have

$$v(z_1,z_2) = \int_0^{2\pi}\int_0^{2\pi} P_{r_1}(\theta_1-t_1)P_{r_2}(\theta_2-t_2)v^*(e^{it_1},e^{it_2})\,\frac{dt_1}{2\pi}\,\frac{dt_2}{2\pi},$$

where $P_r(\theta-t)$ is Poisson kernel for U. We apply Jensen's inequality to get

$$(2.2)\quad |v(z_1,z_2)|^p \le \int_0^{2\pi}\int_0^{2\pi} P_{r_1}(\theta_1-t_1)P_{r_2}(\theta_2-t_2)|v^*(e^{it_1},e^{it_2})|^p\,\frac{dt_1}{2\pi}\,\frac{dt_2}{2\pi}.$$

But the right hand side of (2.2) is a 2-harmonic function and so a 2-harmonic majorant of $|v|^p$. This completes the proof. □

3. A theorem of Hardy and Littlewood

If v is a function in U^2, $v(\cdot,w)$ denotes the function $z \to v(z,w)$ with w fixed and $v_{r,s}$ the function $(z,w) \to v(rz,sw)$. Throughout this paper, $C(\cdots)$ denotes a positive constant depending only on the argument $(\cdots)$ and it may vary from occurence to occurence even in the proof of the same theorem.

LEMMA 3.1. [1] *For* $\alpha > 1$

$$\int_0^{2\pi} |1-re^{i\theta}|^{-\alpha}\,d\theta = O(1-r)^{-\alpha+1},\quad (r \to 1).$$

LEMMA 3.2. *If* $v \in h^p(U^2)$, $p > 1$, *then*

$$|v(z,w)| \le C(p)\,||v(z,\cdot)||_p(1-|w|)^{-1/p}$$

and

$$|v(z,w)| \le C(p)\,||v(\cdot,w)||_p(1-|z|)^{-1/p}.$$

Proof. For a fixed z, we have

$$v(z,x) = \int_0^{2\pi} P_s(\eta-t)v^*(z,e^{it})\,\frac{dt}{2\pi},$$

where $w = se^{i\eta}$ and $v^*(z,e^{it}) = \lim_{s\to 1} v^*(z,se^{it})$. Using Hölder's inequality, we have

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

$$|v(z,w)| \le \left(\int_0^{2\pi} |v^*(z,e^{it})|^p \frac{dt}{2\pi}\right)^{1/p} \left(\int_0^{2\pi} \left(\frac{1-|w|^2}{|e^{it}-w|^2}\right)^q \frac{dt}{2\pi}\right)^{1/q}$$

$$\le ||v(z,\cdot)||_p (1-|w|^2) \left(\int_0^{2\pi} \frac{1}{|e^{it}-se^{i\eta}|^{2q}} \frac{dt}{2\pi}\right)^{1/q} ,$$

where q is the conjugate index of p. By Lemma 3.1 there exists a constant $C(p)$ such that

$$\int_0^{2\pi} \frac{1}{|e^{it}-se^{i\eta}|^{2q}} \frac{dt}{2\pi} \le C(p)(1-|w|)^{1-2q} .$$

Hence

$$|v(z,w)| \le C(p)||v(z,\cdot)||_p (1-|w|)^{-1/p} . \qquad \square$$

LEMMA 3.3. *If* $v \in h^p(U^2)$, $p > 1$, *then*

$$||v(\cdot,w)||_p \le C(p)||v||_p (1-|w|)^{-1/p} .$$

Proof. We use Lemma 3.2 and the monotone convergence theorem (MCT) to get

$$||v(\cdot,w)||_p^p = \lim_{r\to 1} \int_0^{2\pi} |v(re^{i\theta},w)|^p \frac{d\theta}{2\pi}$$

$$\le C(p)(1-|w|)^{-1} \lim_{r\to 1} \int_0^{2\pi} ||v(re^{i\theta},\cdot)||_p^p \frac{d\theta}{2\pi} \qquad \text{(Lemma 3.2)}$$

$$= C(p)(1-|w|)^{-1} \lim_{r\to 1} \int_0^{2\pi} \lim_{s\to 1} \int_0^{2\pi} |v(re^{i\theta},se^{i\eta})|^p \frac{d\eta}{2\pi} \frac{d\theta}{2\pi}$$

$$= C(p)(1-|w|)^{-1} \lim_{\substack{r\to 1\\ s\to 1}} \int_0^{2\pi} \int_0^{2\pi} |v(re^{i\theta},se^{i\eta})|^p \frac{d\eta}{2\pi} \frac{d\theta}{2\pi} \qquad \text{(MCT)}$$

$$= C(p)(1-|w|)^{-1}||v||_p^p .$$

Hence we have

$$||v(\cdot,w)||_p \le C(p)||v||_p (1-|w|)^{-1/p} . \qquad \square$$

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

LEMMA 3.4. *If* $v \in h^p(U^2)$, $p > 1$, *and if*

$$M(\eta) = \sup_{0\le\sigma<1} \left(\int_0^{2\pi} |v(re^{i\theta},\sigma s e^{i\eta})|^p \frac{d\theta}{2\pi}\right)^{1/p},$$

then

$$\int_0^{2\pi} M(\eta)^p \frac{d\eta}{2\pi} \le C(p) M_p(r,s;v)^p .$$

Proof. We use Fubini's theorem and apply the Hardy-Littlewood maximal theorem ('Max") [1, p.11] to the harmonic function $v(z,\cdot)$ to get

$$\begin{aligned}\int_0^{2\pi} M(\eta)^p \frac{d\eta}{2\pi} &= \int_0^{2\pi} \sup_{0\le\sigma<1} \int_0^{2\pi} |v(re^{i\theta},\sigma s e^{i\eta})|^p \frac{d\theta}{2\pi}\frac{d\eta}{2\pi} \\ &\le \int_0^{2\pi}\int_0^{2\pi} \sup_{0\le\sigma<1} |v(re^{i\theta},\sigma s e^{i\eta})|^p \frac{d\theta}{2\pi}\frac{d\eta}{2\pi} \\ &= \int_0^{2\pi}\left(\int_0^{2\pi} \sup_{0\le\sigma<1} |v(re^{i\theta},\sigma s e^{i\eta})|^p \frac{d\eta}{2\pi}\right)\frac{d\theta}{2\pi} \quad \text{(Fubini)} \\ &\le C(p)\int_0^{2\pi}\int_0^{2\pi} |v(re^{i\theta},s e^{i\eta})|^p \frac{d\eta}{2\pi}\frac{d\theta}{2\pi} \quad \text{('Max')} \\ &= C(p) M_p(r,s;v)^p .\end{aligned}$$

□

Now, we can proceed as in the proof of Theorem 2.2 in [7] to prove the following theorem. We give its proof for the sake of completeness. The corresponding theorem on holomorphic functions on the unit disc was proved by Hardy and Littlewood [4,5] and by Flett [2], and on the polydisc U^n, by Kim [7].

THEOREM 3.5. *If* $0 < \alpha < \frac{1}{p}$ *and if* $v \in h^p(U^2)$, $p > 1$, *then* $I^{\alpha,\alpha}v \in h^q(U^2)$ *where* $q = \frac{p}{1-\alpha p}$.

Proof. Set

$$M(re^{i\theta},\sigma w) = \sup_{0\le\rho<1} |v(\rho re^{i\theta},\sigma w)| . \tag{3.1}$$

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

We write $z = re^{i\theta}$ and $w = se^{i\eta}$. By Lemma 3.2

$$(3.2) \qquad |v(\rho z,\sigma w)| \le C(p)\,||v_{r,s}(\cdot,\sigma e^{i\eta})||_p (1-\rho)^{-1/p} .$$

By (3.1) and (3.2), we have

$$(3.3) \qquad \int_0^1 (1-\rho)^{\alpha-1}|v(\rho re^{i\theta},\sigma se^{i\eta})|\,d\rho$$
$$\le C(p)\,||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \int_0^\lambda (1-\rho)^{\alpha-\frac{1}{p}-1}\,d\rho$$
$$+ M(re^{i\theta},\sigma w)\int_\lambda^1 (1-\rho)^{\alpha-1}\,d\rho .$$

If $||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \ge M(re^{i\theta},\sigma w)$, we set $\lambda = 0$ in (3.4). (3.3) is then dominated by $C(\alpha)\,||v_{r,s}(\cdot,\sigma e^{i\eta})||_p$. If $||v_{r,s}(\cdot,\sigma e^{i\eta})||_p$ $< M(re^{i\theta},\sigma w)$, we set

$$\lambda = 1 - \left(\frac{||v_{r,s}(\cdot,\sigma e^{i\eta})||_p}{M(re^{i\theta},\sigma w)}\right)^p .$$

(3.3) is then dominated by

$$C(\alpha,p)\,||v_{r,s}(\cdot,\sigma e^{i\eta})||_p^{\alpha p}\,M(re^{i\theta},\sigma w)^{1-\alpha p} .$$

Hence for any η , we have

$$(3.5) \qquad \int_0^1 (1-\rho)^{\alpha-1}|v(\rho re^{i\theta},\sigma se^{i\eta})|\,d\rho$$

$$\le C(\alpha,p)\left(||v_{r,s}(\cdot,\sigma e^{i\eta})||_p + ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p^{\alpha p}\,M(re^{i\theta},\sigma w)^{1-\alpha p}\right) .$$

Integrating (3.5) with respect to $(1-\sigma)^{\alpha-1}d\sigma$, we get

$$(3.6) \qquad |I^{\alpha,\alpha}v(z,w)| \le C(\alpha,p)\left(\int_0^1 (1-\sigma)^{\alpha-1}||v_{r,s}(\cdot,\sigma e^{i\eta})||_p\,d\sigma\right.$$
$$\left.+\int_0^1 M(re^{i\theta},\sigma w)^{1-\alpha p}||v_{r,s}(\cdot,\sigma e^{i\eta})||_p^{\alpha p}(1-\sigma)^{\alpha-1}\,d\sigma\right) .$$

We take q - means on both sides of (3.6) with respect to $\frac{d\theta}{2\pi}$ and use Minkowski's inequalities in their discrete and continuous forms to get

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

(3.7) $$\left(\int_0^{2\pi} |I^{\alpha,\alpha}v(re^{i\theta},se^{i\eta})|^q \frac{d\theta}{2\pi}\right)^{1/q}$$

$$\le C(\alpha,p)\Big(\int_0^1 (1-\sigma)^{\alpha-1}||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \; d\sigma$$

$$+ \int_0^1 \left(\int_0^{2\pi} M(re^{i\theta},\sigma w)^p \frac{d\theta}{2\pi}\right)^{\frac{1-\alpha p}{p}} ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p^{\alpha p}(1-\sigma)^{\alpha-1}\, d\sigma\Big)$$

$$\le C(\alpha,p)\int_0^1 (1-\sigma)^{\alpha-1}||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \, d\sigma \ .$$

We used the maximal theorem

$$\int_0^2 M(re^{i\theta},\sigma s e^{i\eta})^p \frac{d\theta}{2\pi} \le C(p)||v_{r,s}(\cdot,\sigma e^{i\eta})||_p^p \ ,$$

to get the last inequality in (3.7). Next, we set

(3.8) $$M(\eta) = \sup_{0\le\sigma<1} ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \ .$$

By Lemma 3.3, we have

(3.9) $$||v_{r,s}(\cdot,\sigma e^{i\eta})||_p \le C(p)||v_{r,s}||_p(1-\sigma)^{-1/p} \ .$$

By (3.8) and (3.9), we have as before

$$\int_0^1 ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p(1-\sigma)^{\alpha-1}\, d\sigma$$

$$\le C(p)||v_{r,s}||_p \int_0^\lambda (1-\sigma)^{\alpha-\frac{1}{p}-1}\, d\sigma + M(\eta)\int_\lambda^1 (1-\sigma)^{\alpha-1}\, d\sigma \ .$$

We set $\lambda = 0$ if $M(\eta) \le ||v_{r,s}||_p$ and $\lambda = 1 - \left(\frac{||v_{r,s}||_p}{M(\eta)}\right)^p$, otherwise. We have then for any η,

$$\int_0^1 ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p(1-\sigma)^{\alpha-1}\, d\sigma \le C(\alpha,p)\big(||v_{r,s}||_p + ||v_{r,s}||_p^{\alpha p} M(\eta)^{1-\alpha p}\big) \ .$$

https://doi.org/10.1017/S0004972700010340 Published online by Cambridge University Press

If we take q - means on both sides with respect to $\frac{d\eta}{2\pi}$, we have

$$(3.10)\qquad \left(\int_0^{2\pi}\left(\int_0^1 ||v_{r,s}(\cdot,\sigma e^{i\eta})||_p(1-\sigma)^{\alpha-1}\,d\sigma\right)^q \frac{d\eta}{2\pi}\right)^{1/q}$$
$$\le C(\alpha,p)\left(||v_{r,s}||_p + ||v_{r,s}||_p^{\alpha p}\left(\int_0^2 M(\eta)^p\,d\eta\right)^{1/q}\right)$$
$$\le C(\alpha,p)||v_{r,s}||_p$$

by Lemma 3.4. If we note that $\log\frac{1}{\rho}\sim 1-\rho$ as $\rho\to 1^-$ and we combine (3.7) and (3.10), we have

$$M_q(r,s;I^{\alpha,\alpha}v)\le C(\alpha,p)||v_{r,s}||_p\ .$$

So

$$||I^{\alpha,\alpha}v||_q\le C(\alpha,p)||v||_p\ . \qquad \square$$

4. Proof of the main theorem

By Proposition 2.2, $D^{\frac12,\frac12}v\in h^2(U^2)$. By Theorem 3.5, for any α $(0<\alpha<\frac12)$,

$$I^{\alpha,\alpha}D^{\frac12,\frac12}v = D^{\frac12-\alpha,\frac12-\alpha}v\in h^p(U^2)$$

where $p=\frac{2}{1-2\alpha}$

Taking α arbitrarily close to $\frac12$, we see that $v\in h^p(U^2)$ for any $p>0$. By Proposition 2.3, for each p $(1<p<\infty)$, $|v|^p$ has a 2-harmonic majorant in U^2. If $p\le 1$, then $|v|^p\le |v|^2+1$. By the assertion above, $|v|^2+1$ has a 2-harmonic majorant; so does $|v|^p$. This completes the proof. $\square$

References

[1] P.L. Duren, *Theory of H^p spaces*, (Academic Press, New York, NY. 1970).

[2] T.M. Flett, "The dual of an inequality of Hardy and Littlewood and some related inequalities", *J. Math. Anal. Appl.* 38 (1972), 746-765.

[3] J.B. Garnett, *Bounded analytic functions*, (Academic Press, New York, NY 1981).

[4] G.H. Hardy and J.E. Littlewood, "Some properties of fractional integrals, II", *Math. Z.* 34 (1932), 403-439.

[5] G.H. Hardy and J.E. Littlewood, "Theorems concerning mean values of analytic or harmonic functions", *Quart. J. Math.* 12 (1941), 221-256.

[6] H.O. Kim, "Derivatives of Blaschke products", *Pacific J. Math.*, 114 (1984), 175-189.

[7] H.O. Kim, "On a theorem of Hardy and Littlewood on the polydisc", *Proc. Amer. Math. Soc.* (to appear).

[8] W. Rudin, *Function theory in polydiscs*, (W.A. Benjamin, Inc., New York, NY, 1969).

[9] S. Yamashita, "Dirichlet-finite functions and harmonic majorants", *Illinois J. Math.*, 25 (1981), 626-631.

Department of Applied Mathematics,
Korea Advanced Institute of Science and Technology,
P.O. Box 150,
Cheongryang,
Seoul,
Korea.