Can. J. Math., Vol. XXXIII, No. 3, 1981, pp. 664–670

POWER-RICH AND POWER-DEFICIENT LCA GROUPS

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In [**4**], Edwin Hewitt defined a-rich LCA (i.e., locally compact abelian) groups and classified them by their algebraic structure. In this paper, we study LCA groups with some properties related to a-richness. We define an LCA group G to be *power-rich* if for every open neighbourhood V of the identity in G and for every integer $n > 1$, $\lambda(nV) > 0$, where $nV = \{nx \in G : x \in V\}$ and λ is a Haar measure on G. G is *power-meagre* if for every integer $n > 1$, there is an open neighbourhood V of the identity, possibly depending on n, such that $\lambda(nV) = 0$. G is *power-deficient* if for every integer $n > 1$ and for every open neighbourhood V of the identity such that $\bar{V}$ is compact, $\lambda(n\bar{V}) = 0$. G is *dual power-rich* if both G and $\hat{G}$ are power-rich. We define dual power-meagre and dual power-deficient groups similarly.

If G is an LCA group, $B(G)$, G_0, $T(G)$, $E(G)$ and $\hat{G}$ denote respectively the subgroup of compact elements of G, the component of the identity, the maximal torsion subgroup, the minimal divisible extension of G (topologized in the usual manner so that G is an open subgroup) and the dual group of G. If $f:G \to H$ is a continuous homomorphism between LCA groups, f is an open map (proper map) if f takes open sets of G onto open sets of H (respectively $f(G)$). For every integer $n > 1$, $f_n: G \to G$ is the continuous endomorphism defined by $f_n(g) = ng$ for every $g \in G$. We say that G is *power-open* if f_n is an open map for every $n > 1$, and G is *n-open* if $nG = f_n(G)$ is an open subgroup for every $n > 1$. For every $n > 1$, $G(n)$ denotes the kernel of f_n. $\cong$ denotes topological isomorphism. We use $\times$ to denote topological direct products and $\oplus$ to denote direct sums of discrete abelian groups. J_p denotes the compact abelian group of p-adic integers. Our neighbourhoods of the identity are always open, whether we say it explicitly or not. For brevity, the expression '$n > 1$' shall mean that n is an arbitrary integer greater than one. In general, our reference for topological groups and Haar measure is [**5**], and for discrete abelian groups, it is [**2**].

If G is an LCA group, $G \cong \mathbf{R}^n \times M$, where M contains a compact, open subgroup [**5**, 24.30], and if $G = \mathbf{R}^m \times M_1$ is another decomposition such that M_1 contains a compact, open subgroup, then $m = n$ and $M \cong M_1$ [**1**, Corollary 1]. For brevity, we shall refer to a decomposition

Received December 15, 1979.

https://doi.org/10.4153/CJM-1981-052-6 Published online by Cambridge University Press

of this kind as a normal decomposition of G. We shall frequently use the fact that in any such decomposition, $B(G)$ is always an open subgroup of the second summand M. As we can find no reference for this fact, we prove it as a lemma.

LEMMA 1. *Let $G = \mathbf{R}^n \times M$ be a normal decomposition of G. Then $B(G)$ is a pure, open subgroup of M.*

Proof. There exists a continuous homomorphism f from G onto M such that f is the identity map on M and $\ker f = \mathbf{R}^n$. Let $g = f|B(G)$. Then $B(G)$ is clearly invariant under g and $\ker g$ is trivial. Now g takes $B(G)$ onto $B(G) \cap M$ and is the identity map on $B(G) \cap M$. Hence, by [**5**, 6.22],

$$B(G) = (B(G) \cap M) \times \{0\},$$

so that $B(G) \subseteq M$. As $B(G)$ is the annihilator of the compact component of the identity of $\hat{M}$, $B(G)$ is an open and pure subgroup of M.

1. Power-rich LCA groups. We begin with a characterization of power-rich compact abelian groups.

PROPOSITION 1.1. *The following are equivalent for a compact abelian group G:*

(a) *G is n-open.*

(b) *G is power-open.*

(c) *G is power-rich.*

(d) *$G/G_0 \cong \times A_p$, where A_p is a topological direct product of at most finitely many copies of J_p and finitely many cyclic p-groups for each prime p.*

Proof. Since nG is compact, the equivalence of (a) and (b) follows from [**5**, 5.29]. Next, assuming (b) we shall prove (c). Let V be any neighbourhood of the identity, then nV is open for every $n > 1$, so $\lambda(nV) > 0$ and (c) is proved. We show next that (c) implies (d). By (c), $\lambda(pG) > 0$ for every prime p, so $p(G)$ must be of finite index in the compact group G. Hence, for every prime p, the p-component of the discrete torsion group $(G/G_0)^\wedge$ must be finitely cogenerated [**2**, 25.1], whence (d) follows from duality. Finally, assuming (d) we prove (a). Let $n > 1$. Since $G_0 = \bigcap_{n>1} nG$, it is clear that in the natural one-to-one correspondence between subgroups of G/G_0 and subgroups of G containing G_0, the subgroups nG and $n(G/G_0)$ correspond. From (d), it is clear that $n(G/G_0)$ is open in G/G_0, so nG is open in G. This completes the proof.

COROLLARY 1.1. *A totally disconnected compact abelian group G is power-rich if and only if $G \cong \times A_p$, where A_p is a topological direct product of at most finitely many copies of J_p and finitely many cyclic p-groups for each prime p.*

We now characterize power-rich LCA groups.

https://doi.org/10.4153/CJM-1981-052-6 Published online by Cambridge University Press

THEOREM 1.1. *The following assertions are equivalent for an* LCA *group* G:

(a) G *is power-rich.*

(b) $B(G)$ *contains a compact, open, power-rich subgroup.*

(c) *Every compact, open subgroup of* $B(G)$ *is power-rich.*

(d) G *is power-open.*

Proof. Using Lemma 1 and the argument in [**4**], it is clear that (a), (b), (c) are pairwise equivalent. Clearly (d) implies (a). Finally, we show that (b) implies (d). Let n be any integer greater than one and let V be any open neighbourhood of the identity in G. Let $G = \mathbf{R}^m \times M$ be a normal decomposition of G. Choose $U_1 \times V_1 \subseteq V$ such that U_1, V_1 are open neighbourhoods of the identities of $\mathbf{R}^m$ and M respectively. Let H be a compact, open power-rich subgroup of $B(G)$. Then $V_1 \cap H$ is an open subset of H and since H is power-open [Proposition 1.1], $n(V_1 \cap H)$ is open in H and so in M. Since $\mathbf{R}^m$ is power-open, nU_1 is open in $\mathbf{R}^m$. Hence, $nU_1 \times n(V_1 \cap H)$ is an open neighbourhood of the identity in G contained in

$$nU_1 \times nV_1 = n(U_1 \times V_1) \subseteq nV.$$

Hence, by [**5**, 5.40], f_n is an open map. This completes the proof.

If λ is a Haar measure on an LCA group G, λ is positive on every non-empty open subset, but λ may possibly be positive on some subsets with empty interior. Theorem 1.1 reduces the measure-theoretic problem of power-richness to the structural problem of power-openness. Also, Theorem 1.1 shows that $B(G)$ determines whether G is power-rich or not. We further reduce this problem to a simpler form in the next corollary.

COROLLARY 1.2. *An* LCA *group* G *is power-rich if and only if* $B(G)$ *modulo its component of the identity is power-rich.*

Proof. This follows immediately from Theorem 1.1, Proposition 1.1 and Corollary 1.1.

COROLLARY 1.3. *Let* G *be power-rich. Then every closed subgroup containing* G_0 *is power-rich. In particular, every open subgroup is power-rich.*

Proof. Let H be a closed subgroup containing G_0. Then $B(H) = H \cap B(G)$. Any compact open subgroup K_1 of $B(H)$ is contained in a compact, open subgroup K_2 of $B(G)$. Proposition 1.1 then completes the proof. If H is open, H contains G_0 and the result follows from the first case.

COROLLARY 1.4. *If an* LCA *group* G *is power-rich, then* $E(G)$ *is power-rich.*

Proof. We can assume G contains a compact, open subgroup. Let K

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be a compact, open subgroup of $E(G)$ and let $K_1 = K \cap G$. Since $G_0 = (E(G))_0$, and K_1/G_0 is of finite index in K/G_0, the result follows from Theorem 1.1.

We now characterize dual power-rich LCA groups.

THEOREM 1.2. *The following are equivalent for an* LCA *group G:*
(a) *G is dual power-rich.*
(b) *G and $\hat{G}$ are both power-open.*
(c) *$B(G)$ has a compact, open power-rich subgroup H such that*

$$B(G)/H \cong \oplus A_p,$$

where A_p is a p-group of finite rank for every prime p.

Proof. (a) and (b) are equivalent by Theorem 1.1. We assume (c) and prove (a). Let $\mathbf{R}^m \times M$ be a normal decomposition of G. Since $B(G) \subseteq M$ and $B(\hat{G}) \subseteq \hat{M}$, we can assume G contains a compact, open subgroup. Now (c) implies that G is power-rich (Theorem 1.1). Obviously $A(\hat{G}, H)$ is a compact, open subgroup of $B(\hat{G})$. Since $H \subseteq B(G)$, [**6**, Proposition 2.3] shows that

$$A(\hat{G}, H)/(\hat{G})_0 \cong A(B(G)\hat{\ }, H) \cong (B(G)/H)\hat{\ },$$

which is a compact group of the form described in Proposition 1.1 (d). Hence, $A(\hat{G}, H)$ is power-rich and so $\hat{G}$ is power-rich. Conversely, we show that (a) implies (c). By (a), $B(G)$ contains a compact, open, power-rich subgroup H such that $A(\hat{G}, H)$ is a compact, open, power-rich subgroup of $B(\hat{G})$. Hence,

$$A(\hat{G}, H)/(\hat{G})_0 \cong (B(G)/H)\hat{\ }$$

is a group of the form described in Proposition 1.1 (d), and this implies (c) by duality. This completes the proof.

Remarks 1.1. In the remarks below, we present and discuss some specific classes of power-rich and dual power-rich LCA groups.

(1) Since discrete abelian groups are power-rich, a compact abelian group is dual power-rich if and only if it is power-rich.

(2) A torsion LCA group G is power-rich if and only if G is discrete. For G contains a compact, open subgroup of bounded order which can be power-open if and only if it is discrete.

(3) If G is torsion-free, then G is power-rich if and only if G is dual power-rich. For, suppose G is power-rich. Then f_n is an open map for every n, and the adjoint $\hat{f}_n$ is an open map onto $n\hat{G}$ [**5**, 24.40], so that $n\hat{G}$ is closed for every $n > 1$. But $n\hat{G}$ is dense in $\hat{G}$ [**5**, 24.23], so we must have $n\hat{G} = \hat{G}$. Thus $\hat{G}$ is power-rich. Moreover, $\hat{G}$ is divisible.

(4) Divisible σ-compact LCA groups (which include all connected LCA groups) are power-rich.

(5) In view of Corollary 1.2, it would be of interest to consider LCA groups G for which $G = B(G)$ and G_0 is trivial. Let G be such a group. Then G is power-rich if and only if G is topologically isomorphic with an open subgroup of an LCA group of the form $L \times E(M)$, where L is a discrete, torsion divisible group and M is a compact, totally disconnected power-rich abelian group. G is dual power-rich if and only if G is of the form stated above and every p-component of L is of finite rank. These results are an easy consequence of the results proved earlier and the fact that open subgroups and minimal divisible extensions of power-rich LCA groups are power-rich.

2. Power-meagre and power-deficient LCA groups. We first take up compact abelian groups.

PROPOSITION 2.1. *The following are equivalent for a compact, abelian group G:*

(a) *G is power-meagre.*

(b) *G is power-deficient.*

(c) *$T(\hat{G}) \cong \oplus A_p$, where A_p is a p-group of infinite rank for every prime p.*

(d) *nG is a non-open subgroup for every $n > 1$.*

Proof. Let $n > 1$ and let V be a neighbourhood of the identity such that $\bar{V}$ is compact. By (a), there is a neighbourhood W of the identity such that $\lambda(nW)$ is zero. Since $n\bar{V}$ is a compact subset of the compact group nG, $\lambda(n\bar{V}) > 0$ would imply $\lambda(nG) > 0$, which in its turn would imply that nG is an open subgroup of G. Proposition 1.1 then forces the conclusion $\lambda(nW) > 0$. Hence, we must have $\lambda(n\bar{V}) = 0$. Thus (a) implies (b). Now, assuming (b), we shall prove (c). Let V be an arbitrary neighbourhood of the identity such that $\bar{V}$ is compact and let p be a prime. Then $\lambda(p\bar{V}) = 0$. Since pV is an open subset of the compact group pG, we must have $\lambda(pG) = 0$, so that pG is not an open subgroup of G. Hence, the compact group G/pG is a direct product of infinitely many copies of $\mathbf{Z}(p)$ [**5**, 25.29]. (c) now follows by duality. We next assume (c). By duality, we conclude that nG is of infinite index in G, so that nG is not open in G. This proves (d). Finally, we show that (d) implies (a). Let $n > 1$. By (d), G is not connected so it contains a compact, open subgroup H. Since $nH \subseteq nG$ and nG is not open in G, nH is not open in H. Hence, $\lambda(nH) = 0$. This proves (a).

COROLLARY 2.1. *A totally disconnected compact abelian group G is power-deficient if and only if $\hat{G} = \oplus A_p$, where A_p is a p-group of infinite rank for every prime p.*

In the next two theorems, we characterize power-deficient and dual power-deficient LCA groups.

THEOREM 2.1. *The following are equivalent for an* LCA *group G:*
(a) *Every compact open subgroup of $B(G)$ is power-deficient.*
(b) *G is power-meagre.*
(c) *G is power-deficient.*

Proof. Assuming (a), we prove (b). Suppose G is not power-meagre. Then for some $n > 1$, $\lambda(nU)$ is positive for every neighbourhood U of the identity. Hence, H/nH would be of finite order for every compact, open subgroup H of $B(G)$, contradicting (a). Next, we show that (b) implies (a). Suppose a compact, open subgroup H of $B(G)$ is not power-deficient. Then for some prime p, H/pH would be of finite index implying that G is p-rich. Hence, H must be power-deficient. Assuming (a), we next prove (c). Let V be any neighbourhood of the identity such that $\bar{V}$ is compact. Let $K = \mathbf{R}^1 \times M \times \mathbf{Z}^m$, say, be a compactly generated open subgroup of G containing $\bar{V}$, where M is a compact group. We note that (a) implies that M (necessarily a compact open subgroup of $B(G)$) is power-deficient. Let $n > 1$. Then $nK = \mathbf{R}^1 \times nM \times n\mathbf{Z}^m$ is a closed compactly generated subgroup of G [**6**, Theorem 2.9] containing $n\bar{V}$. Now $\lambda(n\bar{V}) > 0$ would imply that nK is an open subgroup of G [**3**, Section 61, Exercise 3], hence also of K. But every open subgroup of the σ-compact group K must be of countable index, whereas $K/nK \cong (M/nM) \times$ a finite group, is uncountable. Hence, $\lambda(n\bar{V})$ must be zero. Finally, assuming (c), we prove (a). Suppose H is a compact, open subgroup of $B(G)$ which is not power-deficient. Then for some prime p, pH is of finite index in H. Let $G = \mathbf{R}^m \times M$ be a normal decomposition of G and let $V \times H$ be a neighbourhood of the identity in G such that $\bar{V} \times H$ is compact. Then $p(\bar{V} \times H) = p\bar{V} \times pH$ is compact with non-empty interior, since pV and pH are open in $\mathbf{R}^m$ and M respectively. Hence, $\lambda(p\bar{V} \times pH)$ must be positive, which contradicts (c). This completes the proof.

The following corollary is an easy consequence of Theorem 2.1 and Proposition 2.1.

COROLLARY 2.2. *An* LCA *group G is power-deficient if and only if $B(G)$ modulo its component of the identity is power-deficient.*

THEOREM 2.2. *An* LCA *group G is dual power-deficient if and only if $B(G)$ has a compact, open, power-deficient subgroup H such that $B(G)/H \cong \oplus A_p$, where A_p is a p-group of infinite rank for every prime p.*

Proof. Following exactly the line of argument in the proof of Theorem 1.2 and using Proposition 2.1(c) in place of Proposition 1.1(d), the proof is immediate.

Remarks 2.1. We discuss below some specific classes of LCA groups.

https://doi.org/10.4153/CJM-1981-052-6 Published online by Cambridge University Press

(1) It is obvious that no discrete abelian group and no torsion LCA group can be power-deficient.

(2) A torsion-free LCA group $G \cong G_0 \times L$, where L is totally disconnected [**5**, 25.30]. Hence, G is power-deficient if and only if L is non-zero and contains a compact, open subgroup H of the form $\times A_p$, where A_p is a product of infinitely many copies of J_p for every prime p. By Theorem 2.2, one obtains a necessary and sufficient condition on $B(L)/H$ for the dual power-deficiency of G.

(3) A compact abelian group is never dual power-deficient.

(4) Let G be an LCA group such that $G = B(G)$ and G_0 is trivial. If G is power-deficient, one can prove as in Corollaries 1.3 and 1.4 that $E(G)$ and every open subgroup of G is power-deficient. Hence, it is easy to see that G is power-deficient if and only if G is topologically isomorphic with an open subgroup of an LCA group of the form $L \times E(M)$, where L is a discrete, divisible torsion abelian group and M is a compact power-deficient group.

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