Compositio Mathematica **132:** 137–158, 2002.

Linear Forms in two m-adic Logarithms and Applications to Diophantine Problems

YANN BUGEAUD
Université Louis Pasteur, UFR de mathématiques, 7, rue René Descartes, 67084 Strasbourg, France. e-mail: bugeaud@math.u-strasbg.fr

(Received: 21 March 2000; accepted in final form: 30 May 2001)

Abstract. We give sharp, explicit estimates for linear forms in two logarithms, simultaneously for several non-Archimedean valuations. We present applications to explicit lower bounds for the fractional part of powers of rational numbers, and to the Diophantine equation $(x^n - 1)/(x - 1) = y^q$.

Mathematics Subject Classifications (2000). 11J86, 11J61, 11D61.

Key words. Baker's method, Diophantine equation, Linear form in logarithms, non-Archimedean approximation.

1. Introduction

In 1940, Gelfond [14] (see also his book [15]) obtained the first nontrivial effective lower bound for $|b \log \alpha_1 - \log \alpha_2|$, where b, α_1 and α_2 are algebraic numbers, with α_1 and α_2 multiplicatively independent. More than twenty years later, Baker generalized this result to linear forms in an arbitrary number of logarithms of algebraic numbers. His estimates have then been refined by many authors and, to date, the best lower bounds are due to Laurent, Mignotte and Nesterenko [18] in the case of two logarithms and to Baker and Wüstholz [3], Waldschmidt [30] and also Matveev [21] in the general case. In parallel to the Archimedean theory, analogous results have been obtained in a p-adic setting, first by Gelfond [14] and Schinzel [26] in case of two logarithms, and then by Coates, Sprindžuk, van der Poorten and finally Kunrui Yu [31] in the general case. To date, the best known result for two logarithms is due to Bugeaud and Laurent [8], who have provided a sharp upper bound for the p-adic valuation $v_p(\Lambda)$ of $\Lambda = \alpha_1^{b_1} - \alpha_2^{b_2}$, where α_1 and α_2 are algebraic numbers and b_1 and b_2 are positive integers. All these results have many applications, in particular to Diophantine equations, which justify all the efforts made in order to reduce the size of the numerical constants occurring in the estimates. In the case of two Archimedean or non-Archimedean logarithms, the main results of [18] and [8] are very satisfactory and allow us to solve completely Diophantine equations, modulo of course some computer calculations.

At the present time, there is no result treating simultaneously several non-Archimedean places. The main reason is the following: the p-adic case offers a

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

hurdle, which is ignored in the Archimedean setting, namely the radius of convergence of the p-adic exponential function is finite. The current methods (cf., for instance, [8]) rest on analytic techniques and need the introduction of the function $z \mapsto \alpha_1^z$, which has *a priori* no reason to be defined for every z in the ring of p-adic integers, since the p-adic exponential function only converges in the open disk centered at 1 and of radius $p^{-1/(p-1)}$. Thus, we use a trick, which costs roughly a factor p in the upper estimate for $v_p(\Lambda)$ and leads us to fear that treating simultaneously several places would not be possible.

However, in the very particular case when $v_p(\alpha_1 - 1) > 1/(p-1)$, we stay within the disk of convergence without using the trick, and the dependence on p in the upper bound for $v_p(\Lambda)$ is then almost optimal. This observation suggests that it should be possible to obtain a simultaneous estimate for several places q, which all satisfy $v_q(\alpha_1 - 1) > 1/(q-1)$, and thus depend hardly on α_1. In the first part of the present work, we prove such an estimate, which is an extension of Theorem 2 of [6]. In the Archimedean setting, Shorey [28] was the first who noticed that one gets the best possible estimates when the α_i's are all very close to one. This is crucial for numerous applications, especially to Diophantine equations.

Our work is organized as follows. Section 2 is concerned with the statement of our main results, including an explicit upper estimate for $v_p(\Lambda)$ in the case where $\alpha_2 = \pm 1$. The proofs are displayed in Section 3. Applications to explicit lower bounds for the fractional part of powers of rational numbers are given in Section 4, while Section 5 is devoted to the Diophantine equation $(x^n - 1)/(x-1) = y^q$.

2. Linear Forms in m-adic Logarithms

Let $m > 1$ be an integer and write $m = p_1^{u_1} \cdots p_w^{u_w}$, where $p_1 < \cdots < p_w$ are distinct prime numbers and the u_i's are positive integers. Let x be a nonzero integer and let p be a prime. We recall that the p-adic valuation of x, denoted by $v_p(x)$, is the greatest nonnegative integer v such that p^v divides x. Analogously, we define the m-adic valuation of x, which we denote by $v_m(x)$, to be the greatest nonnegative integer v such that m^v divides x. We observe that

$$v_m(x) = \min_{1 \leqslant i \leqslant w} \left[\frac{v_{p_i}(x)}{u_i} \right],$$

where $[\cdot]$ denotes the integer part. Further, if a/b is a nonzero rational number with a and b coprime, we set $v_m(a/b) = v_m(a) - v_m(b)$.

Let x_1/y_1 and x_2/y_2 be two nonzero rational numbers with $x_1/y_1 \neq \pm 1$. Our aim is to provide an upper bound for the m-adic valuation of

$$\Lambda = \left(\frac{x_1}{y_1}\right)^{b_1} - \left(\frac{x_2}{y_2}\right)^{b_2},$$

where b_1 and b_2 are positive integers. To this end, we should add some restrictions on x_1/y_1 and x_2/y_2, namely we assume as in [8] that for all $1 \leqslant i \leqslant w$ we have

$v_{p_i}(x_1/y_1) = v_{p_i}(x_2/y_2) = 0$. No further condition is required when m is prime, and we refer to [8] for that case. The purpose of the present work is to deal with composite m, and our method can be applied only if there exists a positive integer g, coprime with $p_1 \cdots p_w$, such that

$$v_{p_i}\left(\left(\frac{x_1}{y_1}\right)^g - 1\right) \geqslant u_i, \quad v_{p_i}\left(\left(\frac{x_2}{y_2}\right)^g - 1\right) \geqslant 1 \quad \text{for all prime } p_i, 1 \leqslant i \leqslant w \tag{H1}$$

and

$$v_2\left(\left(\frac{x_1}{y_1}\right)^g - 1\right) \geqslant 2, \quad v_2\left(\left(\frac{x_2}{y_2}\right)^g - 1\right) \geqslant 2 \quad \text{if 2 divides } m, \tag{H2}$$

both conditions we shall assume in the sequel of the paper.

With the above notation and hypotheses, we obtain the following extension of Théorème 1 of [8] and of Theorem 1 of [6] in the rational case.

THEOREM 1. *Let $K \geqslant 3$, $L \geqslant 2$, R_1, R_2, S_1, S_2 be positive integers and set*

$$R = R_1 + R_2 - 1, \qquad S = S_1 + S_2 - 1, \qquad N = KL,$$

$$\gamma_1 = \frac{R+g-1}{2R} - \frac{gN}{6R(S+g-1)}, \quad \gamma_2 = \frac{S+g-1}{2S} - \frac{gN}{6S(R+g-1)}.$$

For any $1 \leqslant i \leqslant w$, denote by $p_i^{h_i}$ the greatest power of p_i which divides simultaneously b_1 and b_2 and assume that p_i does not divide $b_2/p_i^{h_i}$. Put $h = \max_{1 \leqslant i \leqslant w} h_i$ and

$$b = \frac{(R-1)b_2 + (S-1)b_1}{2}\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}.$$

Assume that there exist two residue classes c_1 and c_2 modulo g such that

$$\mathrm{Card}\left\{\left(\frac{x_1}{y_1}\right)^r\left(\frac{x_2}{y_2}\right)^s; 0 \leqslant r < R_1, 0 \leqslant s < S_1, m_1 r + m_2 s \equiv c_1 \,\mathrm{modulo}\, g\right\} \geqslant L,$$
$$\mathrm{Card}\left\{rb_2 + sb_1;\, 0 \leqslant r < R_2,\, 0 \leqslant s < S_2, m_1 r + m_2 s \equiv c_2 \,\mathrm{modulo}\, g\right\} > (K-1)L.$$

Under the condition

$$\begin{aligned} &K(L-1)\log m - (1+2w)\log N - (K-1)\log b - \\ &\quad - \gamma_1 LR \max\{|x_1|, |y_1|\} - \gamma_2 LS \max\{|x_2|, |y_2|\} \; > \; 0, \end{aligned} \tag{1}$$

we have

$$v_m(\Lambda) < KL + h - 1/2,$$

As we shall see in the two applications discussed in Sections 4 and 5, our hypothesis (H2) is not very restrictive.

We observe that inequality (1) is exactly the same as inequality (2) in [8] when m is an odd prime. Further, it also contains inequality (5) of [6] in the rational case. The only new parameter appearing when m is composite is the number of its distinct prime factors, but this is at most of the order of $\log m/\log\log m$, thus $(1+2w)\log N$ remains a second-order term. Consequently, the upper bound for $v_m(\Lambda)$ should be of nearly the same quality as in the case m prime. Indeed, this is true, and we now display explicit numerical estimates. As in [8] and in [6], we let $A_1 > 1, A_2 > 1$ be real numbers such that

$$\log A_i \geqslant \max\{\log|x_i|, \log|y_i|, \log m\}, \quad (i = 1, 2)$$

and we put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

In [8], we have provided explicit constants only assuming that x_1/y_1 and x_2/y_2 are multiplicatively independent. However, in the first application we present here, this additional condition has no reason to be satisfied. Thus, the next statements are given without this extra hypothesis.

THEOREM 2. *For all* $\mu \in \{4, 6, 8, 10, 15\}$ *define* $c_1(\mu)$ *and* $c_2(\mu)$ *by the following:*

μ	4	6	8	10	15
$c_1(\mu)$	66.8	46.1	36.9	32	26.1
$c_2(\mu)$	53.6	35.5	27.4	22.9	18

Under hypotheses (H1) *and* (H2)*, and if* m, b_1 *and* b_2 *are relatively prime, we have the upper estimate*

$$v_m(\Lambda) \leqslant \frac{c_1(\mu)\, g}{(\log m)^4}(\max\{\log b' + \log\log m + 0.64, \mu\log m\})^2 \log A_1 \log A_2$$

for all $\mu \in \{4, 6, 8, 10, 15\}$. *If, moreover,* x_1/y_1 *and* x_2/y_2 *are multiplicatively independent, then the above upper bound is true with* $c_2(\mu)$ *in place of* $c_1(\mu)$.

As is apparent in the statement of Theorem 2, we may have some problems when m, b_1 and b_2 have common prime factors. Indeed, the term h occurring in the conclusion of Theorem 1 is then positive, and we see no way for bounding it in the general case where m is composite. We point out that in the two applications of Theorem 2 which we present in the sequel of our work, the assumption on m, b_1 and b_2 is clearly satisfied, since in each case b_1 or b_2 is equal to 1.

Less trouble occurs when m is a power of a prime, as is shown in [8] and in [6], and the trivial bound for h is enough to conclude. For sake of completeness, we give explicit estimates in that case, without any further assumption on b_1 and b_2.

THEOREM 3. *For all* $\mu \in \{4, 6, 8, 10, 15\}$ *define* $c_3(\mu)$ *by the following*

μ	4	6	8	10	15
$c_3(\mu)$	67	46.3	37	32	26.2

With the above notation and under hypotheses (H1) *and* (H2), *if, moreover,* $m = p_1^{u_1}$ *is a power of a prime, then, for all* $\mu \in \{4, 6, 8, 10, 15\}$, *we have the upper estimate*

$$v_m(\Lambda) \leqslant \frac{c_3(\mu)\, g}{(\log m)^4} (\max\{\log b' + \log\log m + 0.64, \mu \log m\})^2 \log A_1 \log A_2.$$

Remark. We observe that the numerical constants are smaller when x_1/y_1 and x_2/y_2 are assumed to be multiplicatively independent. Indeed, as it is clear from the proof of Theorems 2 and 3, our choice of parameters L, K, R and S yields an asymptotic constant 15.46. . ., whence the asymptotic constant obtained in [8] is 64/9.

Remark. The numerical estimates obtained in [8] are less strong for the prime 2 than for the other small primes p, because of the factor $p/(\log p)^4$. However, using Theorem 1, it is possible to slightly refine the bound in that case. Consider for instance $\Lambda = x_1^{b_1} - x_2^{b_2}$ with x_1 and x_2 integers and assume that $x_2 \equiv 1 \bmod 4$ and $x_1 \equiv 1$ or 3 (mod 4). Then $x_1^2 \equiv 1$ mod 8 and we may apply the theorem with $m = 2^3$ to $\Lambda' = (x_1^2)^{b_1} - x_2^{2b_2}$. This trick is used in Sections 4 and 5 below, and allows us to decrease the numerical constant by a factor 3/2.

Further, Theorem 2 could be seen as a first step towards the proof of a conjecture of Philippon [24] on estimates for linear forms in logarithms of rational numbers.

CONJECTURE. *Let* a_1, a_2, b_1 *and* b_2 *be integers with* $a_1, a_2 > 0$ *and* $a_1^{b_1} a_2^{b_2} \neq \pm 1$. *Let* S *be a finite set of places on* $\mathbf{Q}$ *and set* $\mathrm{N}v = p$ *if* v *corresponds to the place* p *and* $\mathrm{N}v = 1$ *if* $v = \infty$. *Then there exists an effectively computable positive constant* C *such that*

$$-\sum_{v \in S} \log |a_1^{b_1} a_2^{b_2} \pm 1|_v \leqslant C \cdot \sum_{v \in S} \log \mathrm{N}v \cdot (\log 3a_1)(\log 3a_2) \max\{\log |3b_1|, \log |3b_2|\}.$$

3. Proofs of Theorems 1, 2 and 3

Proof of Theorem 1. We use the same ideas as in [8] and we consider the $N \times RS$ matrix $\mathbf{M}$, whose coefficients are the numbers

$$\binom{rb_2 + sb_1}{k} \left(\frac{x_1}{y_1}\right)^{\ell r} \left(\frac{x_2}{y_2}\right)^{\ell s},$$

where (k, ℓ) $(0 \leqslant k < K,\ 0 \leqslant \ell < L)$ denotes the index of the lines and (r, s) $(0 \leqslant r < R, 0 \leqslant s < S)$ the index of the columns. In [8], with the notation of that paper, x_1/y_1 and x_2/y_2 should be raised to the power p^t, in order to be in the disk of convergence of the p-adic exponential function. In the rational case, we have $t = 0$ except for $p = 2$. However, as is observed in [6], we can take also $t = 0$ in that case provided that $v_2\big((x_1/y_1)^g - 1\big)$ and $v_2\big((x_2/y_2)^g - 1\big)$ are $\geqslant 2$. In view of our hypothesis (H2), this is indeed the case here, hence, for any prime divisor p_i of m, the corresponding p_i-adic analytic function is well defined, and we can use the method of [8] to estimate $v_{p_i}(\Lambda)$.

Thanks to a zero lemma due to Nesterenko (see Lemme 6 of [8]), we extract from $\mathbf{M}$ a nonsingular square matrix $\mathcal{M}$ of size $N \times N$. Denote by Δ the determinant of $\mathcal{M}$. We shall estimate $|\Delta|$ from below and from above.

- Arithmetic lower bound.

We use the following version of Liouville's inequality. We recall that the logarithmic height of a rational a/b with a and b coprime is $\mathrm{h}(a/b) = \log\max\{|a|, |b|\}$.

LEMMA 1. *Let $P \in \mathbf{Z}[X, Y]$ and $x, y \in \mathbf{Q}$ such that $P(x, y) \neq 0$. Let $v_1, \ldots, v_w$ be distinct places of $\mathbf{Q}$. Then we have*

$$\sum_{i=1}^{w} \log|P(x,y)|_{v_i} \geqslant -\log|P|_1 - (\deg_X P)\mathrm{h}(x) - (\deg_Y P)\mathrm{h}(y),$$

where $|P|_1$ is the maximum of $|P(\xi, \zeta)|$ for ξ and ζ on the unit circle.

Proof. By the maximum principle, we have

$$\log|P(x,y)| \leqslant (\deg_X P)\log\max\{1, |x|\} + (\deg_Y P)\log\max\{1, |y|\} + \log|P|_1.$$

Further, for any non-Archimedean valuation v, we get

$$\log|P(x,y)|_v \leqslant (\deg_X P)\log\max\{1, |x|_v\} + (\deg_Y P)\log\max\{1, |y|_v\}.$$

From the product formula

$$\sum_{v \text{ place on } \mathbf{Q}} \log|P(x,y)|_v = 0,$$

we infer that

$$\begin{aligned}\sum_{i=1}^{w} \log|P(x,y)|_w &\geqslant -\log|P|_1 - \sum_{v \text{ place on } \mathbf{Q}} ((\deg_X P)\log\max\{1, |x|_v\} + \\ &\quad + (\deg_Y P)\log\max\{1, |y|_v\}) \\ &\geqslant -\log|P|_1 - (\deg_X P)\mathrm{h}(x) - (\deg_Y P)\mathrm{h}(y),\end{aligned}$$

as claimed. □

We proceed exactly as in Lemme 11 of [8], except that we apply Lemma 1, and we obtain

$$\begin{aligned}2\sum_{i=1}^{w} \log|\Delta|_{p_i} \geqslant &-N(\log N + (K-1)\log b + \gamma_1 LR\max\{|x_1|, |y_1|\} + \\ &+ \gamma_2 LS\max\{|x_2|, |y_2|\}).\end{aligned} \qquad (2)$$

- Analytic upper bound and completion of the proof.

We already know that Lemme 8 of [8] extends as follows (see [6], Proof of Theorem 1) since we have assumed that g does not divide any p_i, $1 \leqslant i \leqslant w$.

LEMMA 2. *Let* $1 \leqslant i \leqslant w$ *and assume that* $v_{p_i}(\Lambda) \geqslant u_i(N - 1/2) + h_i$. *Then we get*

$$v_{p_i}(\Delta) \geqslant \frac{u_i NK(L-1)}{2} - \frac{N \log N}{\log p_i}.$$

We now complete the proof of Theorem 1. Assume that for all $1 \leqslant i \leqslant w$ we have $v_{p_i}(\Lambda) \geqslant u_i(N - 1/2) + h_i$. Then from Lemma 2 we get

$$\begin{aligned} 2\sum_{i=1}^{w} \log |\Delta|_{p_i} &\leqslant -\sum_{i=1}^{w} (u_i NK(L-1) \log p_i - 2N \log N) \\ &\leqslant 2wN \log N - NK(L-1) \log m. \end{aligned} \tag{3}$$

Combining (2) and (3) and dividing by N, we see that (1) cannot hold. Thus, there exists $1 \leqslant i \leqslant w$ with $v_{p_i}(\Lambda) < u_i(N - 1/2) + h_i$ and, consequently, $v_m(\Lambda) < KL + h - 1/2$. $\square$

Proof of Theorems 2 and 3. For convenience, we keep the notation of [8], hence we set

$$D' = \frac{1}{\log m} \quad \text{and} \quad a_i = \frac{\log A_i}{\log m} \geqslant \max\left\{\frac{|x_i|}{\log m}, \frac{|y_i|}{\log m}, 1\right\} \quad (i = 1, 2).$$

Further, let $B \geqslant D' \log b$ be a positive real that we will fix later. We search parameters K, L, R and S satisfying

$$K(L-1) > (1 + 2w)D' \log N + (K-1)B + \gamma_1 LRa_1 + \gamma_2 LSa_2, \tag{4}$$

since then (1) will be automatically satisfied.

Let k and ℓ be two positive constants and set

$$\begin{aligned} &L = [\ell B] + 2, \quad K = [kgLa_1a_2] + 1, \quad R_1 = gL, \quad S_1 = 1, \\ &R_2 = [\sqrt{g(K-1)La_2/a_1}] + 1, \qquad S_2 = [\sqrt{g(K-1)La_1/a_2}] + 1. \end{aligned}$$

We have the obvious lower bounds

$$R_1S_1 \geqslant gL, \qquad R_2S_2 > g(K-1)L.$$

We shall give an upper estimate for $\gamma_1 LRa_1, \gamma_2 LSa_2$ and $(K-1)B$, which occur in the right-hand side of (4).

LEMMA 3. *We have the upper bounds*

$$\gamma_1 Ra_1 \leqslant \frac{1}{3}\sqrt{g(K-1)La_1a_2} + (gL + g - 1)\frac{a_1}{2} + \frac{ga_2}{6}$$

and

$$\gamma_2 S a_2 \leqslant \frac{1}{3}\sqrt{g(K-1)La_1a_2} + \frac{ga_2}{2} + (gL+g-1)\frac{a_1}{6}.$$

Proof. As in the proof of Lemme 12 of [8], we start from the formula

$$\begin{aligned}\gamma_1 R a_1 &= \frac{(R+g-1)a_1}{2} - \frac{gKLa_1}{6(S+g-1)}\\ &\leqslant \frac{(R_1+g-1+R_2-1)a_1}{2} - \frac{g(K-1)La_1}{6(S_2+g-1)},\end{aligned}$$

and we use the estimates

$$R_1+g-1+R_2-1 \leqslant gL+g-1+\sqrt{g(K-1)La_2/a_1}$$

$$\begin{aligned}\frac{1}{S_2+g-1} &\geqslant \frac{1}{g+\sqrt{g(K-1)La_1/a_2}}\\ &\geqslant \frac{1}{\sqrt{g(K-1)La_1/a_2}} - \frac{1}{(K-1)La_1/a_2}.\end{aligned}$$

The same method yields our upper bound for $\gamma_2 S a_2$. □

We can now reformulate our condition (4) in a numerical relation between the parameters k, ℓ, B, a_1, a_2. Set $\lambda = \ell + (2/B)$ and notice that

$$\ell B < L-1 < L \leqslant \ell B + 2 = \lambda B. \tag{5}$$

We observe that

$$K(L-1) > kgLa_1a_2 \times \ell B > k\ell^2 gB^2 a_1 a_2,$$

and that we have the upper bound

$$(K-1)B \leqslant kgLa_1a_2 \times B \leqslant kgBa_1a_2(\ell B+2) = k\ell gB^2a_1a_2 + 2kgBa_1a_2.$$

Further, Lemma 3 and (5) yield

$$\gamma_1 LRa_1 + \gamma_2 LSa_2 \leqslant gB^2a_1a_2\left(\frac{2}{3}\sqrt{k}\lambda^2 + \frac{4\lambda}{3B} + \frac{2\lambda^2}{3}\right),$$

and we have

$$N = KL \leqslant kgL^2a_1a_2 + L \leqslant k\lambda^2 gB^2a_1a_2 + \lambda B.$$

Replacing the terms occurring in (4) by the above estimates and dividing by $gB^2a_1a_2$, we obtain that the inequality

$$k\ell^2 - k\ell - \frac{2}{3}\lambda^2(\sqrt{k}+1) \geqslant \frac{2k}{B} + \frac{4}{3}\frac{\lambda}{B} + \frac{(1+2w)D'\log(k\lambda^2 gB^2a_1a_2+\lambda B)}{gB^2a_1a_2}, \tag{6}$$

where the right-hand side tends to 0 when B tends to infinity, implies (4).

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

It now remains to compare $B \geqslant D' \log b$ with b', as in Lemme 13 of [8]. Recall that we have set

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1} = \frac{1}{\log m}\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right)$$

and

$$b = \frac{(R-1)b_2 + (S-1)b_1}{2}\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}.$$

LEMMA 4. *We have the upper bound*

$$\log b \leqslant \log b' + \log\log m + \tfrac{3}{2} - \log 2 + \log\left(\frac{\sqrt{k}+1}{k}\right) + \log\left(1 + \frac{1}{K-1}\right).$$

Proof. It has been proved in Lemme 8 of [18] that

$$\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)} \leqslant \exp\left\{-\log(K-1) + \tfrac{3}{2} - \frac{\log(2\pi(K-1)/\sqrt{e})}{K-1} + \frac{\log K}{6K(K-1)}\right\}.$$

Neglecting the negative term

$$-\frac{\log(2\pi(K-1)/\sqrt{e})}{K-1} + \frac{\log K}{6K(K-1)},$$

we get

$$\begin{aligned}
b &\leqslant \frac{(R-1)b_2 + (S-1)b_1}{2(K-1)} e^{3/2} \\
&\leqslant \frac{\sqrt{g(K-1)La_1a_2}}{2(K-1)}\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right)e^{3/2} + \frac{g(L-1)b_2 + b_1}{2(K-1)} e^{3/2} \\
&\leqslant \frac{\sqrt{g(K-1)La_1a_2} + (gL-1)a_1 + a_2}{2(K-1)}\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right)e^{3/2} \\
&\leqslant \frac{\sqrt{kK(K-1)} + K}{2(K-1)k}\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right)e^{3/2}.
\end{aligned}$$

Then it follows that

$$\log b \leqslant \log b' + \log\log m - \log 2 + \tfrac{3}{2} + \log\frac{(\sqrt{k(K-1)} + \sqrt{K})\sqrt{K}}{k(K-1)},$$

as claimed. □

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

Since $x_1/y_1 \neq \pm 1$, our choice $S_1 = 1$ implies that the condition

$$\mathrm{Card}\left\{\left(\frac{x_1}{y_1}\right)^r\left(\frac{x_2}{y_2}\right)^s; 0 \leqslant r < R_1, 0 \leqslant s < S_1, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\right\}$$
$$= \mathrm{Card}\{(r, s); 0 \leqslant r < R_1, 0 \leqslant s < S_1, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\}$$

is trivially fulfilled. Further, if there exists a residue class c modulo g such that

$$\mathrm{Card}\{b_2 r + b_1 s; 0 \leqslant r < R_2, 0 \leqslant s < S_2, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\}$$
$$< \mathrm{Card}\{(r, s); 0 \leqslant r < R_2, 0 \leqslant s < S_2, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\},$$

we have established in [8] the upper bound

$$v_{p_i}(\Lambda) \leqslant gD' \log 2 + 2\sqrt{k}\lambda g B a_1 a_2 + h_i$$

for any prime divisor p_i of m, whence we obviously get

$$v_m(\Lambda) \leqslant gD' \log 2 + 2\sqrt{k}\lambda g B a_1 a_2 + h.$$

Assume now that for any residue class c modulo g we have

$$\mathrm{Card}\{b_2 r + b_1 s; 0 \leqslant r < R_2, 0 \leqslant s < S_2, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\}$$
$$= \mathrm{Card}\{(r, s); 0 \leqslant r < R_2, 0 \leqslant s < S_2, m_1 r + m_2 s \equiv c \,\mathrm{modulo}\, g\}.$$

By Theorem 1, if (6) is satisfied, then we have $v_m(\Lambda) \leqslant KL + h - 1/2$. If we are interested only in the value of the asymptotical constant, we have only to consider the terms of leading degree in B, hence the left-hand side of (6). Arguing as in Subsection 6.2 of [8], we search values of k and ℓ such that $k\ell^2 - k\ell - \frac{2}{3}\lambda^2(\sqrt{k} + 1) > 0$ and the product $k\ell^2$ is minimal. Hence, our choice of parameters cannot yield an asymptotic constant smaller than 15,46..., which is obtained with $k = (26 + 4\sqrt{22})/9$ and $\ell = 3k/\,(3k - 2\sqrt{k} - 2)$.

Summing up all we have proved yet, it follows that if (6) is satisfied, then

$$v_m(\Lambda) \leqslant \max\{KL, gD' \log 2 + 2\sqrt{k}\lambda g B a_1 a_2\} + h.$$

Now, we make explicit this upper bound. Let μ be a positive integer and set

$$B = \frac{1}{\log m} \max\{\log b' + \log\log m + 0.64, \mu \log m\}, \quad \tilde{\lambda} = \ell + \frac{2}{\mu},$$

in such a way that $B \geqslant \mu$ and $\tilde{\lambda} \geqslant \ell + 2/B$. Observe that we may assume that $m \geqslant 3$. Indeed, if we are interested in the 2-adic valuation of Λ, our hypothesis $(H2)$ implies that we may bound $v_4(\Lambda)$. Thus, we get $(1 + 2w)D' \leqslant 2.8$ and we have to find k and ℓ such that

$$k\ell^2 - k\ell - \tfrac{2}{3}(\sqrt{k} + 1)(\tilde{\lambda}^2) \geqslant \frac{2k}{\mu} + \frac{4}{3}\frac{\tilde{\lambda}}{\mu} + \frac{2.8 \log(k\tilde{\lambda}^2\mu^2 + \tilde{\lambda}\mu)}{\mu^2}, \tag{7}$$

provided that

$$\log(k\ell^2\mu^2 + 4k\ell\mu + \ell\mu + 4k + 2) \geqslant 2,$$

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

in which case the last term of the left-hand side of (7) is a decreasing function of the variable μ. We observe that these conditions are satisfied by the triples (k, ℓ, μ) for $\mu \in \{4, 6, 8, 10, 15\}$ given in the following table:

μ	4	6	8	10	15
$k(\mu)$	5.7	5.3	5.2	5.5	5.7
$\ell(\mu)$	2.9	2.6	2.4	2.2	2

Since $K \geqslant [k[\ell\mu + 2] + 1]$, we easily check that

$$\tfrac{3}{2} - \log 2 + \log\left(\frac{\sqrt{k}+1}{k}\right) + \log\left(1 + \frac{1}{K-1}\right) \leqslant 0.64$$

for all our choices of $k(\mu)$ and $\ell(\mu)$. Lemma 4 then shows that the condition $B \geqslant D' \log b$ is satisfied. Consequently, we have the following upper bound for $v_m(\Lambda)$:

$$\begin{aligned} v_m(\Lambda) &\leqslant \max\{KL, gD' \log 2 + 2\sqrt{k}\tilde{\lambda} gBa_1a_2\} + h \\ &\leqslant \max\left\{\left(k\tilde{\lambda}^2 + \frac{\tilde{\lambda}}{\mu}\right) gB^2a_1a_2, \frac{\log 2}{\log 3} gB + 2\sqrt{k}\tilde{\lambda} gBa_1a_2\right\} + h \\ &\leqslant c(\mu) gB^2a_1a_2 + h, \end{aligned}$$

with

$$c(\mu) = \max\left\{k\tilde{\lambda}^2 + \frac{\tilde{\lambda}}{\mu}, \frac{\log 2}{\mu \log 3} + \frac{2\sqrt{k}\tilde{\lambda}}{\mu}\right\}.$$

When m, b_1 and b_2 are coprime, we have $h = 0$. Substituting then the numerical values of k and ℓ given above for any $\mu \in \{4, 6, 8, 10, 15\}$, we get

μ	4	6	8	10	15
$c(\mu)$	66.8	46.1	36.9	32	26.1

as asserted in Theorem 2.

To prove the second statement of Theorem 2, we observe that we have to deal with

$$k\ell^2 - k\ell - \frac{2}{3}(\sqrt{k}+1)(\tilde{\lambda}^2) \geqslant \frac{2k}{\mu} + \frac{4}{3}\frac{\tilde{\lambda}^{3/2}}{\sqrt{\mu}} + \frac{4}{3}\frac{\tilde{\lambda}}{\mu} + \frac{2.8 \log(k\tilde{\lambda}^2\mu^2 + \tilde{\lambda}\mu)}{\mu^2}$$

instead of (7), and to argue as above.

Finally, to prove Theorem 3, we use the trivial upper bound $h \leqslant B^2a_1a_2/\mu$ given in [8]. $\square$

4. Application to Fractional Parts of Powers of Rationals

We denote by $\|\cdot\|$ the distance to the nearest integer. Let $a, b \in \mathbf{Z}$ with $b > 0$, $a/b > 1$ and $a/b \notin \mathbf{Z}$. Obviously, for every integer $k \geqslant 1$, we have

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

$$\left\| \left(\frac{a}{b} \right)^k \right\| \geqslant \frac{1}{b^k}, \tag{8}$$

and there is no reason for this lower bound to be the best possible in terms of k. Indeed, Mahler [20] proved that for any $\varepsilon > 0$, there exists a k_0 depending only on a, b and ε such that for any $k > k_0$ we have $\|(a/b)^k\| > \mathrm{e}^{-\varepsilon k}$. However, the arguments used by Mahler rest on the theorem of Roth–Ridout and the proof does not yield a computable value for k_0. The first effective improvement of (8) was obtained by Baker and Coates [2] and can be stated as follows.

THEOREM BC. *For any relatively prime integers a, b with $a > b \geqslant 2$, there exist effectively computable numbers K and $0 < \eta < 1$, depending on a and b, such that*

$$\left\| \left(\frac{a}{b} \right)^k \right\| \geqslant \frac{1}{b^{\eta k}}$$

for all integers $k \geqslant K$.

Their proof depends on the theory of linear forms in p-adic logarithms and yields a value for η very close to 1.

Another approach allows us to strengthen (8) in an effective way when a/b is assumed to be close to 1. This has been worked out by Beukers [5] who, for instance, showed that $\|(3/2)^k\| \geqslant 2^{-0.9k}$ for all $k > 5000$ (notice that this estimate has been refined by Dubitskas [13] and, very recently, by Habsieger [16]). The method is based on the use of Padé approximants to the functions $z \mapsto (1+z)^{1/k}$ and the results obtained are considerably better than those of Baker and Coates, but they can be applied only to a restricted class of rationals. For instance, according to a remark of Bennett [4], it seems that this approach does not yield any improvement of (8) for $a/b = 4/3$.

Recently, new effective improvements of (8) have been obtained by Corvaja [12], assuming again that a/b is close to 1. His main result is the following.

THEOREM C. *Let $0 < \delta < 1$ be real. There exist effectively computable constants $K = K(\delta)$ and $N = N(\delta)$ such that for any integers $k > K$ and $n > N$ we have*

$$\left\| \left(\frac{N+1}{N} \right)^k \right\| > N^{-\delta k}.$$

Suitable values for K and N are

$$N = \exp\{17\delta^{-2} + 2250\delta\} \quad \text{and} \quad K = 12288\delta^{-2}N^6.$$

Corvaja's proof rests on the Thue principle, and his result yields very strong estimates when N and k are both large.

One of the disadvantages of the method based on p-adic linear forms of logarithms is that the result depends on the prime factors of b. Roughly speaking, if p divides b, we obtain

$$\left\|\left(\frac{a}{b}\right)^k\right\| \geqslant \frac{1}{b^k p^{-\varepsilon k}},$$

for some explicit $\varepsilon > 0$ very small. Thus the estimate closely depends on the prime divisors of b and, hence, should be uniform.

The purpose of the present work is to show that, under some hypothesis, we may apply the theory of linear forms in m-adic logarithms in order to get uniform estimates.

Since we do not want to give too technical statements, we merely present two new results, which illustrate the type of estimates our method yields. For convenience, we write $\log^* x$ for $\max\{\log x, 1\}$.

THEOREM 4. *Let $a \geqslant b \geqslant 1$ and $\ell \geqslant 2$ be integers. For any integer $k \geqslant 1$, we have*

$$\left\|\left(\frac{a\ell+1}{b\ell}\right)^k\right\| \geqslant \frac{1}{b^k \ell^{k(1-1/\tau(a))}},$$

with $\tau(a) = 25000(\log^* a)(\log^* \log^* a)^2$.

Theorem 4 is uniform in the following sense. We fix a and b, which do not need to satisfy a/b close to 1, and we observe that the quality of the improvement of (8) does not depend on ℓ. This is absolutely not the case in the work of Baker and Coates.

THEOREM 5. *Let $u \geqslant v \geqslant 1$ and $\ell \geqslant 2$ be integers. For any integer $k \geqslant 1$, we have*

$$\left\|\left(\frac{\ell^u+1}{\ell^v}\right)^k\right\| \geqslant \frac{1}{\ell^{vk}(1-1/\tau(u/v))},$$

with

$$\tau(u/v) = 25000\left(\log^* \tfrac{u}{v}\right)\left(\log^* \log^* \tfrac{u}{v}\right)^2.$$

Further, in order to point out how the improvements obtained since the paper of Baker and Coates are significant, we compute by the same method effective lower bounds for $\|(3/2)^k\|$ and $\|(4/3)^k\|$.

THEOREM 6. *For any integer $k \geqslant 5$, we have*

$$\left\|\left(\frac{3}{2}\right)^k\right\| \geqslant \frac{1}{2^{0.9995k}},$$

and, for any integer $k \geqslant 2$, we have

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

$$\left\| \left(\frac{4}{3}\right)^k \right\| \geqslant \frac{1}{3^{0.9996k}}.$$

Theorem 6 is far from solving the Waring problem, of course, and it does not yield the up to now best known uniform lower bound for $\|(3/2)^k\|$, which is (see [16])

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geqslant \frac{1}{2^{0.8k}},$$

valid for $k \geqslant 5$.

Proof of Theorems 4 and 5. There exist integers q and r with

$$(a\ell+1)^k = q(b\ell)^k + r \quad \text{and} \quad r = (b\ell)^k \times \left\| \left(\frac{a\ell+1}{b\ell}\right)^k \right\|. \tag{9}$$

Since $\ell \geqslant 2$, we have $r \neq 0$ and the ℓ-adic valuation of $\Lambda = (a\ell+1)^k - r$ is at least equal to k. Further, we note that $r \equiv 1 \bmod \ell$. For $\ell = 2$, we apply Théorème 3 of [8] and we observe that Theorem 4 is true in that case. If ℓ is odd or is divisible by 4, then we may apply Theorem 2 with $m = \ell$. Otherwise, because of hypothesis (H2), we apply it with $m = \ell/2$. Since the first case obviously leads to stronger estimates, we do our computation only for the second one, hence we apply Theorem 2 with $m = \ell/2$, $g = 1$, $x_1/y_1 = a\ell + 1$, $x_2/y_2 = r$, $b_1 = k$ and $b_2 = 1$, and we get

$$\begin{aligned} k \leqslant v_m(\Lambda) \leqslant & \frac{66.8}{(\log m)^4} \log(a\ell+1) \max\{\log m, \log|r|\} \times \\ & \times \left(\max\left\{ \log\left(\frac{k}{\max\{\log m, \log|r|\}} + \frac{1}{\log m} \right) + \log\log m + 0.64, 4\log m \right\} \right)^2. \end{aligned}$$

Assume first that $|r| \geqslant m$. According as the first maximum equals $4\log m$ or not, we get

$$\log|r| \geqslant \frac{k}{1069} \frac{\log^2 m}{\log(a\ell+1)}$$

or

$$\begin{aligned} k\log m \leqslant & \frac{66.8}{(\log m)^2} \frac{\log(a\ell+1)}{\log m} \times \\ & \times \left(\log\left(\frac{k}{\log|r|} + \frac{1}{\log m} \right) + \log\log m + 0.64 \right)^2 \log|r|, \end{aligned}$$

whence, in both cases, we deduce

$$\frac{k}{\log|r|} \leqslant \frac{25000}{\log \ell} (\log^* a)(\log^* \log^* a)^2. \tag{10}$$

Further, if $|r| < m$, then k is bounded:

$$k \leqslant \max\left\{ \frac{1069\log(a\ell+1)}{\log m}, \frac{66.8}{(\log m)^3} \log(a\ell+1) \log^2\left(\frac{k+1}{\log m} \right) \right\}. \tag{11}$$

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

Consequently, from (9), (10) and (11), we get

$$\left\| \left(\frac{a\ell + 1}{b\ell}\right)^k \right\| \geqslant \frac{1}{b^k \ell^{k(1-1/(25000(\log^* a)(\log^* a \log^* a)^2))}}.$$

The proof of Theorem 5 follows the same lines, and we omit it. □

Proof of Theorem 6. Let $k \geqslant 5$ be an integer, and write $3^k = q2^k + r$, with q, r integers and $|r| \leqslant 2^{k-1}$. Since for any integer $n \geqslant 1$ we have $v_2(3^n \pm 1) \leqslant 1 + 2v_2(n)$, we observe that $|r| \neq 1$, thus r and 3 are multiplicatively independent when $k \geqslant 5$. Further, $k \leqslant v_2(3^k - r) \leqslant v_2(9^k - r^2)$ and $r \equiv 1 \pmod 8$, so we apply Theorem 2 with $g = 1$, $m = 8$, $\mu = 4$ to $\Lambda = 9^k - r^2$, and, since $r^2 \geqslant 8$, we get

$$\begin{aligned}\frac{k+1}{3} &\leqslant v_8(\Lambda)\\ &\leqslant \frac{53.6}{(\log 8)^4}(\max\{\log(k/(2\log|r|) + 1/\log 9) + 0.74, 4\log 8\})^2(\log 9)(\log r^2),\end{aligned}$$

whence $k \leqslant 2615 \log|r|$, and

$$\left\| \left(\frac{3}{2}\right)^k \right\| \geqslant 2^{-k} \mathrm{e}^{k/2615} \geqslant \frac{1}{2^{0.9995k}},$$

as claimed.

Let $k \geqslant 3$ be an integer, and write $4^k = q3^k + r$, with q, r integers and $|r| < 3^k/2$. Clearly, we have $r \equiv 1 \pmod 3$. Further, $r \neq 1$, since $v_3(4^k - 1) = 1 + v_3(k) < k$ for $k \geqslant 2$. Finally, using that $v_3(2^n + 1) = v_3(n)$ and $v_3(2^{n+1} - 1) = v_3(n + 1)$ for any odd positive integer n, we can also show that $|r|$ is not a power of 2, thus that 4 and r are multiplicatively independent. We apply Theorem 2 with $g = 1$, $m = 3$, $\mu = 8$ to $\Lambda = 4^k - r$, and we get

$$k \leqslant v_3(\Lambda) \leqslant \frac{27.4}{(\log 3)^4}(\max\{\log(k/\log|r| + 1/\log 4) + 0.74, 8\log 3\})^2 \log 4 \log|r|,$$

whence $k \leqslant 2015 \log|r|$, and

$$\left\| \left(\frac{4}{3}\right)^k \right\| \geqslant 3^{-k} \mathrm{e}^{k/2015} \geqslant \frac{1}{3^{0.9996k}},$$

as claimed. □

5. Application to the Diophantine Equation $(x^n - 1)/(x - 1) = y^q$

To date, the only three known solutions of the Diophantine equation

$$\frac{x^n - 1}{x - 1} = y^q, \qquad \text{in integers } x > 1, y > 1, n > 2, q \geqslant 2. \tag{12}$$

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

are given by $(x, y, n, q) = (3, 11, 5, 2)$, $(7, 20, 4, 2)$ and $(18, 7, 3, 3)$. It is conjectured that these are the only ones, but the question whether (12) has only finitely many solutions remains an open problem. However, we have several partial results. For instance, (12) has been completely solved by Nagell [23] and Ljunggren [19] when n is divisible by 3 or 4 and also when $q = 2$. Recently, several authors have obtained new and interesting results, and (12) is now solved for infinitely many values of x, including all integers $x = z^t$, with $t \geqslant 1$ and $z \leqslant 10^4$ (Theorem BM below, see also [9] for more references). Further, Bugeaud, Mignotte and Roy [11] proved the following criterion which provides a sufficient condition on x ensuring that there is no triple (y, n, q) with (x, y, n, q) being a solution of (12).

THEOREM BMR. *Equation* (12) *has no solution* (x, y, n, q) *where* x *and* y *satisfy the following hypothesis*

Every prime divisor of x *also divides* $y - 1$,

excepted $(18, 7, 3, 3)$. *Consequently, for all other solutions* (x, y, n, q) *of* (12) *with* q *prime, there exists a prime number* p *such that* p *divides* x *and* q *divides* $p - 1$.

Since (12) is completely solved for $q = 2$, Theorem BMR allows us among others to treat (12) for any integer x whose prime factors are of the form $2^a + 1$, with $a \geqslant 0$, for instance for all integers x of the form $2^a 3^b 5^c 17^d$, where a, b, c and d are non-negative integers. Thanks to the main result of the present paper, we are now able to extend Theorem BMR as follows.

THEOREM 7. *Assume that* (x, y, n, q) *is a fourth solution of* (12). *Write* $x = x_1 x_2$, *where* x_1 *is composed by the prime divisors* p_j *of* x *such that* $y \equiv 1 \pmod{p_j}$ *and* x_2 *is composed by the prime divisors* q_k *of* x *such that* $y \not\equiv 1 \pmod{q_k}$. *We define* v *by* $x_1 = x^v$. *If* $x_1 > 1$ *then we have*

$$q \leqslant \frac{532}{v^2} \left(\log \frac{82}{v} \right)^2.$$

Moreover for any $v_0 > 5/6$, *there exists an effectively computable constant* C, *depending only on* v_0, *such that we have either* $v < v_0$ *or* $x \leqslant C$. *In particular, if* $x \geqslant 4.2 \times 10^{11}$, *then we have* $v \leqslant 10/11$.

As is clear from the proof, one can get some other effective statements. For instance, using Theorem BM below and inequalities (20) below, we obtain that there is no new solution to (12) with $q = 3$ and $v > 0.981$, with $q = 5$ and $v > 0.876$, and also with $q = 7$ and $v > 0.838$. Further, we observe that with the notation of Theorem 7, Theorem BMR asserts that $v < 1$.

Theorem 7 allows us now to solve equation (12) for a wide set of integers x, considerably larger than the set S_3 as defined in [6]. To this end, we first recall some definitions from [6]. Let S_1 be the set of all positive integers whose prime factors are all of the form $2^a + 1$, with $a \geqslant 0$. Let S_2 be the set of all positive integers whose prime factors are all of the form $2^a 3^b + 1$, with $a \geqslant 0$ and $b > 0$. We observe that $S_1 \cup S_2$

includes all integers between 2 and 20, except 11. Let S_3 be the set of all integers of the form $x_3 = x_1 x_2 x$, with $x_1 \in S_1$, $x_2 \in S_2$, $x \leqslant (x_1 x_2)^{1/10}$, $\gcd(x, x_1 x_2) = 1$ and, if $x_2 \neq 1$, we moreover assume that $x_3 \equiv \pm 2, \pm 4 \pmod 9$ or $x_3 \equiv \pm 2, \pm 3 \pmod 7$.

THEOREM 8. *Let $x_3 \in S_3$ and $t \geqslant 1$. Then* (12) *has no solution* (x_3^t, y, n, q) *if* $x_3^t \geqslant 4.2 \times 10^{11}$.

Our set S_3 contains and is much bigger than the sets S_3 defined in [27] and in [6]. In particular, we point out that S_3 has positive lower logarithmic density.

As already noticed in [27] and in [6], Theorem 8 can be applied to prove the irrationality of some numbers of Mahler's type. Let us introduce the following notation. Let $g \geqslant 2$ and $h \geqslant 2$ be integers. For any integer $m \geqslant 1$, we define $(m)_h = a_1 \dots a_r$ to be the sequence of digits of m written in h-ary notation, *i.e.* under the form $m = a_1 h^{r-1} + \cdots + a_r$, with $a_1 > 0$ and $0 \leqslant a_i < h$ for $1 \leqslant i \leqslant r$. For a sequence $(n_i)_{i \geqslant 1}$ of non negative integers, we put

$$a_h(g) = 0.(g^{n_1})_h \, (g^{n_2})_h \dots$$

and we call Mahler's numbers the real numbers obtained in this way. It is known that $a_h(g)$ is irrational for any unbounded sequence $(n_i)_{i \geqslant 1}$; see the work of Sander [25] for an account of earlier results in this direction. Sander also considered the case when $(n_i)_{i \geqslant 1}$ is bounded with exactly two elements occurring infinitely many times, which are called, by definition, limit points. As mentioned in [27], his paper contained an incorrect application of a result of Shorey and Tijdeman [29], hence, his Theorem 3 remains unproved. Here, we extend Corollary 1 of [6] as follows.

THEOREM 9. *Let $(n_i)_{i \geqslant 1}$ be a bounded sequence of nonnegative integers which is not ultimately periodic and has exactly two limit points $N_1 < N_2$. Let $g \geqslant 2$ and $h \geqslant 2$ be integers such that $g \neq 1 + h + \cdots + h^{L-1}$ for every integer*

$L \geqslant 2$ if $(N_1, N_2) \neq (0, 1)$. Assume also that (N_1, N_2, g, h) is not equal to $(0, 2, 11, 3)$, $(0, 2, 20, 7)$, $(0, 3, 7, 18)$ or to $(1, 4, 7, 18)$ and that $g^{N_2 - N_1}$ is not equal to $1 + h$ whenever $g^{N_1} < h$. Let t be given by the inequalities $h^{t-1} \leqslant g^{N_1} < h^t$. If $h \in S_3$ and $h^t \geqslant 4.2 \times 10^{11}$ then $a_h(g)$ is irrational.

Before proceeding with the proofs of Theorems 7, 8 and 9, we need to summarize several known results concerning (12).

AUXILIARY RESULTS

THEOREM NL. *The only solutions of* (12) *with $q = 2$ or n divisible by* 3 *or n divisible by* 4 *are given by* $(x, y, n, q) = (3, 11, 5, 2)$*,* $(7, 20, 4, 2)$ *and* $(18, 7, 3, 3)$.

Proof. This is due to Nagell [23] and Ljunggren [19]. □

THEOREM I. *Let* $(x, y, n, 3)$ *be a solution of* (12)*. Then y is not a cube,* $x \equiv 0, \pm 1$ $\pmod 7$ *and* $x \equiv 0, \pm 1 \pmod 9$.

Proof. This is due to Inkeri [17]. □

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

THEOREM BHM. *Let (x, y, n, q) be a solution of* (12) *with $q \geqslant 3$ and $n \neq 3$. Let p be an odd prime divisor of n. Then we have $p \geqslant 29$ or $(p, q) \in \{(17, 17), (19, 19), (23, 23)\}$. Moreover, we have $p \geqslant 101$ if $q = 3$ and $(p, q) \notin \{(29, 5), (29, 19), (29, 23), (31, 23), (37, 5), (37, 7), (37, 11), (67, 5)\}$.*

Proof. This is Théorèmes 2 to 4 of [7]. □

THEOREM BM. *Equation* (12) *has no new solution (x, y, n, q) with $x \leqslant 10^6$, and with $x = z^t$, where $t \geqslant 1$ and $z \leqslant 10^4$ are positive integers.*

Proof. This is due to Bugeaud and Mignotte [9, 10]. □

LEMMA SS. *Let (x, y, n, q) be a solution of Equation* (12). *Let $p_1, \ldots, p_\ell, q_1, \ldots, q_m$ be the distinct prime divisors of x such that $x = x_1 x_2$, where $x_1 = p_1^{u_1} \ldots p_\ell^{u_\ell}$ with $y \equiv 1 \pmod{p_i}$ for $1 \leqslant i \leqslant \ell$ and $x_2 = q_1^{v_1} \ldots q_m^{v_m}$ with $y \not\equiv 1 \pmod{q_j}$ for $1 \leqslant j \leqslant m$. Then we have*

$$x_1^{n+1-2\beta} \leqslant \left(\frac{n+3}{4}\right)^2 \left(2 + \frac{4}{x}\right)^{n-1} q^{\frac{\alpha q}{q-1}} x_2^{n-1+2\beta},$$

where $\alpha = n + 1$ if q does not divide x, $\alpha = 2n$ if q divides x, and $\beta = \max\{1, n/q\}$.

Proof. This is due to Saradha and Shorey [27]. □

Proof of Theorem 7. Let (x, y, n, q) be a solution of (12). By Theorem NL, we may assume that q is an odd prime. Let $p_1, \ldots, p_\ell, q_1, \ldots, q_m$ be the distinct prime divisors of x such that $x = x_1 x_2$, where $x_1 = p_1^{u_1} \ldots p_\ell^{u_\ell}$ with $y \equiv 1 \pmod{p_i}$ for $1 \leqslant i \leqslant \ell$ and $x_2 = q_1^{v_1} \ldots q_m^{v_m}$ with $y \not\equiv 1 \pmod{q_j}$ for $1 \leqslant j \leqslant m$. We have assumed that $x_1 > 1$ and it follows from Theorem BMR that $x_2 > 1$. Since (x, y, n, q) is a solution of (12), the x_1-adic valuation of

$$\Lambda = (1 - x) - \left(\frac{1}{y}\right)^q = -x^n y^{-q} \tag{13}$$

satisfies $v_{x_1}(\Lambda) = n$. As $1 - x$ and y are multiplicatively independent (for a proof, see [27], below inequality (51)), we can apply Theorem 2 to get an upper bound for $v_{x_1}(\Lambda)$. However, in view our hypothesis (H2), the worst case arises when $2 \| x_1$, where, contrary to the other situations, we have to take $g = 2$ or, alternatively, we may estimate $v_{x_1/2}(\Lambda)$. It turns out that the latter approach yields better estimates than the first one. For convenience, we set $m = x_1/2$ or x_1, according as $2 \| x_1$ or not, and we define v' by $m = x^{v'}$.

However, we have to consider separately the case $x_1 = 2$, where we apply Theorem 2 to $\Lambda' = (1 - x)^2 - (1/y)^{2q}$, with $m = 2^3$ and $\mu = 4$. After noticing that $v_2(\Lambda) \leqslant v_2(\Lambda')$, a rapid calculation shows that we get

$$q \leqslant \frac{50}{v^2} \left(\log \frac{8}{v}\right)^2$$

in that case.

Assume now that $x_1 > 2$ and apply Theorem 2 to (13) with $\alpha_1 = 1 - x$, $b_1 = 1$, $\alpha_2 = 1/y$ and $b_2 = q$. Observe that one may rewrite (12) under the form

$$x\frac{x^{n-1}-1}{x-1} = (y-1)\frac{y^q-1}{y-1},$$

and recall that $\gcd\big(y-1, (y^q-1)/(y-1)\big) = q$ or 1, according as $y \equiv 1 \bmod q$ or not. It follows that if a prime power p^u divides x_1, then p^u divides $y-1$, unless $p = q$, in which case $\max\{p, p^{u-1}\}$ divides $y-1$. Thus, in view of our hypothesis on x_1, we deduce that $y - 1 \geqslant \sqrt{x_1}$. Consequently, $\log A_1 = \log x$ and $\log A_2 = 2\log y$ are suitable choices. Theorem 2 with $\mu = 4$ yields the upper bound

$$v_m(\Lambda) \leqslant \frac{107.2}{(\log m)^4}\left(\max\left\{\log\left(\frac{q}{\log x} + \frac{1}{2\log y}\right) + \log\log m + 0.64, 4\log m\right\}\right)^2 \times \\ \times \log x \log y, \tag{14}$$

whence we obviously have

$$v_m(\Lambda) = n. \tag{15}$$

We get from (12) that $n\log x > q\log y$, and we infer from (14) and (15) that

$$q \leqslant 107.2\frac{\log^2 x}{(\log m)^4}\left(\max\left\{\log\left(\frac{q}{\log x} + \frac{1}{2\log y}\right) + \log\log m + 0.64, 4\log m\right\}\right)^2.$$

By the definition of m, it yields

$$q \leqslant \frac{200}{v'^2}\left(\log\frac{50}{v'}\right)^2. \tag{16}$$

Hence, the first statement of the theorem follows from $v' \geqslant (v\log 3)/\log 6$.

In the sequel, in view of Theorem BM, we may assume that $x \geqslant 10^6$. Further, we assume $v \geqslant 5/6$. Then, it follows from (14) a much better estimate than (16), namely we get

$$q \leqslant \frac{1716}{v'^2}. \tag{17}$$

We infer from

$$v' = v - \frac{\log 2}{\log x} = v\left(1 - \frac{\log 2}{v\log x}\right)$$

that $v' \geqslant 0.939v$, whence we deduce from (17) that

$$q \leqslant 2803. \tag{18}$$

Now, we apply the result of Saradha and Shorey given in Lemma SS. We set $\beta = \max\{1, n/q\}$ and we get

$$x^{2vn-n-2\beta+1} \leqslant \left(\frac{n+3}{4}\right)^2\left(2 + \frac{4}{x}\right)^{n-1} q^{2nq/(q-1)}. \tag{19}$$

Assume first that $\beta = n/q$. By Theorem BHM, this is always true when $q \leqslant 29$ and we have $n \geqslant 17$. Further, $q = 3$ (resp. $q = 5$, $q = 7$) implies $n \geqslant 101$ (resp. $n \geqslant 31$, $n \geqslant 29$). Consequently, we get the upper estimates

$$\begin{aligned} &x \leqslant 57.6^{1/(2\nu-5/3)}, \quad \text{for } q = 3,\\ &x \leqslant 128.4^{1/(2\nu-7/5)}, \quad \text{for } q = 5,\\ &x \leqslant 216.6^{1/(2\nu-9/7)}, \quad \text{for } q = 7, \end{aligned} \tag{20}$$

and

$$x \leqslant 2.42^{1/(2\nu-1-2/q)} q^{20/(11(2\nu-1-2/q))}, \qquad \text{for } q \geqslant 11. \tag{21}$$

When $\beta \neq n/q$, then $\beta = 1$ and $q > n$, thus, by Theorem BHM, we have $n \geqslant 29$ and $q \geqslant 31$. We deduce from (19) the upper bound

$$x \leqslant 2.31^{1/(2\nu-30/29)} q^{31/(30(\nu-15/29))}, \tag{22}$$

and we see from (19), (20), (21) and (22) that for any $\nu > 5/6$ we obtain an upper estimate for x, as claimed in the theorem.

To illustrate this statement, we provide an explicit estimate when $\nu = 10/11$. From (20), we have $x \leqslant 4.2 \times 10^{11}$ (obtained for $q = 3$), which bound also follows from (18), (21) and (22), and our last claim is now proved. □

Proof of Theorem 8. Let (x_3^t, y, n, q) be a solution of (12) with $x_3 \in S_3$. By Theorem NL, we may assume that $q \geqslant 3$. Let $p \in S_1 \cup S_2$ be a prime divisor of x_3. There are integers a and b such that $p = 2^a 3^b + 1$. By our assumptions in the case $x_2 > 1$ together with Theorem I, we have $q \neq 3$. Hence q does not divide $p - 1$. This is also true when $x_2 = 1$. Since $y^q \equiv 1 \bmod p$, we deduce that $y \equiv 1 \bmod p$. Hence, one may apply Theorem 7 with $\nu \geqslant 10/11$, and we obtain the result claimed. □

Proof of Theorem 9. Sander ([25], Theorem 2) proved that $a_h(g)$ is irrational if and only if $g^{N_2-N_1} \neq (h^{tL} - 1)/(h^t - 1)$ for every integer $L \geqslant 1$, where t is given in the statement of the theorem. As noticed in [27], we have $(N_1, N_2) = (0, 1)$ or $N_2 - N_1 \geqslant 2$. To the first case corresponds the first condition in the statement of Theorem 9, and it is clear that if $g = 1 + h + \cdots + h^{L-1}$ for an integer $L \geqslant 2$, then $a_h(g)$ is rational. Now, we assume $N_2 - N_1 \geqslant 2$ and $L = 2$, *i.e.* $g^{N_2-N_1} = h^t + 1$. This means that $(g, h, N_2 - N_1, t)$ is a solution (x, y, m, n) of Catalan's equation $x^m - y^n = 1$. We have $t \geqslant 2$ by assumption of the theorem, and we observe that $(g, h, t) = (3, 2, 3)$ is excluded by $g \neq 1 + h + \cdots + h^{L-1}$.

For simplicity, write $N = N_1 - N_2$. Let p be a prime divisor of h of the form $2^a 3^b + 1$ and assume that $p \nmid g - 1$. Since

$$\frac{g^N - 1}{g - 1}(g - 1) = h^t,$$

https://doi.org/10.1023/A:1015825809661 Published online by Cambridge University Press

we obtain that p divides $g^N - 1$. By known results on Catalan's equation, the smallest prime factor of N is at least 10^5 (see for instance [22]), hence a contradiction: p must divide $g-1$. Write $h = h'h''$ with $h' \in S_1 \cup S_2$ and $h'' \leqslant h'^{1/10}$. By Theorem 2, we get the upper bound

$$t = v_{h'}(g^N - 1) \leqslant \frac{66.8}{(\log h')^3} \log \max\{g, h'\} \times \\ \times \left(\max\left\{ \log\left(\frac{N}{\log h'} + \frac{1}{\log h'}\right) + \log\log h' + 0.64, 4\log h' \right\}\right)^2. \tag{23}$$

Combining $g^N \leqslant 2h^t$ with (23), and using the lower estimates $h' > h^{10/11}$ and $h' > (10^4)^{10/11}$ (by Theorem BM), we get $N \leqslant 1070$, which leads to a contradiction with the main result of [22].

Consequently, we have $N_2 - N_1 \geqslant 2$ and $L \geqslant 3$, whence we deduce from Theorem 8 that there is no solution with $h^t \geqslant 4.2 \times 10^{11}$. □

Acknowledgement

I would like to thank the referees for their very careful reading and for their numerous remarks.

References

1. Baker, A.: Linear forms in the logarithms of algebraic numbers, *Mathematika* **12** (1966), 204–216.
2. Baker, A. and Coates, J.: Fractional parts of powers of rationals, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 269–279.
3. Baker, A. and Wüstholz, G.: Logarithmic forms and group varieties, *J. Reine Angew. Math.* **442** (1993), 19–62.
4. Bennett, M.: Fractional parts of powers of rational numbers, *Math. Proc. Cambridge Philos. Soc.* **114** (1993), 191–201.
5. Beukers, F.: Fractional parts of powers of rationals, *Math. Proc. Cambridge Philos. Soc.* **90** (1981), 13–20.
6. Bugeaud, Y.: Linear forms in p-adic logarithms and the Diophantine equation $(x^n-1)/(x-1) = y^q$, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), 373–381.
7. Bugeaud, Y., Hanrot, G. and Mignotte, M.: Sur l'équation diophantienne $(x^n-1)/(x-1) = y^q$, III, *Proc. London Math. Soc.* **84** (2002), 59–78.
8. Bugeaud, Y. and Laurent, M.: Minoration effective de la distance p-adique entre puissances de nombres algébriques, *J. Number Theory* **61** (1996), 311–342.
9. Bugeaud, Y. and Mignotte, M.: Sur l'équation diophantienne $(x^n-1)/(x-1) = y^q$, II, *CR Acad Sci.* Sér **I**, **328** (1999).
10. Bugeaud, Y. and Mignotte, M.: On the Diophantine equation $(x^n-1)/(x-1) = y^q$ with negative x, In: *Proc. Millenial Conference on Number Theory* held in Urbana-Champaign (to appear).
11. Bugeaud, Y., Mignotte, M. and Roy, Y.: On the Diophantine equation $(x^n-1)/(x-1) = y^q$, *Pacific J. Math.* **193** (2000), 257–268.
12. Corvaja, P.: Approximation diophantienne sur la droite, Thèse, Paris, 1995.
13. Dubitskas, A.: A lower bound for the quantity $\|(3/2)^k\|$, *Russian Math. Surveys* **45** (1990), 163–164.

14. Gelfond, A. O.: Sur la divisibilité de la différence des puissances de deux nombres entiers par une puissance d'un idéal premier, *Mat. Sb.* **7** (1940), 7–25.
15. Gelfond, A. O.: *Transcendental and Algebraic Numbers*, Dover, New York, 1960.
16. Habsieger, L.: Explicit lower bounds for $\|(3/2)^k\|$, Preprint.
17. Inkeri, K.: On the Diophantine equation $a(x^n - 1)/(x - 1) = y^m$, *Acta Arith.* **21** (1972), 299–311.
18. Laurent, M., Mignotte, M. and Nesterenko, Yu.: Formes linéaires en deux logarithmes et déterminants d'interpolation, *J. Number Theory* **55** (1995), 285–321.
19. Ljunggren, W.: Noen Setninger om ubestemte likninger av formen $(x^n - 1)/(x - 1) = y^q$, *Norsk. Mat. Tidsskr.* **25** (1943), 17–20.
20. Mahler, K.: On the fractional parts of the powers of a rational number: II, *Mathematika* **4** (1957), 122–124.
21. Matveev, E. M.: Explicit lower estimates for rational homogeneous forms in logarithms of algebraic numbers, *Izv. Akad. Nauk SSSR Ser. Mat.* **62** (1998), 81–136. English: *Izv. Math.* **62** (1998), 723–772.
22. Mignotte, M. and Roy, Y.: Catalan's equation has no new solution with either exponent less than 10651, *Experimental Math.* **4** (1995), 259–268.
23. Nagell, T.: Note sur l'équation indéterminée $(x^n - 1)/(x - 1) = y^q$, *Norsk. Mat. Tidsskr.* **2** (1920), 75–78.
24. Philippon, P.: Quelques remarques sur des questions d'approximation diophantienne, *Bull. Austral. Math. Soc.* **59** (1999), 323–334.
25. Sander, J.W.: Irrationality Criteria for Mahler's numbers, *J. Number Theory* **52** (1995), 145–156.
26. Schinzel, A.: On two theorems of Gelfond and some of their applications, *Acta Arith.* **13** (1967), 177–236.
27. Saradha, N. and Shorey, T.N.: The equation $(x^n - 1)/(x - 1) = y^q$ with x square, *Math. Proc. Cambridge Philos. Soc.* **125** (1999), 1–19.
28. Shorey, T.N.: Linear forms in the logarithms of algebraic numbers with small coefficients I, II, *J. Indian Math. Soc. (N.S.)* **38** (1974), 271–284; ibid. **38** (1974), 285–292.
29. Shorey, T.N. and Tijdeman, R.: New applications of Diophantine approximations to Diophantine equations, *Math. Scand.* **39** (1976), 5–18.
30. Waldschmidt, M.: Minorations de combinaisons linéaires de logarithmes de nombres algébriques, *Canad. J. Math.* **45** (1993), 176–224.
31. Yu, K.: p-adic logarithmic forms and group varieties, II, *Acta Arith.* **89** (1999), 337–378.