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Divisorial Contractions in Dimension Three which Contract Divisors to Compound A_1 Points

MASAYUKI KAWAKITA
Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo 153-8914, Japan. e-mail: kawakita@ms.u-tokyo.ac.jp

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Abstract. We deal with a divisorial contraction in dimension three which contracts its exceptional divisor to a compound A_1 point. We prove that any such contraction is obtained by a suitable weighted blow-up.

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Key words: divisorial contraction.

1. Introduction

The explicit description of divisorial contractions is a beautiful object in itself, and in dimension three it is one of the most important remaining problems. The aim of this paper is to continue the study of this, following my previous paper [Kwk01].

Let $f: (Y \supset E) \to (X \ni P)$ be a divisorial contraction in dimension three which contracts its exceptional divisor E to a point P. The theorem in [Kwk01] is that any such contraction to a smooth point P is obtained by a suitable weighted blow-up. In the proof of this theorem, a numerical game for types of singularities on Y and for dimensions of $\mathcal{O}_X/\mathcal{O}_X(-iE)$'s plays an essential role, and it also works even if P is a Gorenstein singularity. In this paper, we treat the case where P is a compound A_1 point, starting with this game, and prove the following theorem:

THEOREM 1.1 (= Theorem 2.5). *Let Y be a $\mathbb{Q}$-factorial normal variety of dimension three with only terminal singularities, and let $f: (Y \supset E) \to (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor E to a compound A_1 point P. Then f is a weighted blow-up. More precisely, under a suitable analytic identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$, f is one of the following weighted blow-ups:*

(1) General case: *f is the weighted blow-up with its weights* $\mathrm{wt}(x, y, z, w) = (s, 2t - s, t, 1)$, *where s, t are coprime positive integers such that $s \leqslant t \leqslant N/2$.*
(2) Exceptional case: *$N = 3$ and f is the weighted blow-up with its weights* $\mathrm{wt}(x, y, z, w) = (1, 5, 3, 2)$.

Compound A_1 singularities are among the mildest of singularities, except for smooth points, allowed for P and, so there are many ways of obtaining a natural local description of X at P in our case. This makes it difficult to analyze divisorial contractions to X because many contractions may possibly occur.

The hardest part of the theorem is found in the general case. Adding that there exist infinitely many such weighted blow-ups from the choice of an analytic identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$, some difficulties arise in controlling the value of N, which should be large compared to the discrepancy of f. For this, we introduce a special surface $P \in S \subset X$ (Definition 6.5) and reduce the problem to constructing a special surface of which the strict transform on Y has only relatively mild singularities.

Y. Kawamata has produced a description in the case where P is a terminal quotient singularity ([Kwm96]), and A. Corti has described the case where P is an ordinary double point ([Co00, Theorem 3.10]), a special case of Theorem 1.1. He, with M. Mella, has also treated the case where P is analytically isomorphic to $o \in (xy + z^3 + w^3 = 0) \subset \mathbb{C}^4$ ([CM00, Theorem 3.6]). Though every case of the above admits only one or two divisorial contractions, respectively, in this paper we can also see the essence of their proof, comparing discrepancies and using Shokurov's connectedness lemma.

Recently, a study of Mori fiber spaces using the Sarkisov program was made (for example, [Co00], [CPR00] and [CM00]), which requires a precise description of divisorial contractions. I am convinced that Theorem 1.1 can be used to tackle Conjecture 1.2 of [CM00].

2. Statement of Theorem

We will work over the complex number field $\mathbb{C}$. A variety means an irreducible, reduced, separated scheme of finite type over $\operatorname{Spec}\mathbb{C}$. Though our objects are algebraic in themselves and we work in the algebraic category throughout the paper, we often use analytic functions for convenience. This produces no problem when adding higher terms to them if necessary to put them into algebraic functions. Our argument does not depend on the local ring $\mathcal{O}_{X,P}$ itself, but only on a quotient $\mathcal{O}_{X,P}/\mathfrak{m}_P^n$ by a sufficiently large multiple of the maximal ideal $\mathfrak{m}_P \subset \mathcal{O}_{X,P}$. We use basic terminologies in [K$^+$92, Chapters 1, 2].

First we define a divisorial contraction. Here it means a morphism which may emerge in the minimal model program.

DEFINITION 2.1 Let $f\colon Y \to X$ be a morphism with connected fibers between normal varieties. We call f a *divisorial contraction* if it satisfies the following conditions:

(1) Y is $\mathbb{Q}$-factorial with only terminal singularities.
(2) The exceptional locus of f is a prime divisor.

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(3) $-K_Y$ is f-ample.
(4) The relative Picard number of f is one.

We recall the classification of terminal Gorenstein singularities in dimension three.

DEFINITION 2.2. Let $P \in X$ be an algebraic germ (resp. an analytic germ) of a variety (resp. an analytic space) of dimension three. We call P a *cDV* (*compound Du Val*) *point* if a general hyperplane section is normal and has a Du Val singularity at P. The singularity P is said to be cA_n, cD_n, cE_n (*compound* A_n, D_n, E_n) according to the type of Du Val singularity on a general hyperplane section.

THEOREM 2.3 ([R83, Theorem 1.1]). *Let $P \in X$ be an algebraic germ (resp. an analytic germ) of a normal variety (resp. analytic space) of dimension three. Then P is a terminal Gorenstein singularity if and only if P is an isolated cDV point.*

Remark 2.4. (1) Let $P \in X \cong o \in (f = 0) \subset \mathbb{C}^4$ be a terminal Gorenstein singularity in dimension three. We can divide such singularities by rank r of the Hessian matrix of f at o.

– $r = 1$. P is cD_n, cE_6, cE_7, or cE_8.
– $r = 2$. P is cA_n with $n \geqslant 2$.
– $r = 3$. P is cA_1, but is not an ordinary double point.
– $r = 4$. P is an ordinary double point.

(2) If P is an isolated cA_1 point, we have an analytic identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ for some $N \geqslant 2$. This N depends only on $P \in X$ itself.

It is now time to state the theorem precisely.

THEOREM 2.5. *Let Y be a $\mathbb{Q}$-factorial normal variety of dimension three with only terminal singularities, and let $f: (Y \supset E) \to (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor E to a compound A_1 point P. Then f is a weighted blow-up. More precisely, under a suitable analytic identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$, f is one of the following weighted blow-ups.*

(1) General case: *f is the weighted blow-up with its weights* $\mathrm{wt}(x, y, z, w) = (s, 2t - s, t, 1)$*, where s, t are coprime positive integers such that $s \leqslant t \leqslant N/2$.*
(2) Exceptional case: *$N = 3$ and f is the weighted blow-up with its weights* $\mathrm{wt}(x, y, z, w) = (1, 5, 3, 2)$.

Remark 2.6. Consider an analytic germ of a cA_1 point $o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ $(N \geqslant 2)$ and blow-up this with weights as in Theorem 2.5. Then the exceptional locus of this weighted blow-up is irreducible, and the weighted blown-up analytic space has actually only terminal singularities.

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As in [Kwk01], in our argument we often identify prime divisors on different varieties if they are the same as valuations.

NOTATION 2.7. Let X be a normal variety and let E be an algebraic valuation, that is, a valuation of the function field of X which is obtained as an exceptional divisor of some birational morphism $f\colon Y \to X$ from a normal variety Y. Let D be a $\mathbb{Q}$-divisor on X or $D = \mu\mathsf{M}$, where μ is a rational number and M is a linear system of finite dimension on X which has no base points of codimension one.

(1) Let Z be a normal variety which is birational to X. D_Z denotes the strict transform of D on Z.
(2) $\mathcal{O}_X(iE)$ $(i \in \mathbb{Z})$ denotes $f_*\mathcal{O}_Y(iE)$.
(3) Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. $\alpha_{K_X+D}(E)$ denotes the discrepancy of E with respect to $K_X + D$, that is, $K_Y = f^*(K_X + D) + \alpha_{K_X+D}(E)E +$ (others).
(4) Assume that D is $\mathbb{Q}$-Cartier. $m_D(E)$ denotes the multiplicity of E with respect to D, that is, $f^*D = D_Y + m_D(E)E +$ (others).

3. Review of Singular Riemann–Roch Technique

In this section we review some of the numerical results in [Kwk01, § 4] obtained by using the singular Riemann–Roch formula. Let Y be a $\mathbb{Q}$-factorial normal variety of dimension three with only terminal singularities, and let $f\colon (Y \supset E) \to (X \ni P)$ be an algebraic germ of a divisorial contraction which contracts its exceptional divisor E to a Gorenstein point P. We fix this situation throughout this section.

Let $K_Y = f^*K_X + aE$ and let r be the global Gorenstein index of Y, that is, the smallest positive integer such that rK_Y is Cartier. Because a and r are coprime by [Kwk01, Lemma 4.3], we can take an integer e such that $ae \equiv 1$ modulo r.

Let

$$I = \left\{ Q : \text{type}\, \frac{1}{r_Q}(1, -1, b_Q) \right\}$$

be the set of fictitious singularities of Y, that is, terminal quotient singularities obtained by flat deformations of non-Gorenstein singularities of Y. Then $(\mathcal{O}_{Y_Q}(E_Q))_Q \cong (\mathcal{O}_{Y_Q}(eK_{Y_Q}))_Q$, where (Y_Q, E_Q) is the deformed pair near Q from (Y, E). We note that b_Q is coprime to r_Q and that e is also coprime to r_Q because r divides $ae - 1$. Hence, $v_Q = \overline{eb_Q}$ is coprime to r_Q. Here $\bar{}$ denotes the smallest residue modulo r_Q, that is, $\bar{j} = j - \lfloor j/r_Q \rfloor r_Q$, where $\lfloor\ \rfloor$ denotes the round down, that is, $\lfloor j \rfloor = \max\{k \in \mathbb{Z} | k \leqslant j\}$. Replacing b_Q with $r_Q - b_Q$ if necessary, we may assume that $v_Q \leqslant r_Q/2$. With this description, r is one if I is empty, and otherwise r is the lowest common multiple of $\{r_Q\}_{Q \in I}$. We define $J = \{(r_Q, v_Q)\}_{Q \in I}$. Moreover, we set

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$$d(i) = \dim_{\mathbb{C}} \mathcal{O}_X(iE)/\mathcal{O}_X((i-1)E) \quad (i \in \mathbb{Z}),$$
$$D(i) = \sum_{0 \leqslant j < i} d(-j) = \dim_{\mathbb{C}} \mathcal{O}_X/\mathcal{O}_X(-iE) \quad (i \geqslant 0).$$

We note that $d(i) = 0$ if $i \geqslant 1$, and that $d(0) = 1$.

PROPOSITION 3.1 ([Kwk01, Proposition 4.4]).

(1) $rE^3 \in \mathbb{Z}_{>0}$.
(2) $aE^3 = 2 - \sum_{Q \in I} \dfrac{v_Q(r_Q - v_Q)}{r_Q}$.
(3) $D(i) = i^2 + B_i - i^2 B_1$ $(0 \leqslant i \leqslant a)$, *where* $B_i = \sum_{Q \in I} \dfrac{\overline{iv_Q}(r_Q - \overline{iv_Q})}{2r_Q}$.
(4) *If* $a \geqslant 2$, *then* $D(2) = 4 - \sum_{Q \in I} v_Q \in \{1, 2, 3, 4\}$.

THEOREM 3.2 ([Kwk01, Theorem 4.5]). *Assume that* $a \geqslant 2$. *Then according the value of* $D(2)$, *exactly one of the cases given in Table I holds.*

Set $r_1 = 1$, $r_2 = r$ *in the case III, and consider the cases II-b and III. In these cases we have*

$$D(i)\begin{cases} = N_i = i & (0 \leqslant i \leqslant \min\{r_1, a\}) \\ = N_i > i & (\min\{r_1, a\} < i \leqslant \min\{r_2, a\}) \\ > N_i > i & (\min\{r_2, a\} < i \leqslant a), \end{cases}$$

where N_i *is the number of elements in the set*

$$I_i = \{(s, t) \in \mathbb{Z}^2_{\geqslant 0} | s + r_1 t < i\}.$$

4. First Step to Proof

In this section we take the first step to the proof of Theorem 2.5. We keep numerical data in Section 3.

- $K_Y = f^* K_X + aE$,
- r : the Gorenstein index,
- e : an integer such that $ae \equiv 1$ modulo r,
- $I = \{Q : \text{type } \frac{1}{r_Q}(1, -1, b_Q)\}$: the set of fictitious singularities of Y,

Table I.

Case	$D(2)$	J	a
I	1	$\{(7, 3)\}$ or $\{(3, 1), (5, 2)\}$	2
II-a	2	$\{(r, 2)\}$	2 or 4
II-b	2	$\{(r_1, 1), (r_2, 1)\}$ $(r_1 \leq r_2)$	$(r_1 + r_2)/r_1 r_2 E^3$
III	3	$\{(r, 1)\}$	$(1 + r)/rE^3$
IV	4	$\emptyset$	2
		In this case f is the blow-up along a smooth point P.	

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- $v_Q = \overline{eb_Q}$ $(v_Q \leqslant r_Q/2)$,
- $J = \{(r_Q, v_Q)\}_{Q\in I}$,
- $d(i) = \dim_{\mathbb{C}} \mathcal{O}_X(iE)/\mathcal{O}_X((i-1)E) \quad (i \in \mathbb{Z})$,
- $D(i) = \dim_{\mathbb{C}} \mathcal{O}_X/\mathcal{O}_X(-iE) \quad (i \geqslant 0)$.

Additionally, we define an integer $N \geqslant 2$ as

$$P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4.$$

First we construct a tower of normal varieties.

CONSTRUCTION 4.1. We construct birational morphisms $g_i\colon X_i \to X_{i-1}$ between normal factorial varieties, closed subvarieties $Z_i \subset X_i$, and prime divisors F_i on X_i inductively, and define positive integers n, m, with the following procedure.

(1) Define X_0 as X and Z_0 as P.
(2) (a) If Z_{i-1} is a point, we define g_i as the blow-up of X_{i-1} along Z_{i-1}.
  (b) If Z_{i-1} is a curve, we define $b_i\colon \mathrm{Bl}_{Z_{i-1}}(X_{i-1}) \to X_{i-1}$ as the blow-up of X_{i-1} along Z_{i-1}, and define $b'_i\colon X_i \to \mathrm{Bl}_{Z_{i-1}}(X_{i-1})$ as a resolution of singularities near $b_i^{-1}(Z_{i-1})$. Precisely, b'_i is a proper morphism which is isomorphic over $\mathrm{Bl}_{Z_{i-1}}(X_{i-1})\backslash b_i^{-1}(Z_{i-1})$, and X_i is smooth near $(b_i \circ b'_i)^{-1}(Z_{i-1})$. We note that b'_i is isomorphic at the generic point of the center of E on $\mathrm{Bl}_{Z_{i-1}}(X_{i-1})$. We define $g_i = b_i \circ b'_i\colon X_i \to X_{i-1}$.
(3) Define Z_i as the center of E on X_i with the reduced induced closed subscheme structure, and F_i as the only g_i-exceptional prime divisor on X_i which contains Z_i.
(4) We stop this process when $Z_n = F_n$. This process must terminate after finite steps like [Kwk01, Construction 3.1] and thus we get the sequence $X_n \to \cdots \to X_0$.
(5) We define $m \leqslant n$ as the largest integer such that Z_{m-1} is a point.
(6) We define g_{ji} $(j \leqslant i)$ as the induced morphism from X_i to X_j.

Remark 4.2. $Z_i \subseteq F_i$ $(1 \leqslant i \leqslant n)$ is exactly one of the cases given in Table II.

Table II.

Case	i	Z_i	$Z_i \subseteq F_i$
P_1	$1 \leq i < m$	point	the vertex point $\in Q_0(\subset \mathbb{P}^3)$
P_2			a nonvertex point $\in Q_0(\subset \mathbb{P}^3)$
P_3			a point $\in Q(\subset \mathbb{P}^3)$
P_4			a point $\in \mathbb{P}^2$
C_1	$i = m < n$	curve	a curve$\subset Q_0(\subset \mathbb{P}^3)$
C_2			a curve $\subset Q(\subset \mathbb{P}^3)$
C_3			a curve $\subset \mathbb{P}^2$
C'	$m < i < n$		a curve $\subset F_i$
S	$i = n$	surface	the surface $= F_i$

Q_0 (resp. Q) in Table II denotes the cone $(xy + z^2 = 0)$ (resp. the smooth quadratic $(xy + zw = 0)) \subset \mathbb{P}^3$ with homogeneous coordinates x, y, z, w.

Remark 4.3. We remark that $\mathcal{O}_X(-iE) = \mathcal{O}_X(-iF_n)$ for any i because E and F_n are the same as valuations.

From the next lemma, we have only to prove that F_n equals, as valuations, the only exceptional divisor obtained by a weighted blow-up of X emerging in Theorem 2.5.

LEMMA 4.4 ([Kwk01, Lemma 3.4]). *Let $f_i\colon Y_i \to X$ with $i = 1, 2$ be projective birational morphisms between normal varieties. Assume that E_i, the exceptional locus of f_i, is an anti-f_i-ample prime divisor for each i, and that E_1 and E_2 are the same as valuations.*

Second we evaluate various discrepancies and multiplicities.

NOTATION 4.5. (1) We define a positive integer $l \leqslant m$ as the largest integer satisfying that $l = 1$ or that Z_{l-1} is of type P_1.

(2) For curves Z_i $(m \leqslant i < n)$, we define the degree d_i of Z_i as follows.

(a) In cases C_1 and C_2, d_i denotes the degree of Z_i considered as a subvariety in P^3 as in Remark 4.2.

(b) In case C_3, d_i denotes the degree of Z_i considered as a subvariety in P^2 as in Remark 4.2.

(c) In case C', d_i denotes the degree of the finite morphism $Z_i \to Z_{i-1}$.

NOTATION 4.6. Let M be a general f-very ample linear system of finite dimension on Y. We define positive rational numbers μ, c_i by the following equations.

$$K_Y + \mu\mathsf{M} = f^*(K_X + \mu\mathsf{M}_X),$$

$$g_{0n}^*(\mu\mathsf{M}_X) = \mu\mathsf{M}_{X_n} + \sum_{1 \leqslant i \leqslant n} c_i(g_{in}^* F_i) + (\text{others}).$$

Remark 4.7. (1) Because M is a general f-very ample linear system on Y, for any algebraic valuation G we have $\alpha_{K_X+\mu\mathsf{M}_X}(G) = \alpha_{K_Y}(G)$.

(2) Putting $G = F_i$ in (4.7.1), we obtain

$$\alpha_{K_X}(F_i) - \sum_{1 \leqslant j \leqslant i} c_j m_{F_j}(F_i) = \alpha_{K_Y}(F_i) \begin{cases} > 0 & (i < n) \\ = 0 & (i = n), \end{cases}$$

since Y has only terminal singularities.

We give an evaluation for c_i's.

PROPOSITION 4.8.

(1) $1 > c_1$ *except the case* $n = 1$.
(2) $c_n > \alpha_{K_{X_{n-1}}}(F_n)$ *except the case* $n = 1$.
(3) (a) *If* Z_i *is a point of type* P_1 *or* P_4, *then* $c_i \geqslant c_{i+1}$.
(b) *If* Z_i *is a point of type* P_2, *then* $2c_i \geqslant c_{i+1}$.
(c) *If* Z_i *is a curve of type* C_3 *or* C', *then* $c_i \geqslant d_i c_{i+1}$.
(d) *If* Z_i *is a curve of type* C_1, *then* $2c_i \geqslant d_i c_{i+1}$.
(4) *If* Z_i *is of type* P_3 *or* C_2, *then* $c_i \geqslant 1$.

Proof. (1) Putting $i = 1$ into Remark 4.7.2, we have $1 - c_1 > 0$.

(2) We use Remark 4.7. Because

$$\begin{aligned}
&K_{X_n} + \mu \mathsf{M}_{X_n} \\
&\quad = g_n^*(K_{X_{n-1}} + \mu \mathsf{M}_{X_{n-1}}) + (\alpha_{K_{X_{n-1}}}(F_n) - c_n)F_n + (\text{others}) \\
&\quad = g_n^*(g_{0,n-1}^*(K_X + \mu \mathsf{M}_X) + \alpha_{K_Y}(F_{n-1})F_{n-1} + (\text{others})) + \\
&\qquad + (\alpha_{K_{X_{n-1}}}(F_n) - c_n)F_n + (\text{others}) \\
&\quad = g_{0n}^*(K_X + \mu \mathsf{M}_X) + \\
&\qquad + (\alpha_{K_{X_{n-1}}}(F_n) - c_n + \alpha_{K_Y}(F_{n-1})m_{F_{n-1}}(F_n))F_n + (\text{others}),
\end{aligned}$$

we have

$$\alpha_{K_{X_{n-1}}}(F_n) - c_n + \alpha_{K_Y}(F_{n-1})m_{F_{n-1}}(F_n) = \alpha_{K_Y}(F_n) = 0.$$

Hence

$$c_n - \alpha_{K_{X_{n-1}}}(F_n) = \alpha_{K_Y}(F_{n-1})m_{F_{n-1}}(F_n) > 0.$$

(3a) We will prove (3) with the same idea. Let l be a general line on $F_i \cong Q_0 \subset \mathbb{P}^3$ or $\cong \mathbb{P}^2$ through Z_i, and let l' be its strict transform on X_{i+1}. Then,

$$0 \leqslant (\mu \mathsf{M}_{X_{i+1}} \cdot l')_{X_{i+1}} = -c_{i+1}(F_{i+1} \cdot l')_{X_{i+1}} - c_i(F_i \cdot l)_{X_i} = -c_{i+1} + c_i.$$

(3b) Let c be a general conic on $F_i \cong Q_0 \subset \mathbb{P}^3$ through Z_i, and let c' be its strict transform on X_{i+1}. Then,

$$0 \leqslant (\mu \mathsf{M}_{X_{i+1}} \cdot c')_{X_{i+1}} = -c_{i+1}(F_{i+1} \cdot c')_{X_{i+1}} - c_i(F_i \cdot c)_{X_i} = -c_{i+1} + 2c_i.$$

(3c) Let l be a general line on $F_i \cong \mathbb{P}^2$ in case C_3 and a general fiber of $F_i \to Z_{i-1}$ in case C', and let l' be its strict transform on X_{i+1}. Then,

$$0 \leqslant (\mu \mathsf{M}_{X_{i+1}} \cdot l')_{X_{i+1}} = -c_{i+1}(F_{i+1} \cdot l')_{X_{i+1}} - c_i(F_i \cdot l)_{X_i} = -d_i c_{i+1} + c_i.$$

(3d) Let c be a general conic on $F_i \cong Q_0 \subset \mathbb{P}^3$, and let c' be its strict transform on X_{i+1}. Then,

$$0 \leqslant (\mu \mathsf{M}_{X_{i+1}} \cdot c')_{X_{i+1}} = -c_{i+1}(F_{i+1} \cdot c')_{X_{i+1}} - c_i(F_i \cdot c)_{X_i} = -d_i c_{i+1} + 2c_i.$$

(4) Our proof is a generalization of the proof of [Co00, Theorem 3.10] using Shokurov's connectedness lemma ([K+92, Theorem 17.4]). Let H be a general hyperplane section on X through P, and let L be a general hyperplane section on X_{i-1} through Z_{i-1} such that $Z_i \not\subseteq L_{X_i} \cap F_i$, and that $L_{X_i} \cap F_i$ consists of two lines $l_1 + l_2$ on $F_i \cong Q \subset \mathbb{P}^3$, which are fibers of two rulings of $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then

$$\begin{aligned} &g_{0i}^*(K_X + \mu \mathsf{M}_X + \alpha_{K_Y}(F_{i-1})H) + g_i^* L \\ &\quad = K_{X_i} + \mu \mathsf{M}_{X_i} + L_{X_i} + 0F_{i-1\,X_i} + c_i F_i + (\text{others}), \end{aligned}$$

where we omit the term $\alpha_{K_Y}(F_{i-1})H$ if $i = 1$. Because

$$\begin{aligned} &\alpha_{(g_{0i}^*(K_X+\mu\mathsf{M}_X+\alpha_{K_Y}(F_{i-1})H)+g_i^*L)}(F_n) \\ &\quad = -m_{(\alpha_{K_Y}(F_{i-1})g_{0i}^*H+g_i^*L)}(F_n) \\ &\quad \leqslant -m_L(F_n) = -1, \end{aligned}$$

we have

$$Z_i \subseteq \mathrm{LLC}(X_i,\ g_{0i}^*(K_X + \mu\mathsf{M}_X + \alpha_{K_Y}(F_{i-1})H) + g_i^* L),$$

where LLC denotes the locus of log canonical singularities for a log pair, that is, the union of centers of all algebraic valuations with discrepancies $\leqslant -1$. Moreover,

$$L_{X_i} \cap F_i \subseteq \mathrm{LLC}(X_i,\ g_{0i}^*(K_X + \mu\mathsf{M}_X + \alpha_{K_Y}(F_{i-1})H) + g_i^* L).$$

Since $Z_i \not\subseteq L_{X_i} \cap F_i \cong l_1 + l_2$, using the connectedness lemma for two small contractions in the analytic category contracting l_1, l_2 respectively, we obtain

$$\mathbb{P}^1 \times \mathbb{P}^1 \cong F_i \subseteq \mathrm{LLC}(X_i,\ g_{0i}^*(K_X + \mu\mathsf{M}_X + \alpha_{K_Y}(F_{i-1})H) + g_i^* L),$$

that is, $c_i \geqslant 1$. □

We have a refined restriction as a corollary of preceding results.

COROLLARY 4.9. (1) *If $a = 1$, then f is the usual blow-up of X along P.*

(2) *Assume that $a \geqslant 2$, that is, $n \geqslant 2$. Then,*

- (a) *Case I never occurs.*
- (b) *Neither case P_3 nor case C_2 occurs.*
- (c) *Exactly one of cases P_2 and C_1 occurs.*
- (d) *$m < n$.*
- (e) *$\forall d_i = 1$.*
- (f) *$2 > 2c_1 \geqslant \cdots \geqslant 2c_l \geqslant c_{l+1} \geqslant \cdots \geqslant c_n > 1$.*

Proof. (1) This comes from Lemma 4.4.

(2a) Since $a = 2$ in case I, we have $n = 2$ and

- Z_1 is a point of type P_1 and $N \geqslant 4$, or
- Z_1 is a curve.

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In both cases, a general hyperplane section on X through P has multiplicity one along F_2, which means that $\mathcal{O}_X(-2E) = \mathcal{O}_X(-2F_2) \subsetneq \mathfrak{m}_P$. This is a contradiction.

(2b) Propositions 4.8.1, 4.8.3a, and 4.8.4 imply this.

(2c) If neither case P_2 nor case C_1 occurs, then from Proposition 4.8 we have

$$1 > c_1 \geqslant \left(\prod_{m \leqslant i < n} d_i\right) c_n > \left(\prod_{m \leqslant i < n} d_i\right) \alpha_{K_{X_{n-1}}}(F_n).$$

This is a contradiction.

(2d–f) We obtain them by considering the following inequalities as in the proof of (2c)

$$2 > 2c_1 \geqslant \left(\prod_{m \leqslant i < n} d_i\right) c_n > \left(\prod_{m \leqslant i < n} d_i\right) \alpha_{K_{X_{n-1}}}(F_n).$$

$m < n$ comes from $\alpha_{K_{X_{n-1}}}(F_n) = 1$ and 2c. □

Remark 4.10. Because Corollaries 4.9.2c and 2e, $F_l \cong Q_0$ and $N \geqslant 2l + 1$ if $n \geqslant 2$. We define l_0 as the unique line on $F_l \cong Q_0 \subset \mathbb{P}^3$ containing Z_l.

The problem is reduced to investigating cases II-a, II-b, and III, which will be done in the following sections. As the final part of this section, we give some information for these remaining cases.

COROLLARY 4.11. (1) $Z_{i+1} \not\subseteq F_{iX_{i+1}} \cap F_{i+1}$.

(2) $a = n + m - l$.

(3) (a) *In cases* II-a *and* II-b, $Z_1 \subset F_1 \cong Q_0$ *in* $\mathbb{P}^3$ *and it is a point*.

(b) *In case* III, $Z_1 \subset F_1 \cong Q_0$ *in* $\mathbb{P}^3$ *and it is a line*.

Proof. (1) This is trivial since $\mathfrak{m}_P \neq \mathcal{O}_X(-2E) = \mathcal{O}_X(-2F_n)$.

(2) This comes from 1 and Corollary 4.9.2c.

(3) $F_1 \cong Q_0$ comes from $a \geqslant 2$ and Corollary 4.9.2b. We know the shape of $Z_1 \subset F_1 \cong Q_0 \in \mathbb{P}^3$ from the equation below.

$$\begin{aligned} 5 - D(2) &= \dim_{\mathbb{C}} \mathcal{O}_X(-2E)/\mathfrak{m}_P^2 \\ &= \dim_{\mathbb{C}} \mathrm{Im}\,[(v \in \mathfrak{m}_P | Z_1 \subseteq \mathrm{div}(v)_{X_1}) \to \mathfrak{m}_P/\mathfrak{m}_P^2] \\ &= \dim_{\mathbb{C}} \{v \in \Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) | v = 0 \text{ or } \mathrm{Z}_1 \subseteq \mathrm{div}(v)\}, \end{aligned}$$

where the second equality comes from $\mathfrak{m}_P \neq \mathcal{O}_X(-2E)$. □

5. Exceptional Case

In this section, we treat the exceptional case, which corresponds to case II-a, and our aim is the following.

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PROPOSITION 5.1. *Assume that f is of type* II-a. *Then f is a weighted blow-up of exceptional type.*

Throughout this section we assume that f is of type II-a and struggle with Proposition 5.1. We note that $1 < m < n$ by the assumption and Corollaries 4.9.2d and 4.11.3a, and that $N \geqslant 3$ by Remark 4.10.

First we restate the conclusion.

LEMMA 5.2. *The following imply Proposition* 5.1.

(1) $(n, m, l) = (3, 2, 1)$.
(2) *Z_2 is a curve which intersects the strict transform of l_0 on X_2.*

Proof. Though analytic functions seem to emerge in this proof, we stay in the algebraic category by adding higher terms to them if necessary, as we have said in the first paragraph in Section 2. First we prove a claim on an analytic description.

CLAIM 5.3. *There exists an identification*

$$P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$$

satisfying the following conditions.

(1) $l_0 = F_1 \cap \operatorname{div}(y)_{X_1} \cap \operatorname{div}(z)_{X_1}$.
(2) $Z_1 = l_0 \cap \operatorname{div}(w)_{X_1}$.
(3) $Z_2 = F_2 \cap \operatorname{div}(z)_{X_2}$.

Proof. It is trivial that we can choose an identification satisfying 1. Then by 1, $Z_1 = l_0 \cap \operatorname{div}(w + tx)_{X_1}$ for some $t \in \mathbb{C}$. Because $xy + z^2 + w^N = xy' + z^2 + (w')^N$ for $w' = w + tx$ and $y' = y + (w^N - (w + tx)^N)/x$, by replacing y, w with y', w' we may assume (2) moreover. Then $Z_2 = F_2 \cap \operatorname{div}(z + tx^2)_{X_2}$ for some $t \in C$ by 5.2.2 and Corollaries 4.9.2e and 4.11.1. Because $xy + z^2 + w^N = xy' + (z')^2 + w^N$ for $z' = z + tx^2$ and $y' = y - 2txz - t^2x^3$, by replacing y, z with y', z' we may assume (3) moreover. □

Second we prove that F_3 equals, as valuations, an exceptional divisor obtained by a weighted blow-up of X.

CLAIM 5.4. *Under the identification in Claim* 5.3, *F_3 equals, as valuations, an exceptional divisor obtained by the weighted blow-up of X with its weights* $\mathrm{wt}(x,y,z,w) = (1, 5, 3, 2)$.

Proof. First we remark that $x, z/x, w/x \in \mathcal{O}_{X_1,Z_1}$ generate local coordinates of X_1 at Z_1, that $y/x = -((z/x)^2 + x^{N-2}(w/x)^N)$, and that F_3 equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X_1 with its weights $\mathrm{wt}(x, z/x, w/x) = (1, 2, 1)$. Thus we obtain

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$$(m_{\mathrm{div}}(x)(F_3), m_{\mathrm{div}}(y)(F_3), m_{\mathrm{div}}(z)(F_3), m_{\mathrm{div}}(w)(F_3)) = (1, 5, 3, 2).$$

Since any $v \in \mathcal{O}_{X,P}$ has an expansion of a formal series $v = v_1(x, z, w) + v_2(y, z, w)$, it is sufficient to prove that for any $i \geqslant 0$,

$$v = \sum_{(p,q,r,s)\in I_i} c_{pqrs} x^p y^q z^r w^s \in \mathcal{O}_X(-(i+1)F_3) \quad (c_{pqrs} \in \mathbb{C})$$

implies $v = 0$, where

$$I_i = \{(p, q, r, s) \in \mathbb{Z}^4_{\geqslant 0} | p + 5q + 3r + 2s = i,\ p \text{ or } q = 0\}.$$

However, by replacing v with $x^j v$ for a sufficiently large j, we have only to show that for any $i \geqslant 0$,

$$v = \sum_{(p,q,r)\in J_i} c_{pqr} x^p z^q w^r \in \mathcal{O}_X(-(i+1)F_3) \quad (c_{pqr} \in \mathbb{C})$$

implies $v = 0$, where $J_i = \{(p, q, r) \in \mathbb{Z}^3_{\geqslant 0} | p + 3q + 2r = i\}$.

Take any $v = \sum_{(p,q,r)\in J_i} c_{pqr} x^p z^q w^r$ contained in $\mathcal{O}_X(-(i+1)F_3)$. Then $v = \sum_{(p,q,r)\in J_i} c_{pqr} x^{p+q+r} (z/x)^q (w/x)^r$. Because F_3 equals, as valuations, the exceptional divisor obtained by the weighted blow-up of X_1 with its weights $\mathrm{wt}(x, z/x, w/x) = (1, 2, 1)$, it is enough to show that the weight of any monomial $x^{p+q+r}(z/x)^q(w/x)^r$ ($(p, q, r) \in J_i$) with respect to its weights $\mathrm{wt}(x, z/x, w/x) = (1, 2, 1)$ equals i. But this is trivial by a direct calculation $(p + q + r) + 2q + r = p + 3q + 2r = i$. □

Only the proof of $N = 3$ remains. Because of Lemma 4.4 and properties of toric geometry, we have only to show the following claim.

CLAIM 5.5. *Consider an analytic germ of a cA_1 point $o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ ($N \geqslant 4$) and blow-up this with its weights $\mathrm{wt}(x,y,z,w) = (1,5,3,2)$. Then the exceptional locus of this weighted blow-up is irreducible, and the weighted blown-up analytic space is normal and has a nonterminal singularity.*

Proof. Direct calculation shows that its exceptional locus is isomorphic to $(xy + z^2 = 0) \subset \mathbb{P}(1, 5, 3, 2)$ with weighted homogeneous coordinates x, y, z, w, which is irreducible, and that all singularities on the obtained analytic space are one terminal quotient singularity of type $\frac{1}{5}(-1, 3, 2)$ and one nonterminal singularity isomorphic to $o \in (xy + z^2 + w^{2N-6} = 0) \subset \mathbb{C}^4/\mathbb{Z}_2(1, 1, 1, -1)$. □

Now our problem is proving Lemma 5.2.1-2, which will be shown in Lemmas 5.8.1 and 5.9. We show all the possible cases.

LEMMA 5.6. *$a = 4$, and the tower $X_n \to \cdots \to X_0$ is exactly one of the following.*

(1) $(n, m, l) = (3, 2, 1)$, $N \geqslant 3$, $r = 5$.
(2) $(n, m, l) = (4, 2, 2)$, $N \geqslant 5$, $r = 5$.
(3) $(n, m, l) = (4, 3, 3)$, $N \geqslant 7$, $r = 7$.

Proof. Though $a = 2$ or 4 in case II-a, $a = 2$ is impossible because $n \geqslant 3$. Hence $a = 4$. By Corollary 4.11.2, it is trivial that the values of n, m, l in (1)-(3) cover all the possibilities for $a = 4$ and $1 < m < n$.

Now we calculate the value of r in each case using Proposition 3.1.3. Because $a = 4$ and $J = \{(r, 2)\}$ $(r \geqslant 5)$, Proposition 3.1.3 implies that

$$D(3) = 3 + \max\{0, 6 - r\},$$
$$D(4) = 4 + \max\{0, 8 - r\}.$$

Thus we have only to the next claim.

CLAIM 5.7. (1) *In case 5.6.1,* $D(3) = 4$.
(2) *In case 5.6.2,* $D(3) = 4$.
(3) *In case 5.6.3,* $D(4) = 5$.

Proof. We will express $\mathcal{O}_X(-iE)$'s in each case under a suitable identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$.

(1) As in Claim 5.3, we may assume that

$$l_0 = F_1 \cap \operatorname{div}(y)_{X_1} \cap \operatorname{div}(z)_{X_1} \quad \text{and} \quad Z_1 = l_0 \cap \operatorname{div}(w)_{X_1}.$$

Then

$$\mathcal{O}_X(-2E) = (y, z, w) + \mathfrak{m}_P^2,$$
$$\mathcal{O}_X(-3E) = (v, y) + (z, w)\mathfrak{m}_P + \mathfrak{m}_P^3,$$

where $v = t_z z + t_w w + t_{x^2} x^2$ for some $t_z, t_w, t_{x^2} \in \mathbb{C}$ such that t_z or t_w is nonzero. This implies (1).

(2) We may assume that $l_0 = F_2 \cap \operatorname{div}(y)_{X_2} \cap \operatorname{div}(z)_{X_2}$. Then

$$\mathcal{O}_X(-2E) = (x, y, z) + \mathfrak{m}_P^2,$$
$$\mathcal{O}_X(-3E) = (y, z) + (x)\mathfrak{m}_P + \mathfrak{m}_P^3.$$

This implies (2).

(3) We may assume that $l_0 = F_3 \cap \operatorname{div}(y)_{X_3} \cap \operatorname{div}(z)_{X_3}$. Then

$$\mathcal{O}_X(-2E) = (x, y, z) + \mathfrak{m}_P^2,$$
$$\mathcal{O}_X(-3E) = (x, y, z) + \mathfrak{m}_P^3,$$
$$\mathcal{O}_X(-4E) = (y, z) + (x)\mathfrak{m}_P + \mathfrak{m}_P^4.$$

This implies (3). □

We exclude cases 5.6.2-3, which shows 5.2.1. Moreover, we determine the values of c_i's in case 5.6.1.

LEMMA 5.8. (1) *Neither case* 5.6.2 *nor case* 5.6.3 *occurs.*
(2) *In case* 5.6.1, $c_1 = 4/5$, $c_2 = 8/5$, $c_3 = 8/5$.

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Proof. We note that $m_E(F_i) \in \frac{1}{r}\mathbb{Z}$ for any i. Using Remark 4.7.2, for any i we have

$$\alpha_{K_X}(F_i) - \sum_{1 \leqslant j \leqslant i} c_j = \alpha_{K_Y}(F_i) = \alpha_{f^*K_X+4E}(F_i) = \alpha_{K_X}(F_i) - 4m_E(F_i).$$

Hence $\sum_{1 \leqslant j \leqslant i} c_j = 4m_E(F_i) \in \frac{4}{r}\mathbb{Z}$, and thus $\forall c_i \in \frac{4}{r}\mathbb{Z}$.

But on the other hand, c_i's satisfy the relations in Remark 4.7.2 and Corollary 4.9.2f. Using them we know that there is no possibility for such c_i's in cases 5.6.2-3, and that 5.8.2 is the only possibility in case 5.6.1. $\square$

Now it is sufficient to deal with only case 5.6.1. Lemma. 5.2.2 comes from the following lemma, and therefore we finish the proof of Proposition 5.1. Let l_0' be the strict transform of l_0 on X_2.

LEMMA 5.9. (1) *Let M_{F_1} be the linear system on $F_1 \cong Q_0$ obtained by the total pull-back of M_{X_1} with the inclusion map $F_1 \hookrightarrow X_1$. Then M_{F_1} is a zero-dimensional linear system consisting of some multiple of l_0.*

(2) *Let M_{F_2} be the linear system on $F_2 \cong \mathbb{P}^2$ obtained by the total pull-back of M_{X_2} with the inclusion map $F_2 \hookrightarrow X_2$. Then M_{F_2} is a zero-dimensional linear system consisting of some multiple of Z_2.*

Proof. (1) Let c be the multiplicity of M_{F_1} along l_0, and let l be a general line on $F_1 \cong Q_0 \subset \mathbb{P}^3$. Then,

$$c/2 = (cl_0 \cdot l)_{F_1} \leqslant (\mu\mathsf{M}_{F_1} \cdot l)_{F_1} = -c_1(F_1 \cdot l)_{X_1} = 4/5.$$

On the other hand,

$$\begin{aligned} -c/2 = (cl_0' \cdot l_0')_{F_{1X_2}} &\leqslant (\mu\mathsf{M}_{X_2} \cdot l_0')_{X_2} \\ &= -c_2(F_2 \cdot l_0')_{X_2} - c_1(F_1 \cdot l_0)_{X_1} \\ &= -c_2 + c_1 = -4/5. \end{aligned}$$

By these two inequalities, we obtain $c = 8/5$ and $(cl_0 \cdot l)_{F_1} = (\mu\mathsf{M}_{F_1} \cdot l)_{F_1}$. This shows (1).

(2) Because Corollary 4.9.2e tells that Z_2 is a line on $F_2 \cong \mathbb{P}^2$, we know that g_3 induces an isomorphism $F_{2X_3} \cong F_2 \cong \mathbb{P}^2$. Let $\mathsf{M}_{F_{2X_3}}$ be the linear system on $F_{2X_3} \cong \mathbb{P}^2$ obtained by the total pull-back of M_{X_3} with the inclusion map $F_{2X_3} \hookrightarrow X_3$. It is enough to prove that $\mathsf{M}_{F_{2X_3}} = \emptyset$.

Let l be a general line on $F_2 \cong \mathbb{P}^2$, and let l' be the strict transform of l on X_3. Then

$$(\mu\mathsf{M}_{F_{2X_3}} \cdot l')_{F_{2X_3}} = -c_3(F_3 \cdot l')_{X_3} - c_2(F_2 \cdot l)_{X_2} = -c_3 + c_2 = 0,$$

which shows that $\mathsf{M}_{F_{2X_3}} = \emptyset$. $\square$

6. General Case

In this section we treat the remaining general case, which corresponds to Cases II-b and III, and our aim is the following, which terminates the proof of Theorem 2.5.

PROPOSITION 6.1. *Assume that f is of type II-b or III. Then f is a weighted blow-up of general type.*

Throughout this section, except for Definition 6.5 and Proposition 6.6, we assume that f is of type II-b or III and struggle with Proposition 6.1. We set $(r_1, r_2) = (1, r)$ in case III in this section because we want to treat both cases II-b and III simultaneously.

First we restate the conclusion.

LEMMA 6.2. *The following imply Proposition 6.1.*

(1) $l = m$.
(2) $N \geqslant 2a$.
(3) *There exists an identification* $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ *satisfying that* $z \in \mathcal{O}_X(-aE)$.

Proof. We use the same idea as that in the proof of Lemma 5.2. First we note that $a = n$ by (1) and Corollary 4.11.2. By (3) we have an identification $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ satisfying that $z \in \mathcal{O}_X(-nE)$. Moreover, by (1) we may assume that

$$Z_m = F_m \cap \operatorname{div}(y)_{X_m} \cap \operatorname{div}(z)_{X_m} \subset F_m \cong Q_0 \subset \mathbb{P}^3.$$

We have

$$(m_{\operatorname{div}(x)}(F_n), m_{\operatorname{div}(z)}(F_n), m_{\operatorname{div}(w)}(F_n)) = (m, n, 1).$$

CLAIM 6.3. *Under the above identification, F_n equals, as valuations, an exceptional divisor obtained by the weighted blow-up of X with its weights* $\operatorname{wt}(x, y, z, w)$ $(m, 2n - m, n, 1)$.

Proof. First we remark that $z/w^m, w \in \mathcal{O}_{X_m, Z_m}$ generate local coordinates of X_m at the generic point of Z_m, that $x/w^m \in \mathcal{O}^{\times}_{X_m, Z_m}$, that $y/w^m = -(x/w^m)^{-1}((z/w^m)^2 + w^{N-2m})$, and that F_n equals, as valuations, the exceptional divisor dominating Z_m obtained by the weighted blow-up of X_m along Z_m with its weights $\operatorname{wt}(z^m/w, w) = (n - m, 1)$. Thus, we obtain

$$(m_{\operatorname{div}(x)}(F_n), m_{\operatorname{div}(y)}(F_n), m_{\operatorname{div}(z)}(F_n), m_{\operatorname{div}(w)}(F_n)) = (m, 2n - m, n, 1),$$

considering 6.2.2 also. Since any $v \in \mathcal{O}_{X,P}$ has an expansion of a formal series $v = v_1(x, z, w) + v_2(y, z, w)$, it is sufficient to prove that for any $i \geqslant 0$,

$$v = \sum_{(p,q,r,s) \in I_i} c_{pqrs} x^p y^q z^r w^s \in \mathcal{O}_X(-(i+1)F_n) \quad (c_{pqrs} \in \mathbb{C})$$

implies $v = 0$, where

$$I_i = \{(p, q, r, s) \in \mathbb{Z}^4_{\geqslant 0} | mp + (2n - m)q + nr + s = i, \ p \text{ or } q = 0\}.$$

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However, by replacing v with $x^j v$ for a sufficiently large j, we have only to show that for any $i \geqslant 0$,

$$v = \sum_{(p,q,r)\in J_i} c_{pqr} x^p z^q w^r \in \mathcal{O}_X(-(i+1)F_n) \quad (c_{pqr} \in \mathbb{C})$$

implies $v = 0$, where $J_i = \{(p, q, r) \in \mathbb{Z}^3_{\geqslant 0} | mp + nq + r = i\}$.

Take any $v = \sum_{(p,q,r)\in J_i} c_{pqr} x^p z^q w^r$ contained in $\mathcal{O}_X(-(i+1)F_n)$. Then

$$v = \sum_{(p,q,r)\in J_i} c_{pqr} (x/w^m)^p (z/w^m)^q w^{mp+mq+r}.$$

We remark that $x/w^m \in \mathcal{O}^\times_{X_m, Z_m}$. Because F_n equals, as valuations, the exceptional divisor dominating Z_m which is obtained by the weighted blow-up of X_m along Z_m with its weights $\mathrm{wt}(z/w^m, w) = (n-m, 1)$, it is enough to show that the weight of any monomial $(z/w^m)^q w^{mp+mq+r}$ $((p, q, r) \in J_i)$ with respect to its weights $\mathrm{wt}(z/w^m, w) = (n-m, 1)$ equals i. But this is trivial by a direct calculation $(n-m)q + (mp + mq + r) = mp + nq + r = i$. □

There remains only proving that m, n are coprime. Because of Lemma 4.4 and the properties of toric geometry, we have only to show the following claim:

CLAIM 6.4. *Consider an analytic germ of a cA_1 point $o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ $(N \geqslant 2n)$ and blow-up this with its weights $\mathrm{wt}(x, y, z, w) = (m, 2n-m, n, 1)$, where m, n are positive integers with $m < n$ and are not coprime. Then the exceptional locus of this weighted blow-up is irreducible, and the weighted blown-up analytic space is normal and has a nonterminal singularity.*

Proof. Direct calculation shows that its exceptional locus is isomorphic to $(xy + z^2 = 0)$ or $(xy + z^2 + w^{2n} = 0) \subset \mathbb{P}(m, 2n-m, n, 1)$ with weighted homogeneous coordinates x, y, z, w, which is irreducible, and that the obtained analytic space is singular along the line $(xy + z^2 = w = 0) \subset \mathbb{P}(m, 2n-m, n, 1)$. Normality is easy. □

Our problem is proving 6.2.1-3. For this we introduce one definition, which also makes sense in more general situation as in Section 3.

DEFINITION 6.5. An algebraic surface $P \in S \subset X$ is said to be *special of type s*, where s is a positive integer, if it satisfies the following conditions.

(1) S is normal and has a Du Val singularity of type A_s at P.
(2) $f^*S = S_Y + aE$.

A special surface has beautiful properties.

PROPOSITION 6.6. *Let $P \in S \subset X$ be a special surface of type s, and let f_S be the induced morphism from S_Y to S. Then S_Y is normal and $K_{S_Y} = f_S^* K_S$. Especially, the minimal resolution of S factors through S_Y.*

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Proof. It is sufficient to show that S_Y is normal and that $K_{S_Y} = f_S^* K_S$, because these imply the last part of the statement. We will prove them simultaneously.

Let $\nu\colon \widetilde{S_Y} \to S_Y$ be the normalization of S_Y. First we calculate the dualizing sheaf ω_{S_Y} on S_Y. Let $Y^o \subseteq Y$ be the Gorenstein locus of Y. We remark that $Y \setminus Y^o$ is a finite set. By the adjunction formula, we obtain that

$$\begin{aligned}\omega_{S_Y}|_{Y^o \cap S_Y} &= \omega_Y(S_Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_{S_Y}|_{Y^o \cap S_Y}\\ &= f_S^*(\omega_X(S) \otimes_{\mathcal{O}_X} \mathcal{O}_S)|_{Y^o \cap S_Y} = f_S^* \omega_S|_{Y^o \cap S_Y}.\end{aligned}$$

On the other hand, we know that ω_{S_Y} is (S_2), that $S_Y \setminus (Y^o \cap S_Y) \subseteq S_Y$ is of codimension greater than one, and that $f_S^* \omega_S$ is invertible. Thus we obtain $\omega_{S_Y} = f_S^* \omega_S$, and our problem is reduced to only proving that ν is isomorphism.

Second we calculate the dualizing sheaf $\omega_{\widetilde{S_Y}}$ on $\widetilde{S_Y}$. Grothendieck duality tells that

$$\begin{aligned}\omega_{\widetilde{S_Y}} &= \mathcal{H}om_{\mathcal{O}_{S_Y}}(\nu_* \mathcal{O}_{\widetilde{S_Y}}, \omega_{S_Y})\\ &= \mathcal{H}om_{\mathcal{O}_{S_Y}}(\nu_* \mathcal{O}_{\widetilde{S_Y}}, f_S^* \omega_S)\\ &= \mathcal{H}om_{\mathcal{O}_{S_Y}}(\nu_* \mathcal{O}_{\widetilde{S_Y}}, \mathcal{O}_{S_Y}) \otimes \mathcal{O}_{\widetilde{S_Y}} \nu^* f_S^* \omega_S,\end{aligned}$$

where the remark that ω_S is invertible induces the third equality.

Because S is canonical, the above equation shows that the conductor ideal sheaf $\mathcal{H}om_{\mathcal{O}_{S_Y}}(\nu_* \mathcal{O}_{\widetilde{S_Y}}, \mathcal{O}_{S_Y}) \subseteq \mathcal{O}_{\widetilde{S_Y}}$ has to equal $\mathcal{O}_{\widetilde{S_Y}}$. Hence ν is isomorphism. □

We come back to cases II-b and III treated in this section. In our situation, the type of any special surface must be higher.

LEMMA 6.7. *Let $P \in S \subset X$ be a special surface of type s. Then $s \geqslant r_1 + r_2 - 1$.*

Proof. First we give easy statements about a Du Val singularity of type A_s.

CLAIM 6.8. *Let $P \in S$ be an algebraic germ (resp. an analytic germ) of a Du Val singularity of type A_s ($s \geqslant 1$), let $f_S\colon (S_Y \supset E) \to (S \ni P)$ be a nonisomorphic partial resolution factored through by the minimal resolution of S, and let C be a general hyperplane section on S through P.*

(1) *C has its multiplicity one along every prime component of E, that is, $f_S^* C = C_{S_Y} + E$.*
(2) *The set $C_{S_Y} \cap E$ consists of two points, say Q_1, Q_2. These Q_1, Q_2 are Du Val singularities of types A_{s_1}, A_{s_2} with $s_1 + s_2 < s$ ($s_1, s_2 \geqslant 0$). Here we define a Du Val singularity of type A_0 as a smooth point.*
(3) *For $i = 1, 2$, the local intersection number $(C_{S_Y} \cdot E)_{S_Y, Q_i}$ equals $1/(s_i + 1)$.*

Proof. Let $f\colon (T \supset F) \to (S \ni P)$ be the minimal resolution of S, and let $g\colon T \to S_Y$ be the induced morphism. $F = \sum_{1 \leqslant i \leqslant s} F_i$ is a chain of (-2)-curves F_i's. We order the indices i's so that they are compatible with the order of F_i's in this chain. It is fundamental to see that $f^* C = C_T + F$ and that C_T intersects F exactly at a point,

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say P_1, on $F_1 \backslash F_2$ and at a point, say P_2, on $F_s \backslash F_{s-1}$ transversally, where we omit $\backslash F_2$ and $\backslash F_{s-1}$ if $s = 1$. Let s_1 (resp. s_2) be the smallest nonnegative integer such that F_{s_1+1} (resp. F_{s-s_2}) is not contracted by g. Then $Q_i = g(P_i)$ $(i = 1, 2)$ is a Du Val singularity of type s_i, and $C_{S_Y} \cap E$ consists of Q_1, Q_2. Because $g^*g(F_{s_1+1}) = (s_1 + 1)^{-1}F_1 +$ (others) (resp. $g^*g(F_{s-s_2}) = (s_2 + 1)^{-1}F_s +$ (others)), we have $(C_{S_Y} \cdot E)_{S_Y, Q_1} = 1/(s_1 + 1)$ (resp. $(C_{S_Y} \cdot E)_{S_Y, Q_2} = 1/(s_2 + 1)$). □

We begin to prove Lemma 6.7. We keep the notation $f_S : S_Y \to S$ in Proposition 6.6. Let H be a general hyperplane section on X through P. Then $P \in C = H|_S \subset S$ is also a general hyperplane section on S through P. Because $m_P \neq \mathcal{O}_X(-2E)$, we have $f^*H = H_Y + E$ and $f_S^*C = H_Y|_{S_Y} + E|_{S_Y}$. The support of $E|_{S_Y}$ is exactly the exceptional locus of f_S, and f_S is factored through by the minimal resolution of S by Proposition 6.6. Thus by Claim 6.8.1, we obtain that $E|_{S_Y}$ is reduced and that $H_Y|_{S_Y} = C_{S_Y}$, the strict transform of C on S_Y.

We calculate the intersection number of C_{S_Y} and $E|_{S_Y}$ around $f_S^{-1}(P)$.

$$\begin{aligned}(C_{S_Y} \cdot E|_{S_Y})_{S_Y} &= (H_Y \cdot E \cdot S_Y)_Y \\ &= ((f^*H - E) \cdot E \cdot (f^*S - aE))_Y \\ &= aE^3 = (1/r_1) + (1/r_2),\end{aligned}$$

where the last equality comes from Proposition 3.1.2.

By Claim 6.8.2, the set $C_{S_Y} \cap E|_{S_Y}$ consists of two points, say Q_1, Q_2, and thus

$$(C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_1} + (C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_2} = (1/r_1) + (1/r_2).$$

We may assume that

$$(C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_1} \geqslant (C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_2}.$$

Considering the set I and Claim 6.8.3, we know that

$$(C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_1} = 1/r_1 \quad \text{and} \quad (C_{S_Y} \cdot E|_{S_Y})_{S_Y, Q_2} = 1/r_2,$$

and that the local Gorenstein indices of Q_1, Q_2 are r_1, r_2. Therefore by Claims 6.8.2-3, we obtain that Q_1, Q_2 are Du Val singularities of types A_{r_1-1}, A_{r_2-1} with $(r_1 - 1) + (r_2 - 1) < s$, that is, $r_1 + r_2 \leqslant s + 1$. □

Remark 6.9. The above proof tells that Y has exactly two non-Gorenstein singularities in case II-b.

We obtain an upper-bound of the value of a.

LEMMA 6.10. $r_1 + r_2 \geqslant 2a$.

Proof. $a(r_1r_2E^3) = r_1 + r_2$ by Theorem 3.2. Thus we have only to show that $a \neq r_1 + r_2$ because of Proposition 3.1.1. $m_E(F_1) \in \mathbb{Z}$ (resp. $(1/r_1)\mathbb{Z}$, $(1/r_2)\mathbb{Z}$) when the center of F_1 on Y is not a non-Gorenstein point (resp. is the non-Gorenstein point of index r_1, is the non-Gorenstein point of index r_2). Like the proof of Lemma 5.8, we obtain $c_1 \in a\mathbb{Z}$ (resp. $a/r_1\mathbb{Z}$, $a/r_2\mathbb{Z}$). By this and Proposition 4.8.1 we have a (resp. a/r_1, a/r_2) < 1, which implies that $a \neq r_1 + r_2$. □

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Combining Lemmas 6.7 and 6.10, we obtain a corollary.

COROLLARY 6.11. *Let $P \in S \subset X$ be a special surface of type s. Then $s \geqslant 2a-1$.*

Now we will prove 6.2.1-3 by constructing special surfaces.

LEMMA 6.12. *There exists an identification $P \in X \cong o \in (xy+z^2+w^N=0) \subset \mathbb{C}^4$ satisfying that $m_{\mathrm{div}(w)}(E)=1$ and that $z+p(w) \in \mathcal{O}_X(-aE)$ for some $p(w) \in \bigoplus_{i=1}^{a-1} \mathbb{C}w^i \subset \mathbb{C}[w]$.*

Proof. We express $\mathcal{O}_X(-iE)$'s explicitly using the above claim. □

CLAIM 6.13. (1) *Take an identification $P \in X \cong o \in (xy+z^2+w^N=0) \subset \mathbb{C}^4$ satisfing that $m_{\mathrm{div(w)}}(E)=1$. Then for $1 \leqslant i \leqslant \min\{r_1, a\}$,*

$$\mathcal{O}_X(-iE) = (x_i, y_i, z_i) + (w^i)$$

for some $x_i = x + p_i^x(w)$, $y_i = y + p_i^y(w)$, $z_i = z + p_i^z(w)$ ($p_i^x(w)$, $p_i^y(w)$, $p_i^z(w) \in \bigoplus_{j=1}^{i-1} \mathbb{C}w^j \subset \mathbb{C}[w]$).

(2) *Assume $r_1 < a$.*

(a) *In 1, x_{r_1}, y_{r_1} or $x_{r_1} - y_{r_1} - 2z_{r_1} \notin \mathcal{O}_X(-(r_1+1)E) + (w^{r_1})$.*

(b) *In 1, assume that $x_{r_1} \notin \mathcal{O}_X(-(r_1+1)E) + (w^{r_1})$. Under this situation, for $r_1 \leqslant i \leqslant a$,*

$$\mathcal{O}_X(-iE) = (y_i, z_i) + \sum_{(j,k) \in \cup_{s \geqslant i} J_s} (x_{r_1}^j w^k)$$

for some $y_i = y + p_i^y(x_{r_1}, w)$, $z_i = z + p_i^z(x_{r_1}, w)$ ($p_i^y(x_{r_1}, w), p_i^z(x_{r_1}, w) \in \bigoplus_{s=1}^{i-1} \bigoplus_{(j,k) \in J_s} Cx_{r_1}^j w^k \subset \mathbb{C}[x_{r_1}, w]$), where

$$J_i = \{(s,t) \in \mathbb{Z}^2_{\geqslant 0} | r_1 s + t = i\}.$$

Proof. (1) We will construct x_i, y_i, z_i inductively starting with $x_1 = x$, $y_1 = y$, $z_1 = z$. Assume that we have constructed x_i, y_i, z_i ($1 \leqslant i < \min\{r_1, a\}$). There exists a surjective map λ_i,

$$\lambda_i : ((x_i, y_i, z_i) + (w^i))/(\mathfrak{m}_P(x_i, y_i, z_i) + (w^{i+1}))$$
$$\twoheadrightarrow \mathcal{O}_X(-iE)/\mathcal{O}_X(-(i+1)E).$$

By $i < \min\{r_1, a\}$ and Theorem 3.2, $d(-i) = D(i+1) - D(i) = 1$. Since $m_{\mathrm{div(w)}}(E) = 1$, we know that w^i generates $\mathcal{O}_X(-iE)/\mathcal{O}_X(-(i+1)E)$, and that $x_i + t_x w^i$, $y_i + t_y w^i$, $z_i + t_z w^i \in \mathrm{Ker}\,\lambda_\mathrm{i}$ for some $t_x, t_y, t_z \in \mathbb{C}$. Hence, it is enough to put $x_{i+1} = x_i + t_x w^i$, $y_{i+1} = y_i + t_y w^i$, $z_{i+1} = z_i + t_z w^i$.

(2a) As in the above proof, using $x_{r_1}, y_{r_1}, z_{r_1}$ in 1, we have a surjective map λ_{r_1},

$$\lambda_{r_1} : ((x_{r_1}, y_{r_1}, z_{r_1}) + (w^{r_1}))/(\mathfrak{m}_P(x_{r_1}, y_{r_1}, z_{r_1}) + (w^{r_1+1}))$$
$$\twoheadrightarrow \mathcal{O}_X(-r_1 E)/\mathcal{O}_X(-(r_1+1)E).$$

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Dividing by (w^{r_1}), we have another surjective map $\overline{\lambda}_{r_1}$,

$$\overline{\lambda}_{r_1}: (x_{r_1}, y_{r_1}, z_{r_1})/\mathfrak{m}_P(x_{r_1}, y_{r_1}, z_{r_1}) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_X(-r_1 E)/(\mathcal{O}_X(-(r_1+1)E) + (w^{r_1})).$$

By Theorem 3.2 and $m_{\mathrm{div}}(w)(E) = 1$, $\dim_C \mathcal{O}_X(-r_1 E)/\mathcal{O}_X(-(r_1+1)E + (w^{r_1})) = d(-r_1) - 1 = 1$. Hence $\dim_C \operatorname{Ker} \lambda_{r_1} = 3 - 1 = 2$, which shows (2a).

(2b) We will prove (2b) as in the proof of (1), constructing y_i, z_i inductively starting with y_{r_1}, z_{r_1} in 1. Assume that we have constructed y_i, z_i ($r_1 \leqslant i < a$). There exists a surjective map λ_i,

$$\lambda_i: \left((y_i, z_i) + \sum_{(j,k)\in \cup_{s \geqslant i} J_s} (x_{r_1}^j w^k))/(\mathfrak{m}_P(y_i, z_i) + \sum_{(j,k)\in \cup_{s \geqslant i+1} J_s} (x_{r_1}^j w^k) \right) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{O}_X(-iE)/\mathcal{O}_X(-(i+1)E).$$

We know that x_{r_1}, w^{r_1} generate $\mathcal{O}_X(-r_1 E)/\mathcal{O}_X(-(r_1+1)E)$ because of the proof of (2a). Thus any nonzero element in $\bigoplus_{(j,k)\in J_i} \mathbb{C} x_{r_1}^j w^k \subset \mathbb{C}[x_{r_1}, w]$, which always decomposes into a product of $w^{i - \lfloor \frac{i}{r_1} \rfloor r_1}$ and $\lfloor \frac{i}{r_1} \rfloor$ linear combinations of x_{r_1}, w^{r_1}, has exactly its multiplicity i along E. On the other hand, by Theorem 3.2 and Lemma 6.10, we have $d(-i) = N_{i+1} - N_i$, which is the number of elements in J_i. Thus $\{x_{r_1}^j w^k\}_{(j,k)\in J_i}$ generate $\mathcal{O}_X(-iE)/\mathcal{O}_X(-(i+1)E)$, and that $y_i + t_i^y, z_i + t_i^z \in \operatorname{Ker} \lambda_i$ for some $t_i^y, t_i^z \in \bigoplus_{(j,k)\in J_i} \mathbb{C} x_{r_1}^j w^k \subset \mathbb{C}[x_{r_1}, w]$. Hence, it is enough to put $y_{i+1} = y_i + t_i^y, z_{i+1} = z_i + t_i^z$. □

We will construct an identification in Lemma 6.12 using Claim 6.13. It is easy to see that we can take an identification in 6.13.1. Lemma 6.12 is trivial if $a \leqslant r_1$ by Claim 6.13.1. If $r_1 < a$, by Claim 6.13.2a and an equation $xy + z^2 + w^N = (x - y - 2z)y + (y + z)^2 + w^N$, we may assume that $x_{r_1} \notin \mathcal{O}_X(-(r_1+1)E)$ in the construction of $x_{r_1}, y_{r_1}, z_{r_1}$ in 6.13.1. Then by Claim 6.13.2b, we obtain that

$$z_a = z + p_a^z(x + p_{r_1}^x(w), w) \in \mathcal{O}_X(-aE).$$

We express z_a as

$$z_a = z + p(w) + q(x, w)x(p(w) \in \bigoplus_{i=1}^{a-1} \mathbb{C} w^i \subset \mathbb{C}[w], q(x, w) \in \mathbb{C}[x, w]).$$

Thus, it is sufficient to replace y, z with

$$y' = y - 2q(x, w)z - q(x, w)^2 x, z' = z + q(x, w)x$$

because $xy + z^2 + w^N = xy' + (z')^2 + w^N$. □

Corollary 6.11, Lemma 6.12, and the following lemma induce 6.2.1-3, which terminates the proof of Proposition 6.1 and therefore also the proof of Theorem 2.5 completely.

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LEMMA 6.14. (1) *Under the identification* $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ *in Lemma 6.12, assume* $N < 2a$ *or* $p(w) \neq 0$. *Then there exists a special surface of type* s *with* $s < 2a - 1$.
(2) *Under the identification* $P \in X \cong o \in (xy + z^2 + w^N = 0) \subset \mathbb{C}^4$ *in Lemma 6.12, assume* $N \geqslant 2a$, $p(w) = 0$, *and* $l < m$. *Then there exists a special surface of type* $2a - 3$.

Proof. (1) Take a surface $P \in S = \mathrm{div}(z + p(w) + cw^a)$ for a general $c \in \mathbb{C}$. Then $P \in S \cong o \in (xy + (p(w) + cw^a)^2 + w^N = 0) \subset \mathbb{C}^3$, which is a Du Val singularity of type A_s, where

$$s = \min\{2a, a + \mathrm{ord}\, p(w), \mathrm{ord}\, (p(w)^2 + w^N)\} - 1.$$

Here $\mathrm{ord}\, q(w) = \sup\{i \in \mathbb{Z}_{\geqslant 0} | w^i \text{ divides } q(w)\} \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$. We remark that $s < 2a - 1$ if $N < 2a$ or $p(w) \neq 0$. Because $z + p(w) \in \mathcal{O}_X(-aE)$ and $m_{\mathrm{div}}(w)(E) = 1$, the multiplicity of S along E equals a. Thus $P \in S \subset X$ is special of type s.

(2) We may assume that $l_0 = F_l \cap \mathrm{div}(y)_{X_l} \cap \mathrm{div}(z)_{X_l}$. Since $l < m$, Z_l is a point on l_0 except the vertex point of $F_l \cong Q_0$. Thus $Z_l = l_0 \cap \mathrm{div}(tx + w^l)_{X_l}$ for some $t \in \mathbb{C}$. We note that $tx + w^l \in \mathcal{O}_X(-(l+1)F_{l+1}) \subseteq \mathcal{O}_X(-(l+1)E)$ because $Z_l \in \mathrm{div}(tx + w^l)_{X_l}$. Take a surface $P \in S = \mathrm{div}(z + w^{a-l-1}(tx + w^l) + cw^a)$ for a general $c \in \mathbb{C}$. Then $P \in S \cong o \in (xy + (w^{a-l-1}(tx + w^l) + cw^a)^2 + w^N = 0) \subset \mathbb{C}^3$, which is a Du Val singularity of type A_{2a-3}. Because $z \in \mathcal{O}_X(-aE)$, $tx + w^l \in \mathcal{O}_X(-(l+1)E)$, and $m_{\mathrm{div(w)}}(E) = 1$, the multiplicity of S along E equals a. Thus $P \in S \subset X$ is special of type $2a - 3$. □

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References

[Co00] Corti, A.: Singularities of linear systems and 3-fold birational geometry, In: *Explicit Birational Geometry of 3-folds*, London Math. Soc. Lecture Notes Ser. 281, Cambridge Univ. Press, 2000, pp. 259–312.

[CM00] Corti, A. and Mella, M.: Birational geometry of terminal quartic 3-folds I, Preprint.

[CPR00] Corti, A., Pukhlikov A., and Reid, M.: Fano 3-fold hypersurfaces, In: *Explicit Birational Geometry of 3-folds*, London Math. Soc. Lecture Notes Ser. 281, Cambridge Univ. Press, 2000, pp. 175–258.

[Cu88] Cutkosky, S.: Elementary contractions of Gorenstein threefolds, *Math. Ann.* **280**, (1988) 521–525.

[Kwk01] Kawakita, M.: Divisorial contractions in dimension three which contract divisors to smooth points, to appear in *Invent. Math.*

[Kwm96] Kawamata, Y.: Divisorial contractions to 3-dimensional terminal quotient singularities, In: *Higher-Dimensional Complex Varieties*, de Gruyter, Berlin, 1996, pp. 241–246.

https://doi.org/10.1023/A:1016334006624 Published online by Cambridge University Press

[K+92] Kollar, J. *et al*.: Flips and abundance for algebraic threefolds, *Astérisque* **211**, (1992).

[R83] Reid, M.: Minimal models of canonical 3-folds, *Adv. Stud. Pure Math.* **1** (1983), 131–180.

[R87] Reid, M.: Young person's guide to canonical singularities, *Proc. Sympos. Pure Math.* **46** (1987), 345–414.