J. Austral. Math. Soc. Ser. B 27 (1985), 88–110

STEADY STATES OF THE REACTION-DIFFUSION EQUATIONS. PART III: QUESTIONS OF MULTIPLICITY AND UNIQUENESS OF SOLUTIONS

J. G. BURNELL[1], A. A. LACEY[2] and G. C. WAKE[1]

(Received 19 September 1983; revised 3 April 1984)

Abstract

In earlier papers (Parts I and II) existence and uniqueness of the solutions to a coupled pair of nonlinear elliptic partial differential equations with linear boundary conditions was considered. These equations arise when material is undergoing an exothermic chemical reaction which is sustained by the diffusion of a reactant. In this paper we establish the existence of multiple solutions for many different values of the parameters not considered in the earlier parts. It is shown that the case, also omitted in earlier parts, with perfect thermal and mass transfer on the boundary (the double-Dirichlet case) does have a unique solution for sufficiently large values of the exothermicity or an equivalent parameter. The methods of solution provide specific bounds on the region of existence of multiple solutions.

1. Introduction

In our two earlier papers (Burnell, Lacey and Wake [1, 2]) we investigated properties of the steady-state solutions of the equations governing the diffusion of a reactant which is undergoing an exothermic reaction. In Part I (Burnell, Lacey and Wake [1]) we showed this involved discussing properties of the solutions of the equations

$$\nabla^2 u + \lambda(1+v)e^u = 0 \quad \text{in } \Omega, \tag{1a}$$

$$\nabla^2 v - \lambda\alpha(1+v)e^u = 0 \quad \text{in } \Omega, \tag{1b}$$

with

$$\partial u/\partial n + \mu u = 0 \quad \text{on } \partial\Omega, \tag{2a}$$

$$\partial v/\partial n + \nu v = 0 \quad \text{on } \partial\Omega, \tag{2b}$$

[1] Mathematics Department, Victoria University, Private Bag, Wellington, New Zealand. Please address all correspondence to Dr. Wake at this address.

[2] Mathematics Department, Heriot-Watt University, Edinburgh EH14 4AS, Scotland.

where $\Omega \subseteq \mathbf{R}^n$ is an open bounded set and $\lambda, \alpha, \mu, \nu$ are positive constants, u is a dimensionless temperature rise over the ambient and $1 + v$ is a dimensionless concentration of the diffusing reactant.

In Parts I and II we established the following properties.

THEOREM 1. (1) *The equations* (1), (2) *have at least one solution for all values of the parameters, except when* $\mu < \nu = \infty$. *For the case* $\mu < \nu = \infty$ *and* $n \leqslant 3$ *there exists* λ^* (*which depends on* α *and* μ) *such that, for* $\lambda < \lambda^*$, (1), (2) *have at least one solution*; *and, for* $\lambda > \lambda^*$, (1), (2) *have no solutions* (*this holds for any* α *and any* $\mu < \infty$).

(2) *For all cases, except when* $\mu < \nu = \infty$, *the solution to* (1), (2) *is unique for sufficiently small values of* λ. *Further, excluding the case* $\mu = \nu = \infty$, *the equations* (1), (2) *have a unique solution for sufficiently large values of* λ.

(3) *If* $\mu = \nu$ *then, for* α *sufficiently small, there are values of* λ *for which* (1), (2) *have at least two solutions.*

(4) *Except for the case* $\mu < \nu = \infty$ *then, for any solution* (u, v) *of* (1), (2),

$$\begin{aligned} &(\forall x \in \bar{\Omega}) \quad -1 \leqslant v(x) \leqslant 0, \\ &(\forall x \in \bar{\Omega}) \quad 0 \leqslant u(x) \leqslant 1/\alpha, \quad \text{for } \nu \leqslant \mu, \\ &\qquad\qquad\quad\; 0 \leqslant u(x) \leqslant \nu/\alpha\mu, \quad \text{for } \mu < \nu < \infty. \end{aligned}$$

(5) *If there is an interval* (a, b) *such that, for* $\lambda \in (a, b)$, *the solution to* (1), (2) *is unique, then the solution branch* $(\lambda, (u(\lambda), v(\lambda)))$ *is a continuous function of* λ *on* (a, b).

The purpose of this paper is to provide results regarding the multiplicity and uniqueness of solutions to (1) and (2) for those cases that were omitted in Parts I and II. Specifically we show that (1), (2) has multiple solutions for all $\mu, \nu < \infty$, when α is sufficiently small, and that the solution of (1), (2) when $\mu = \nu = \infty$ is unique for sufficiently large λ; and for a fixed λ, the solution is unique for sufficiently large α. In the second case the equations (1), (2) are equivalent to a single nonlinear equation and, in fact, we prove in Section 3 a general uniqueness result for such an equation. Using the proofs of these results, in Section 5, we can obtain explicit bounds on the values of λ for which multiple solutions exist.

2. Results on multiple solutions

In this section we shall discuss conditions under which the equations (1), (2) have multiple solutions for some appropriate (fixed) values of the parameters.

Firstly we shall show that when α is small, there are values of λ for which there exist solutions of (1), (2) "near" the minimal solution for the case with $\alpha = 0$.

When $\alpha = 0$, equations (1), (2) reduce to the single equation

$$\nabla^2 u + \lambda e^u = 0 \quad \text{in } \Omega, \tag{3}$$

$$\partial u/\partial n + \mu u = 0 \quad \text{on } \partial\Omega. \tag{4}$$

It is well known that the equations (3), (4) have solutions for values of λ less than or equal to some λ_0 (see for instance Keller and Cohen [3]).

THEOREM 1. *Suppose that, for $\lambda \leqslant \lambda_0$, u_λ is the corresponding minimal solution to* (3), (4). *Then there exists $\alpha_0 > 0$ such that, for $\alpha \leqslant \alpha_0$ and $\lambda \leqslant \lambda_0$,* (1), (2) *has a solution (u, v) with*

$$(\forall x \in \bar{\Omega}) \quad -1 \leqslant v(x) \leqslant 0, 0 \leqslant u(x) \leqslant u_\lambda(x).$$

PROOF. To prove this result we shall make use of Schauder's fixed point theorem. First, define a map T: $C(\bar{\Omega}) \times C(\bar{\Omega}) \to C(\bar{\Omega}) \times C(\bar{\Omega})$ by

$$T(u, v) = \big(\lambda K_\mu(F(u, v)), -\lambda\alpha K_\nu(F(u, v))\big), \tag{5}$$

where K_σ is the Green's function operator for $(-\nabla^2, \partial/\partial n + \sigma)$ and F: $C(\bar{\Omega}) \times C(\bar{\Omega}) \to C(\bar{\Omega})$ is given by

$$F(u, v)(x) = (1 + v(x))e^{u(x)}. \tag{6}$$

In Part I of this paper (Burnell, Lacey and Wake [1]) we showed (see Section 2, Proposition 5) that if (u, v) is a fixed point of T then (u, v) is a solution of (1), (2).

Now suppose $\lambda \leqslant \lambda_0$, then let

$$D = \big\{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}) \colon (\forall x \in \bar{\Omega})\ 0 \leqslant u(x) \leqslant u_\lambda(x),\ -1 \leqslant v(x) \leqslant 0\big\}.$$

For $(u, v) \in D$, let $(w, z) = T(u, v)$. Then since $u_\lambda = \lambda K_\mu(e^{u_\lambda})$,

$$(w - u_\lambda) = \lambda K_\mu(F(u, v) - e^{u_\lambda}).$$

Also, since $-1 \leqslant v \leqslant 0$,

$$\begin{aligned} F(u, v) - e^{u_\lambda} &= (1 + v)e^u - e^{u_\lambda} \\ &\leqslant (1 + v)e^{u_\lambda} - e^{u_\lambda} \\ &= ve^{u_\lambda}. \leqslant 0. \end{aligned}$$

Thus, as K_μ is a positive operator,

$$w - u_\lambda \leqslant \lambda K_\mu(0) = 0,$$

so

$$w \leqslant u_\lambda.$$

Since $(u, v) \in D$, $(1 + v)e^u \geqslant 0$; hence by the maximum principle, $w \geqslant 0$. Therefore $(\forall x \in \bar{\Omega})$, $0 \leqslant w(x) \leqslant u_\lambda(x)$.

Also, it is clear that $z \leqslant 0$ and

$$\|z\|_0 = \|\lambda\alpha K_\nu(F(u, v))\|_0 \leqslant \lambda\alpha\|K_\nu\|_0\|F(u, v)\|_0$$

where $\|K_\nu\|_0$ is the norm of the operator K_ν (with respect to the supremum norm on $C(\overline{\Omega})$). Now

$$\|F(u, v)\|_0 = \sup_{x \in \Omega} |(1 + v(x))e^{u(x)}|$$
$$\leqslant \sup_{x \in \Omega} |e^{u_\lambda(x)}| = \|e^{u_\lambda}\|_0.$$

Hence $\|z\|_0 \leqslant \lambda\alpha\|K_\nu\|_0\|e^{u_\lambda}\|_0$. It is known (Keller and Cohen [3]) that if U_0 is the minimal solution of (3), (4) when $\lambda = \lambda_0$ then, for $\lambda \leqslant \lambda_0$, we have $u_\lambda \leqslant U_0$.

So, if α_0 is chosen so that

$$\lambda_0\alpha_0\|K_\nu\|_0\|e^{U_0}\|_0 < 1$$

then, for $\alpha \leqslant \alpha_0$, $\lambda \leqslant \lambda_0$,

$$\|z\|_0 < 1.$$

Therefore

$$(\forall x \in \overline{\Omega}) - 1 \leqslant z(x) \leqslant 0.$$

This then means that, for $\alpha \leqslant \alpha_0$ and $\lambda \leqslant \lambda_0$, $T(D) \subseteq D$.

Now, it is clear that D is a closed convex subset of $C(\overline{\Omega}) \times C(\overline{\Omega})$. Also, using the known properties of K_μ and K_ν it is easily shown (see Burnell, Lacey and Wake [1]) that T is a compact map. Hence the Schauder fixed point theorem ensures that there exists $(u, v) \in D$ such that

$$T(u, v) = (u, v),$$

when $\alpha \leqslant \alpha_0$ and $\lambda \leqslant \lambda_0$. Consequently, for $\alpha \leqslant \alpha_0$ and $\lambda \leqslant \lambda_0$, there exists a solution (u, v) of (1), (2) with

$$(\forall x \in \overline{\Omega}) \quad 0 \leqslant u(x) \leqslant u_\lambda(x), -1 \leqslant v(x) \leqslant 0. \qquad Q.E.D.$$

In order to prove that the equations (1), (2) have at least 2 solutions for certain values of the parameters we make use of properties of the Leray-Schauder degree and some results from Part I (Burnell, Lacey and Wake [1]). For easy reference we shall summarise these results here.

THEOREM 2. (1) *Suppose X is a normed vector space and D is an open bounded subset of X containing* 0. *Further, suppose S: $\overline{D} \to X$ is a compact map with $0 \notin (I - S)(\partial D)$. Then the Leray-Schauder degree of $I - S$ at* 0 *on D is an integer which satisfies*:

(a) *if $S = 0$ then $d(I, D, 0) = 1$*;

(b) *if $d(I - S, D, 0) \neq 0$ then there exists $x \in D$ such that $x - Sx = 0$*;

(c) *if* $H\colon \overline{D} \times [0,1] \to X$ *satisfies*:

(i) $(\forall s \in [0,1])\ H(\cdot, s)\colon \overline{D} \to X\colon x \mapsto H(x, s)$ *is compact*;

(ii) $(\forall \varepsilon > 0)(\exists \delta > 0)$ *such that whenever* $|s_1 - s_2| < \delta$,

$$(\forall x \in \overline{D})\|H(x, s_1) - H(x, s_2)\| < \varepsilon;$$

(iii) $(\forall s \in [0,1])\ 0 \notin (I - H(\cdot, s))(\partial D)$,

then $d(I - H(\cdot, s), D, 0)$ *is independent of* $s \in [0, 1]$;

(d) *If* $E \subseteq D$ *is closed and* $0 \notin (I - S)(E)$ *then*

$$d(I - S, D, 0) = d(I - S, D \setminus E, 0).$$

(2) *If* T *is the map defined by* (5), *and* $\mu, \nu < \infty$ *then, for any* $\alpha > 0$ *and* $\lambda \geqslant 0$, T *is compact and*

$$d(I - T, B(2 + m(\alpha)), 0) = 1,$$

where

$$m(\alpha) = \begin{cases} 1/\alpha, & \text{for } \mu \geqslant \nu, \\ \nu/(\alpha\mu), & \text{for } \mu < \nu. \end{cases}$$

and

$$B(r) = \left\{(w, z) \in C(\overline{\Omega}) \times C(\overline{\Omega})\colon \|w\|_0 + \|z\|_0 < r\right\}.$$

(3) *Suppose* $\mu, \nu < \infty$. *Then, for sufficiently large values of* λ, (1), (2) *has a unique solution* (u, v) *and for some constant* A, *independent of* λ,

$$(\forall x \in \overline{\Omega})\left(\nu/\mu - A(\lambda\alpha)^{-1/2}\right)/\alpha \leqslant u(x) \leqslant 1/\alpha \quad \text{if } \mu \geqslant \nu,$$

$$\left(1 - A(\lambda\alpha)^{-1/2}\right)\nu/(\alpha\mu) \leqslant u(x) \leqslant \nu/(\alpha\mu) \quad \text{if } \mu < \nu.$$

In order to define the Leray-Schauder degree of $I - T$ we need to find an open set $D \subseteq C(\overline{\Omega}) \times C(\overline{\Omega})$ with $0 \notin (I - T)(\partial D)$. Further we would like to show that there is a solution to (1), (2) in D which is different from the solution that exists as a consequence of Theorem 1. This is the purpose of the next result. For the rest of this section let u_λ be the solution to (3), (4) defined in Theorem 1.

PROPOSITION 3. *Fix* λ_1 *so that* $0 < \lambda_1 < \lambda_0$ *and fix* $\mu, \nu < \infty$. *Then there exist constants* C, α_1, *and a function* w *such that, for* $\lambda_1 < \lambda$ *and* $0 < \alpha < \alpha_1$: *whenever* (u, v) *is a solution of* (1), (2) *with*

$$(\forall x \in \overline{\Omega}) \qquad Cw(x) \leqslant u(x),$$

we have

$$(\forall x \in \overline{\Omega}) \qquad Cw(x) < u(x)$$

and

$$\|u_\lambda\|_0 < C\|w\|_0.$$

PROOF. Here we will only present the proof for the case when $\mu > \nu$, and will discuss the case $\mu \leqslant \nu$ at the end of this proof.

Choose a nonnegative function $f \in C^2(\bar{\Omega})$ which has its nonempty support in Ω. Let w be the unique (positive) solution of

$$\left.\begin{aligned} \nabla^2 w + f = 0 \quad & \text{in } \Omega, \\ \frac{\partial w}{\partial n} + \mu w = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{7}$$

Choose C so that it satisfies the inequalities

$$(\forall x \in \bar{\Omega}) \ \frac{\nu}{\mu} e^{Cw(x)} - \frac{Cf(x)}{\lambda_1} \geqslant 1, \tag{8}$$

and, for $\lambda_1 \leqslant \lambda \leqslant \lambda_0$,

$$C\|w\|_0 > \|u_\lambda\|_0. \tag{9}$$

Since $(\forall \lambda \in [\lambda_1, \lambda_0]) u_\lambda \leqslant u_{\lambda_0}$ on $\bar{\Omega}$, and $(\forall x \in \bar{\Omega}) w(x) > 0$, it is easily seen that such a choice of C is possible. Also, choose α_1 so that, on $\bar{\Omega}$,

$$\alpha_1 C w e^{Cw} < 1. \tag{10}$$

Then it follows that

$$Cw < 1/\alpha_1. \tag{11}$$

Now, suppose $\lambda > \lambda_1$, $\alpha \in (0, \alpha_1]$ and (u, v) is a solution of (1), (2) with $(\forall x \in \bar{\Omega})\ Cw(x) \leqslant u(x)$. Then $h = \alpha u + v$ satisfies the equation

$$\left.\begin{aligned} \nabla^2 h = 0 \quad & \text{in } \Omega, \\ \frac{\partial h}{\partial n} + \mu h = (\mu - \nu) v \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{12}$$

Since $(\forall x \in \bar{\Omega}) - 1 \leqslant v(x) \leqslant 0$, it follows from the maximum principle that

$$(\forall x \in \bar{\Omega}) \qquad -1 + \nu/\mu \leqslant h(x) \leqslant 0. \tag{13}$$

Using (1) we then see that $\varphi = u$ is the unique solution of

$$\left.\begin{aligned} \nabla^2 \varphi + \lambda(1 + h - \alpha\varphi) e^u = 0 \quad & \text{in } \Omega, \\ \frac{\partial \varphi}{\partial n} + \mu\varphi = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{14}$$

We shall now show that Cw is a strict lower solution for the equation (14):

$$\begin{aligned} \nabla^2(Cw) + \lambda(1 + h - \alpha Cw) e^u &= -Cf + \lambda(1 + h - \alpha Cw) e^u \\ &\geqslant \lambda\left[-Cf/\lambda + (\nu/\mu - \alpha_1 Cw) e^{Cw}\right] \\ &\geqslant \lambda\left[-Cf/\lambda_1 + (\nu/\mu) e^{Cw} - \alpha_1 C w e^{Cw}\right] \\ &> \lambda[1 - 1] = 0 \quad \text{in } \Omega. \end{aligned}$$

Also,

$$\frac{\partial(Cw)}{\partial n} + \mu(Cw) = 0 \quad \text{on } \partial\Omega.$$

Hence Cw is a strict lower solution for (14). Now

$$\nabla^2(1/\alpha) + \lambda(1 + h - \alpha(1/\alpha))e^u = \lambda h e^u \leqslant 0 \quad \text{in } \Omega,$$

and

$$\frac{\partial(1/\alpha)}{\partial n} + \mu(1/\alpha) = \mu/\alpha > 0 \quad \text{on } \partial\Omega,$$

so $\varphi(x) = 1/\alpha$ is an upper solution for (14). It then follows from (11) and the theory of upper and lower solutions that the unique solution $\varphi = u$ of (14) satisfies

$$(\forall x \in \bar{\Omega}) \qquad Cw(x) < u(x) < 1/\alpha.$$

For the case when $\mu \leqslant \nu$ the proof is essentially the same. The differences are:

(a) (13) becomes $(\forall x \in \bar{\Omega}) \quad 0 \leqslant h(x) \leqslant \nu/\mu - 1;$ (13′)

(b) (8) becomes $e^{Cw} - Cf/\lambda_1 \geqslant 1;$ (8′)

(c) $\varphi(x) = \nu/(\alpha\mu)$ is used as an upper solution for (14). *Q.E.D.*

This then leads us to our main result concerning multiple solutions.

THEOREM 4. *Suppose μ, $\nu < \infty$ and $\lambda_1 \in (0, \lambda_0)$. Then there exists $\alpha_2 > 0$ such that, for $\alpha \in (0, \alpha_2]$ and $\lambda \in [\lambda_1, \lambda_0]$, (1),(2) has at least two distinct solutions.*

PROOF. We present the proof for the case when $\mu \geqslant \nu$ only, as the proof for the other case, $\mu < \nu$, is similar and is left to the reader. Let C, w be as in Proposition 3 ((7), (8) and (9)), and choose α_2 so that it satisfies

$$\left.\begin{array}{r} \alpha_2 \leqslant \min\{\alpha_0, \alpha_1\}, \\ \text{and } (\forall x \in \bar{\Omega}) \quad \alpha_2 Cw(x) < \frac{1}{2}(\nu/\mu), \end{array}\right\} \tag{15}$$

where α_0 and α_1 are given in Theorem 1 and Proposition 3 respectively.

Now, by Theorem 2(3), there exists $\lambda_2 > \lambda_0$ such that (1), (2) has a unique solution (U, V) when $\lambda = \lambda_2$ and $\alpha = \alpha_2$, and

$$\left.\begin{array}{r} (\forall x \in \bar{\Omega}) \quad U(x) \geqslant \left(\nu/\mu - A(\lambda_2\alpha_2)^{-1/2}\right)/\alpha_2 \\ \text{also } (\forall x \in \bar{\Omega}) \quad \left(\nu/\mu - A(\lambda_2\alpha_2)^{-1/2}\right)/\alpha_2 > Cw(x). \end{array}\right\} \tag{16}$$

(This last inequality follows from (15), by choosing λ_2 large enough.) Fix $\alpha \in (0, \alpha_2]$ and $\lambda \in [\lambda_1, \lambda_0]$, and set

$$B = \{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}): \\ (\forall x \in \bar{\Omega}) Cw(x) < u(x) < 1/\alpha + 1, \text{and } -2 < v(x) < 1\}.$$

Now, it is clear that B is an open subset of $C(\bar{\Omega}) \times C(\bar{\Omega})$ and

$$\bar{B} = \{(u, v) \in C(\bar{\Omega}) \times C(\bar{\Omega}):$$
$$(\forall x \in \bar{\Omega}) Cw(x) \leqslant u(x) \leqslant 1/\alpha + 1 \text{ and } -2 \leqslant v(x) \leqslant 1\}.$$

Let T be the map defined by (5) and suppose $H: \bar{B} \times [0,1] \to C(\bar{\Omega}) \times C(\bar{\Omega})$ is the homotopy given by

$$H((u, v), t) = (t + (1 - t)\lambda_2/\lambda) T(u, v).$$

We shall now show that H satisfies the conditions of Theorem 2(1)(c). First, it is clear that, for each $t \in [0,1]$, $H(\cdot, t)$ is compact since T is a compact map (see Theorem 1 of Section 1). Also as $\bar{B}$ is a bounded set, there exists a constant M such that

$$(\forall (u, v) \in \bar{B}) \quad \|T(u, v)\|_0 \leqslant M.$$

Hence

$$(\forall t, s \in [0,1]), (\forall (u, v) \in \bar{B}) \|H((u, v), t) - H((u, v), s)\|_0$$
$$= |t - s| |1 - \lambda_2/\lambda| \|T(u, v)\|_0 \leqslant M |1 - \lambda_2/\lambda| |t - s|.$$

So, it is easily seen that conditions (i) and (ii) of Theorem 2(1)(c) are satisfied.

Now, as noted in Theorem 1, if (u, v) is a fixed point of T then (u, v) satisfy (1), (2). Similarly, if $H((u, v), t) = (u, v)$ for some $t \in [0,1]$ then (u, v) satisfy

$$\left.\begin{aligned} \nabla^2 u + (\lambda t + (1 - t)\lambda_2)(1 + v) e^u = 0 \quad & \text{in } \Omega, \\ \nabla^2 v - (\lambda t + (1 - t)\lambda_2)\alpha(1 + v) e^u = 0 \quad & \text{in } \Omega, \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} \partial u/\partial n + \mu u = 0 \quad & \text{on } \partial\Omega, \\ \partial v/\partial n + \nu v = 0 \quad & \text{on } \partial\Omega. \end{aligned}\right\} \tag{18}$$

Suppose, for some $t \in [0,1]$ and $(u, v) \in \bar{B}$, $H((u, v), t) = (u, v)$. Then, since $\lambda t + (1 - t)\lambda_2 \in [\lambda_1, \lambda_2]$ and

$$(\forall x \in \bar{\Omega}) \quad Cw(x) \leqslant u(x),$$

it follows from Proposition 3 that

$$(\forall x \in \bar{\Omega}) \quad Cw(x) < u(x).$$

Also, by Section 1, Theorem 1(4),

$$(\forall x \in \bar{\Omega}) \quad u(x) \leqslant 1/\alpha, -1 \leqslant v(x) \leqslant 0.$$

Hence $(u, v) \in B$.

Consequently, we see that, for $t \in [0,1]$,

$$(0,0) \notin (I - H(\cdot, t))(\partial B).$$

So, the conditions of Theorem 2(1)(c) are satisfied. Therefore

$$\begin{aligned} d(I - T, B, 0) &= d(I - H(\cdot, 1), B, 0) \\ &= d(I - H(\cdot, 0), B, 0). \end{aligned} \tag{19}$$

Now, for $(u, v) \in B$,

$$H((u, v), 0) = \big(\lambda_2 K_\mu((1+v)e^u), -\lambda_2 \alpha K_\nu((1+v)e^u)\big).$$

So the fixed points of $H(\cdot, 0)$ are the solutions of (1), (2) with $\lambda = \lambda_2$. Then it follows from (16) and Theorem 1(4) that $(U, V) \in B$. Also, by Theorem 2(1)(d), (2),

$$d(I - H(\cdot, 0), B, 0) = d(I - H(\cdot, 0), B(2 + m(\alpha)), 0) = 1,$$

and, so, by Theorem 3(1)(b), there exists $(u_1, v_1) \in B$ which satisfy equation (1), (2).

Since $\alpha_2 \leqslant \alpha_0$ and $\lambda \leqslant \lambda_0$, it follows from Theorem 1 that there exists a solution (u_2, v_2) of (1), (2) such that

$$(\forall x \in \bar{\Omega}) \quad u_2(x) \leqslant u_\lambda(x).$$

Then, as $(u_1, v_1) \in B$, it follows from (9) that

$$(\forall x \in \bar{\Omega}) \quad \|u_2\|_0 \leqslant \|u_\lambda\|_0 < C\|w\|_0 \leqslant \|u_1\|_0.$$

Hence $u_1 \neq u_2$ and (u_1, v_1) and (u_2, v_2) are distinct solutions of (1), (2). *Q.E.D.*

3. Uniqueness of solutions for large λ

As stated in Section 1, Theorem 1, we investigated in Part II (Burnell, Lacey and Wake [2]) situations for which the equations (1), (2) will have a unique solution. Since completing that paper we have succeeded in extending the results given there. It is the purpose of the next two sections to look at these extensions. In particular we shall prove that if $\mu = \nu = \infty$ then (1), (2) have a unique solution for sufficiently large values of λ; and if α is large enough then (1), (2) will have a unique solution for all values of the other parameters except when $\mu < \nu = \infty$.

Firstly we shall look at the case when $\mu = \nu = \infty$. Here we see that the harmonic function $h = \alpha u + v$ is zero and (1), (2) reduce to the single equation

$$\left.\begin{aligned} \nabla^2 u + \lambda(1 - \alpha u)e^u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}\right\} \tag{20}$$

In fact we shall consider the more general equation

$$\left.\begin{aligned} \nabla^2 u + \lambda f(u) &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{21}$$

where $f\colon \mathbf{R} \to \mathbf{R}\colon t \mapsto (1 - t)g(t)$ and $g\colon \mathbf{R} \to \mathbf{R}$ satisfies

(a) $(\forall t \in [0, 1]) 1 \geqslant g(t) \geqslant \gamma_0 > 0$,

(b) g is a C^2 function.

(c) $(\forall t \in \mathbf{R})\ g(t) > 0$.

Now, it follows easily from the maximum principle that any solution of (21), u say, satisfies

$$(\forall x \in \overline{\Omega}) \qquad 0 \leqslant u(x) \leqslant 1.$$

Further, $u = 0$ and $u(x) = 1$ are lower and upper solutions of (21) respectively; hence, for any $\lambda \geqslant 0$, (21) has at least one solution. The first step in showing that (21) has a unique solution for sufficiently large values of λ is to find bounds on the solution of (21) when λ is large. For this purpose, we require properties of the solution of the following initial value problem,

$$\left.\begin{aligned} &z'' + f(z) = 0 \quad \text{on } (0, \infty), \\ &z(0) = 0, \qquad z'(0) = \sqrt{2\int_0^1 f(t)\,dt}\,. \end{aligned}\right\} \tag{22}$$

PROPOSITION 1. *Equation* (22) *has a unique solution* z, *and* $(\forall s \in [0, \infty))$

$$1 - \exp\left(-\sqrt{\gamma_0}\, s\right) \leqslant z(s) < 1.$$

PROOF. Now, (22) has at most one solution. Suppose (22) has a solution z; then

$$z'(s)^2 = z'(0)^2 - 2\int_0^{z(s)} f(t)\,dt. \tag{23}$$

If there exists $s_1 \in (0, \infty)$ for which $z(s_1) = 1$ then (from (23)) $z'(s_1) = 0$. Hence, on $[0, s_1]$, z satisfies

$$\begin{gathered} z'' + f(z) = 0, \\ z(s_1) = 1, \qquad z'(s_1) = 0; \end{gathered}$$

but the function $z_1(s) = 1$ is the unique solution of this equation. It then follows from (23) that $z'(s) > 0$ on $(0, \infty)$, and so we must have

$$(\forall s \in [0, \infty)) \quad 0 \leqslant z(s) < 1.$$

This then means that (22) is equivalent to the problem

$$\begin{gathered} z'' + f_1(z) = 0 \quad \text{on } (0, \infty), \\ z(0) = 0, \qquad z'(0) = \sqrt{2\int_0^1 f(s)\,ds}\,, \end{gathered}$$

where f_1 is a suitable bounded function which agrees with f on $[0, 1]$. And, from the theory of ordinary differential equations, we know that this problem has a solution defined on $[0, \infty)$.

Thus we have shown that (22) has a unique solution z which satisfies $(\forall s \in [0, \infty))$ $0 \leqslant z(s) < 1$. Therefore, $(\forall s \in (0, \infty))$ $z''(s) < 0$; so z' is decreasing on $(0, \infty)$. Clearly this means that $z'(s) \to 0$ as $s \to \infty$. It then follows from (23) that $z(s) \to 1$ as $s \to \infty$.

Consider the boundary-value problem

$$\varphi'' + (1-\varphi)g(z) = 0 \quad \text{in } (0,\infty),$$
$$\varphi(0) = 0, \qquad \varphi(s) \to 1 \quad \text{as } s \to \infty;$$

clearly $\varphi = z$ is the unique solution of this problem. If we take the function $\varphi_1(s) = 1 - \exp(-\sqrt{\gamma_0}\, s)$ then

$$\varphi_1'' + (1-\varphi_1)g(z) = \left(-\gamma_0 + g(z)\right)\exp\left(-\sqrt{\gamma_0}\, s\right) > 0$$

and

$$\varphi_1(0) = 0, \qquad \varphi_1(s) \to 1 \quad \text{as } s \to \infty.$$

Hence φ_1 is a lower solution for this problem. Similarly $\varphi_2(s) = 1$ is an upper solution; therefore we must have

$$(\forall s \in [0,\infty)) \quad 1 - \exp\left(-\sqrt{\gamma_0}\, s\right) \leqslant z(s) < 1. \qquad Q.E.D.$$

Now $\partial\Omega$ is of class $C^{2+\sigma}$, hence $\partial\Omega$ satisfies a uniform interior sphere condition. That is, there exists $\rho > 0$ such that, for each $x \in \partial\Omega$, there is an open ball of radius ρ, B_x, such that $B_x \subseteq \Omega$ and $x \in \partial B_x$. Then we have:

THEOREM 2. *Suppose u is a solution of* (21); *then there exist constants A, B such that, for λ sufficiently large,*

$$1 - \exp(-A\lambda^{1/2}) \leqslant u(x) \leqslant 1 \quad \textit{when } d(x,\partial\Omega) \geqslant \rho/2,$$
$$z(\lambda^{1/2}y) - B/\lambda^{1/2} \leqslant u(x) \leqslant z(\lambda^{1/2}y) + B/\lambda^{1/2} \quad \textit{when } y = d(x,\partial\Omega) \leqslant \rho/2,$$

where z is the unique solution of (22) *and $d(x,\partial\Omega)$ is the distance from x to the boundary $\partial\Omega$.*

PROOF. The theorem is proved by constructing suitable upper and lower solutions for the equation

$$\left.\begin{aligned} \nabla^2\varphi + \lambda(1-\varphi)g(u) &= 0 \qquad \text{in } \Omega,\\ \varphi &= 0 \qquad \text{on } \partial\Omega, \end{aligned}\right\} \tag{24}$$

where u is a solution of (21).

Choose $k > 0$ so that the solution w of

$$\begin{aligned} \nabla^2 w + k &= 0 \qquad \text{in } \Omega,\\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

satisfies $(\forall x \in \bar{\Omega})\ |\nabla w(x)| \leqslant \gamma_0/2$. Let us now show that $u_1 = 1 - \exp[-\lambda^{1/2}w]$ is a lower solution for (24) for λ sufficiently large:

$$\begin{aligned} \nabla^2 u_1 + \lambda(1-u_1)g(u) &= \sqrt{\lambda}(\nabla^2 w)e^{-\sqrt{\lambda}w} - \lambda|\nabla w|^2 e^{-\sqrt{\lambda}w} + \lambda e^{-\sqrt{\lambda}w}g(u)\\ &= \sqrt{\lambda}e^{-\sqrt{\lambda}w}\left[-k - \sqrt{\lambda}|\nabla w|^2 + \sqrt{\lambda}g(u)\right]\\ &> \sqrt{\lambda}e^{-\sqrt{\lambda}w}\left[-k - \gamma_0\sqrt{\lambda}/2 + \gamma_0\sqrt{\lambda}\right]\\ &> 0 \end{aligned}$$

if $\sqrt{\lambda} > 2k/\gamma_0$. Also $u_1 = 0$ on $\partial\Omega$; hence u_1 is a lower solution for (24) if $\lambda > 4k^2/\gamma_0^2$.

Now, $u_2(x) = 1$ is an upper solution for (24) and $(\forall x \in \bar{\Omega})\ u_1(x) < u_2(x)$. Therefore (24) has a solution φ which satisfies $(\forall x \in \bar{\Omega})\ u_1(x) \leqslant \varphi(x) \leqslant 1$, if $\lambda > 4k^2/\gamma_0^2$. Since $\varphi = u$ is the unique solution of (24), we find that, for $\lambda > 4k^2/\gamma_0^2$,

$$(\forall x \in \bar{\Omega}) \quad 1 - \exp\left(-\lambda^{1/2} w(x)\right) \leqslant u(x) \leqslant 1.$$

If $\Omega' = \{x \in \Omega:\ d(x, \partial\Omega) \geqslant \rho/2\}$ then Ω' is closed and bounded. Also, it follows from the maximum principle that there exists a constant A such that

$$(\forall x \in \bar{\Omega}) \quad w(x) \geqslant A > 0.$$

Thus, when $\lambda > 4k^2/\gamma_0^2$,

$$1 - \exp(-A\lambda^{1/2}) \leqslant u(x) \leqslant 1 \quad \text{if } d(x, \partial\Omega) \geqslant \rho/2. \tag{25}$$

Now, let us find upper and lower solutions which allow us to show that the second inequality holds. Let $T_\rho = \{x \in \Omega:\ d(x, \partial\Omega) < \rho/2\}$; then T_ρ is open and bounded. We define a function w on T_ρ as follows:

$$w(x) = z\left(\lambda^{1/2} d(x, \partial\Omega)\right).$$

We then see that on T_ρ,

$$\nabla^2 w(x) = \lambda z''(\lambda^{1/2} y) + \lambda^{1/2} (\nabla \cdot \boldsymbol{\eta}) z'(\lambda^{1/2} y), \tag{26}$$

where $y = d(x, \partial\Omega)$ and $\boldsymbol{\eta}$ is the unit normal vector to the surface $\{y:\ d(y, \partial\Omega) = d(x, \partial\Omega)\}$. Since $\partial\Omega$ is of class $C^{2+\sigma}$ there exists a constant M such that

$$(\forall x \in T_\rho) \quad |\nabla \cdot \boldsymbol{\eta}(x)| \leqslant M.$$

We now show that the function $w(x) - B/\lambda^{1/2}$ is a lower solution for the equation

$$\left.\begin{aligned} \nabla^2\varphi + \lambda(1 - \varphi) g(u) &= 0 \quad \text{in } T_\rho, \\ \varphi &= u \quad \text{on } \partial T_\rho, \end{aligned}\right\} \tag{27}$$

if λ is sufficiently large.

Using properties of z from Proposition 1 we have, on T_ρ,

$$\begin{aligned} &\nabla^2\left(w - B/\lambda^{1/2}\right) + \lambda\left(1 - \left(w - B/\lambda^{1/2}\right)\right) g(u) \\ &\quad = \lambda z''(\lambda^{1/2} y) + \lambda^{1/2} (\nabla \cdot \boldsymbol{\eta}) z'(\lambda^{1/2} y) + \lambda\left(1 - z(\lambda^{1/2} y) + B/\lambda^{1/2}\right) g(u) \\ &\quad = \lambda\left(1 - z(\lambda^{1/2} y)\right)\left[g(u) - g\left(z(\lambda^{1/2} y)\right)\right] \\ &\qquad + \lambda^{1/2} (\nabla \cdot \boldsymbol{\eta}) z'(\lambda^{1/2} y) + \lambda^{1/2} B g(u) \\ &\quad \geqslant \lambda^{1/2}\left[-\lambda^{1/2} \exp\left(-(\gamma_0\lambda)^{1/2} y\right)\left|g(u) - g\left(z(\lambda^{1/2} y)\right)\right| - Mz'(0) + B\gamma_0\right]. \end{aligned}$$

Now, since $g(u) - g(z(\lambda^{1/2}y))$ is a bounded function of x and, when $y = d(x, \partial\Omega) = 0$, $g(u) - g(z(\lambda^{1/2}y)) = 0$, it follows that there exists a constant M_1 with $\lambda^{1/2}\exp(-(\gamma_0\lambda)^{1/2}y)|g(u) - g(z(\lambda^{1/2}y))| < M_1$ for sufficiently large λ and for $x \in T_\rho$. Thus if $B = (M_1 + Mz'(0))/\gamma_0$ then for λ sufficiently large,

$$\nabla^2(w - B/\lambda^{1/2}) + \lambda(1 - (w - B/\lambda^{1/2}))g(u) \geqslant 0 \quad \text{on } T_\rho.$$

Also, when $x \in \partial T_\rho$ we either have

(a) $d(x, \partial\Omega) = 0$ in which case $w(x) - B/\lambda^{1/2} = z(0) - B/\lambda^{1/2} = -B/\lambda^{1/2} < u(x)$, since $u \geqslant 0$ on $\bar{\Omega}$; or

(b) $d(x, \partial\Omega) = \rho/2$ in which case, if $\lambda^{1/2}e^{-A\lambda^{1/2}} < B$ and $\lambda > 4k^2/\gamma_0^2$,

$$\begin{aligned} w(x) - B/\lambda^{1/2} &= z(\lambda^{1/2}\rho/2) - B/\lambda^{1/2} \\ &< 1 - B/\lambda^{1/2} \quad \text{by Proposition 1} \\ &< 1 - e^{-A\lambda^{1/2}} \\ &\leqslant u(x) \quad \text{by (25).} \end{aligned}$$

Therefore, when λ is sufficiently large, $w - B/\lambda^{1/2}$ is a lower solution for (27).

A similar argument shows that the function $w(x) + B/\lambda^{1/2}$ is an upper solution for (27) when λ is sufficiently large. Consequently, the unique solution $\varphi = u$ of (27) must satisfy

$$(\forall x \in T_\rho) \quad z(\lambda^{1/2}y) - B/\lambda^{1/2} \leqslant u(x) \leqslant z(\lambda^{1/2}y) + B/\lambda^{1/2}, \tag{28}$$

where $y = d(x, \partial\Omega)$. *Q.E.D.*

We shall later require the following result. For the rest of this paper we shall let z be the unique solution of (22).

LEMMA 3. *Suppose ψ is the unique solution of*

$$\left.\begin{aligned} \frac{d^2\psi}{ds^2} + f'(z(s))\psi = 0, &\qquad s \in (0, \infty), \\ \psi(0) = 0, &\qquad \psi'(0) = 1. \end{aligned}\right\} \tag{29}$$

Then there exists $X > 0$ and $\Pi_1 < 0$ such that

$$\psi'(X) > 0$$

and $(\forall s \geqslant X) f'(z(s)) \leqslant \Pi_1$.

PROOF. Firstly, the unique solution of (29) is

$$\psi(s) = z'(0)z'(s)\int_0^s \frac{dt}{z'(t)^2}.$$

Also, using (22), (23) and Proposition 1

$$z'(s)^2 = 2\int_{z(s)}^1 f(t)\,dt \leqslant 2\int_{1-\exp(-\sqrt{\gamma_0}s)}^1 f(t)\,dt$$
$$\leqslant 2\exp\left(-\sqrt{\gamma_0}\,s\right).$$

Hence

$$\int_0^s \frac{dt}{z'(t)^2} \geqslant \frac{1}{2\sqrt{\gamma_0}}\left(\exp\left(\sqrt{\gamma_0}\,s\right) - 1\right) \to \infty \qquad \text{as } s \to \infty.$$

Accordingly, using L'Hôpital's rule on the above gives that $\psi(s) \to \infty$ as $s \to \infty$. The result then follows from the properties of z given in Proposition 1. *Q.E.D.*

The approach that we now take is to assume that (21) does not have a unique solution for some value of λ, and to then show that this leads to a contradiction if λ is sufficiently large. If (21) does not have a unique solution then the minimal and maximal solutions of (21), u_m and u_M respectively must satisfy (see Keller and Cohen [3])

$$(\forall x \in \bar{\Omega}) \quad u_m(x) \leqslant u_M(x).$$

Since $\nabla^2(u_M - u_m) + \lambda[f(u_M) - f(u_m)] = 0$, it follows from the mean value theorem that there exists a function w_0 such that:

(a) $(\forall x \in \bar{\Omega}) \quad u_m(x) \leqslant w_0(x) \leqslant u_M(x)$;

(b) the function $f'(w_0)$ is continuous;

(c) there is a positive function φ which satisfies

$$\begin{aligned} \nabla^2\varphi + \lambda f'(w_0)\varphi &= 0 \quad \text{in } \Omega, \\ \varphi &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{30}$$

Now, for λ sufficiently large and $s \in (0, X]$, the open set $\Omega_s = \{x \in \Omega\colon d(x, \partial\Omega) > s/\lambda^{1/2}\}$ has a $C^{2+\sigma}$ boundary $S_s = \{x \in \bar{\Omega}\colon d(x, \partial\Omega) = s/\lambda^{1/2}\}$. Then define a function I: $[0, X] \to \mathbf{R}$ by

$$I(s) = \int_{S_s} \varphi\, dS$$

where φ is a positive solution of (30) and X is as found in Lemma 3. The function I then satisfies the differential equation given in the following Proposition.

PROPOSITION 4. *If λ is sufficiently large then I is a C^2 function and*

$$\frac{d}{ds}\left(\frac{dI}{ds} + \frac{a_2(s)}{\lambda^{1/2}} I\right) + \left(f'(z(s)) + \frac{a_1(s)}{\lambda^{1/2}}\right) I = 0, \qquad s \in (0, X),$$

where a_2 is a C^1 function. Further, there exist constants A_1, A_2 (independent of λ) such that

$$(\forall s \in (0, X]) \quad |a_1(s)| \leqslant A_1,\ |a_2(s)| \leqslant A_2.$$

PROOF. Now, for $s \in [0, X]$ and $x \in S_s$, let $\mathbf{n}(x)$ denote the unit normal vector to the surface S_s at the point x. Since S_s is of class $C^{2+\sigma}$, the function $\mathbf{n}$ is of class $C^{2+\sigma}$. Then $\mathbf{n}$ can be extended to the whole of $\bar{\Omega}$ so that it remains a $C^{2+\sigma}$ function. Consequently, making use of the various forms of the divergence theorem, we have:

$$\begin{aligned} I(s) &= \int_{S_s} (\varphi\mathbf{n} \cdot \mathbf{n})\, dS \\ &= \int_{\Omega_s} \nabla \cdot (\varphi\mathbf{n})\, dx \\ &= \int_{\Omega_s} [(\nabla\varphi) \cdot \mathbf{n} + \varphi(\nabla \cdot \mathbf{n})]\, dx. \end{aligned}$$

Then, using the fact that, on S_s, $s = \lambda^{1/2} d(x, \partial\Omega)$,

$$\begin{aligned} \frac{dI}{ds} &= \frac{d}{ds}\int_{\Omega_s} [\nabla\varphi \cdot \mathbf{n} + \varphi(\nabla \cdot \mathbf{n})]\, dx \\ &= -\lambda^{-1/2}\int_{S_s} [\nabla\varphi \cdot \mathbf{n} + \varphi(\nabla \cdot \mathbf{n})]\, dS \\ &= \lambda^{-1/2}\int_{S_s} \frac{\partial\varphi}{\partial n}\, dS - \lambda^{-1/2}\int_{S_s} \varphi(\nabla \cdot \mathbf{n})\, dS. \end{aligned} \tag{31}$$

Also, using (30), and the divergence theorem again

$$\int_{S_s} \frac{\partial\varphi}{\partial n}\, dS = \int_{\Omega_s} \nabla^2\varphi\, dx = \int_{\Omega_s} -\lambda f'(w_0)\varphi\, dx, \tag{32}$$

and so by considering $\int_{S_{s+h}} (\partial\varphi/\partial n)\, dS - \int_{S_s} (\partial\varphi/\partial n)\, dS$ we obtain

$$\frac{d}{ds}\int_{S_s} \frac{\partial\varphi}{\partial n}\, dS = \lambda^{1/2}\int_{S_s} f'(w_0)\varphi\, dS.$$

Therefore using (31) we obtain

$$\frac{d}{ds}\left[\frac{dI}{ds} + \lambda^{-1/2}\int_{S_s} \varphi(\nabla \cdot \mathbf{n})\, dS\right] = -\int_{S_s} f'(w_0)\varphi\, dS.$$

Since $(\forall x \in \bar{\Omega})\ 0 \leqslant u_m(x) \leqslant w_0(x) \leqslant u_M(x) \leqslant 1$, $f'(w_0)$ is bounded on Ω. In particular, for $s \in (0, X]$, there exist $x_1, x_2, \in S_s$ such that

$$(\forall x \in S_s) \quad f'(w_0(x_1)) \leqslant f'(w_0(x)) \leqslant f'(w_0(x_2)).$$

Thus, since φ is positive on Ω, we have, for $s > 0$,

$$f'(w_0(x_1)) \leqslant \left(\int_{S_s} f'(w_0)\varphi\, dS\right) \Big/ \left(\int_{S_s} \varphi\, dS\right) \leqslant f'(w_0(x_2)).$$

https://doi.org/10.1017/S0334270000004781 Published online by Cambridge University Press

If λ is sufficiently large then Theorem 2 holds and for $x \in S_s$, $d(x, \partial\Omega) = s/\lambda^{1/2} \leqslant \rho/2$. Since $u_m(x)$ and $u_M(x)$ satisfy (28),

$$z(s) - B/\lambda^{1/2} \leqslant w_0(x) \leqslant z(s) + B/\lambda^{1/2}.$$

Consequently, as f'' is bounded on [0, 1], there exists a constant A_1 such that, for $i = 1, 2$,

$$|f'(z(s)) - f'(w_0(x_i))| \leqslant A_1/\lambda^{1/2}.$$

Let $a_1(s) = \lambda^{1/2}(\int_{S_s} f'(w_0)\varphi\, dS/(\int_{S_s} \varphi\, dS) - f'(z(s)))$; then $|a_1(s)| \leqslant A_1$ and a_1 is a C^1 function. Then $\int_{S_s} f'(w_0)\varphi\, dS = [f'(z(s)) + a_1(s)/\lambda^{1/2}]I(s)$. Similar arguments show that

$$\int_{S_s} \varphi(\nabla \cdot \mathbf{n})\, dS = a_2(s)I(s),$$

where a_2 is a C^1 function and, for $s \in (0, X]$

$$|a_2(s)| \leqslant A_2$$

(A_2 is independent of λ). Therefore

$$\frac{d}{ds}\left(\frac{dI}{ds} + a_2(s)I/\lambda^{1/2}\right) + \left(f'(z(s)) + a_1(s)/\lambda^{1/2}\right)I = 0, \qquad s \in (0, X).$$

Q.E.D.

Now, if φ is a solution of (30) which is positive on Ω then the above result shows that $I'(0) > 0$. Hence, it is possible to choose such a φ so that I satisfies the differential equation of Proposition 4 with

$$I(0) = 0, \qquad I'(0) = 1,$$

and further, $(\forall s > 0)\ I(s) > 0$.

LEMMA 5. *Suppose λ is sufficiently large and I is the unique solution of the differential equation in Proposition 4 with the initial conditions*

$$I(0) = 0, \qquad I'(0) = 1.$$

Then, there exist constants Γ, Γ_1 (independent of λ) such that

$$(\forall s \in [0, X]) \quad I(s) \leqslant \Gamma$$
$$I'(s) \leqslant \Gamma_1.$$

PROOF. Now, I satisfies the integro-differential equation

$$\frac{dI}{ds} + \left(a_2(s)/\sqrt{\lambda}\right)I(s) = -\int_0^s \left(f'(z(t)) + a_1(t)/\sqrt{\lambda}\right)I(t)\, dt + 1 \qquad (33)$$

with $I(0) = 0$. Since $0 \leqslant z \leqslant 1$ on $\mathbf{R}$, there exists a constant $M_1 > 1$ such that

$$(\forall s \in \mathbf{R}) \quad |f'(z(s))| < M_1.$$

We now show that, for λ sufficiently large,

$$(\forall s \in [0, X]) \quad 0 \leqslant I(s) < e^{m_1 s}$$

where $m_1 = (1 + \sqrt{5 + 4M_1})/2$.

Let $I_1(s) = e^{m_1 s}$, since $m_1^2 - m_1 - (1 + M_1) = 0$ we have

$$I_1' - I_1 = (1 + M_1)\int_0^s I_1(t)\,dt + m_1 - 1$$

also

$$I_1(0) = 1$$

and

$$I_1'(0) = m_1 > 2.$$

Consequently $I_1(s) > I(s)$ for s in some neighbourhood of 0. Suppose there exists a point $S \in (0, X]$ such that

$$(\forall s \in [0,)) \quad I(s) < I_1(s) \quad \text{and} \quad I(S) = I_1(S);$$

then, since $I(s) > 0$ for $s \in (0, X]$, $I'(S) \geqslant I_1'(S)$. However, from (33), if λ is sufficiently large,

$$\begin{aligned} I'(S) &= -\left(a_2(S)/\sqrt{\lambda}\right)I(S) - \int_0^S \left[f'(z(t)) + a_1(t)/\sqrt{\lambda}\right]I(t)\,dt + 1 \\ &< \left(A_2/\sqrt{\lambda}\right)I_1(S) + \int_0^S \left(M_1 + A_1/\sqrt{\lambda}\right)I_1(t)\,dt + m_1 - 1 \\ &< I_1(S) + (M_1 + 1)\int_0^S I_1(t)\,dt + m_1 - 1 \\ &= I_1'(S). \end{aligned}$$

This contradiction means that

$$(\forall s \in [0, X]) \quad I(s) < I_1(s) = e^{m_1 s}.$$

The first inequality in the lemma follows by taking $\Gamma = \exp(m_1 X)$. The second inequality can then be easily derived from (33). *Q.E.D.*

Using these last three results we can show that the function I tends to the solution of (29) as $\lambda \to \infty$. In particular we have:

PROPOSITION 6. *There exists a constant $L > 0$ such that, for λ sufficiently large,*

$$\frac{dI}{ds}(X) \geqslant L.$$

PROOF. From (29) and (33) we see that, for λ sufficiently large,

$$\frac{d}{ds}(\psi - I) + \int_0^s f'(z(t))(\psi(t) - I(t))\,dt = \left(a_2(s)/\sqrt{\lambda}\right)I(s) - \int_0^s \left(a_1(t)/\sqrt{\lambda}\right)I(t)\,dt \tag{34}$$

with $\psi(0) - I(0) = 0$.

Letting $J = \psi - I$ and $J_1(s) = (M/\sqrt{\lambda M_1})\exp(\sqrt{M_1}\,s)$, where $M = (A_2 + XA_1)\Gamma$ and M_1 is an upper bound for $|f'(z)|$ as given in Lemma 6, similar arguments to those of Lemma 5 show that, for λ sufficiently large

$$(\forall s \in [0, X]) \quad J(s) \leqslant J_1(s).$$

Therefore $J(s) \to 0$ uniformly on $[0, X]$ as $\lambda \to \infty$; and so $I(s) \to \psi(s)$ uniformly on $[0, X]$ as $\lambda \to \infty$.

Using this and the boundedness of a_1, a_2 and I, it follows from (34) that

$$(\forall s \in [0, X]) \quad \frac{dI}{ds}(s) \to \frac{d\psi}{ds}(s) \qquad \text{as } \lambda \to \infty.$$

From Lemma 3 we have $(d\psi/ds)(X) > 0$; so for λ sufficiently large

$$\frac{dI}{ds}(X) > \frac{d\psi}{ds}(X)/2. \qquad Q.E.D.$$

We shall use the equation that φ satisfies and the inequalities of Theorem 2 to show that $(dI/ds)(X) \to l$ as $\lambda \to \infty$ and $l \leqslant 0$.

Now, suppose $x \in \Omega_X$ and $d(x, \partial\Omega) = s/\lambda^{1/2} \leqslant \rho/2$; then, by Theorem 2 and the definition of w_0, $|z(s) - w_0(x)| < B/\lambda^{1/2}$. Since $z([X, \infty)) \subseteq (0,1)$ and f' is continuous, it follows that there exists $\delta > 0$ such that

$$(\forall t \in [X, \infty)) \quad |z(t) - r| < \delta \Rightarrow |f'(z(t)) - f'(r)| < \Pi_1/2.$$

In particular $(\forall t \in [X, \infty))\ |z(t) - r| < \delta \Rightarrow f'(r) < 0$. So, if λ is sufficiently large then $|z(s) - w_0(x)| < \delta$ and $f'(w_0(x)) < 0$. That is, for λ sufficiently large, if $x \in \Omega_X$ with $d(x, \partial\Omega) \leqslant \rho/2$ then $f'(w_0(x)) < 0$.

It follows from Theorem 2 that, for $x \in \Omega_X$ with $d(x, \partial\Omega) \geqslant \rho/2$, we have

$$1 - \exp(-A\lambda^{1/2}) < w_0(x) \leqslant 1.$$

Since $f'(1) < 0$, it follows that for λ sufficiently large we have, for any $x \in \Omega_X$ with $d(x, \partial\Omega) \geqslant \rho/2$,

$$f'(w_0(x)) < 0.$$

Consequently we see that for λ sufficiently large,

$$(\forall x \in \Omega_X) \quad f'(w_0(x)) < 0.$$

Therefore, by (32),

$$\int_{S_X} \frac{\partial\varphi}{\partial n}\,dS = -\lambda\int_{\Omega_X} f'(w_0)\varphi\,dx > 0;$$

and by (31)

$$\int_{S_X} \frac{\partial \varphi}{\partial n} dS = -\lambda^{1/2} \frac{dI}{ds}(X) - \int_{S_X} \varphi(\nabla \cdot \mathbf{n})\, dS$$
$$= -\lambda^{1/2}\left[\frac{dI}{ds}(X) + \lambda^{-1/2} a_2(X) I(X)\right].$$

Consequently, we either have

(a) for sufficiently large λ, $(dI/ds)(X) < 0$; or

(b) $(dI/ds)(X) \to 0$ as $\lambda \to \infty$.

In either case we have a contradiction of the result in Proposition 6.

Hence the assumption that (21) has at least two solutions for arbitrarily large values of λ was false. That is (21) has a unique solution for sufficiently large λ.

4. Uniqueness of solutions for large α

In this section we shall show that the equations (1), (2) have a unique solution for all values of λ if α is large enough. Of course we have to exclude the case $\mu < \nu = \infty$, since we showed in Part II (Burnell, Lacey and Wake [2]) that (1), (2) always has multiple solutions for any value of α, if $\mu < \nu = \infty$ and $n = 1$.

Firstly we note that (1), (2) have a solution if and only if the equations

$$\begin{aligned} &\nabla^2 v - \lambda\alpha(1 + v)e^{(1/\alpha)(h-v)} = 0 && \text{in } \Omega, \\ &\frac{\partial v}{\partial n} + \nu v = 0 && \text{on } \partial\Omega, \end{aligned} \tag{35}$$

$$\begin{aligned} &\nabla^2 h = 0 && \text{in } \Omega, \\ &\frac{\partial h}{\partial n} + \mu h = (\mu - \nu) v && \text{on } \partial\Omega, \end{aligned} \tag{36}$$

have a solution. This follows since (u, v) satisfy (1), (2) if and only if $(v, \alpha u + v)$ satisfy (35), (36). Thus we need only show that (35), (36) has a unique solution for α sufficiently large.

Define a map S: $C(\bar{\Omega}) \times C(\bar{\Omega}) \to C(\bar{\Omega}) \times C(\bar{\Omega})$ as follows: $S(v, h) = (v', h')$ where v', h' are the unique solutions of the equations

$$\begin{aligned} &\nabla^2 v' - \lambda\alpha v' = \lambda\alpha(1 + v)e^{(1/\alpha)(h-v)} - \lambda\alpha v && \text{in } \Omega, \\ &\frac{\partial v'}{\partial n} + \nu v' = 0 && \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} &\nabla^2 h' = 0 && \text{in } \Omega, \\ &\frac{\partial h'}{\partial n} + \mu h' = (\mu - \nu) v' && \text{on } \partial\Omega, \end{aligned}$$

https://doi.org/10.1017/S0334270000004781 Published online by Cambridge University Press

respectively. Clearly (v, h) is a fixed point of S if and only if (v, h) is a solution of (35), (36). Further, it follows from Section 1, Theorem 1 and the maximum principle that if (v, h) is a fixed point of S then

$$(\forall x \in \bar{\Omega}) \quad -1 \leqslant v(x) \leqslant 0,$$
$$0 \leqslant h(x) \leqslant \nu/\mu - 1 \quad \text{if } \mu < \nu,$$
$$-1 + \nu/\mu \leqslant h(x) \leqslant 0 \quad \text{if } \mu \geqslant \nu.$$

We prove that (1), (2) have a unique solution for large α by showing that two fixed points of S must satisfy a certain inequality. We derive this inequality from the following result.

Proposition 1. *There exists a constant A (independent of λ, α) such that if*:
(a) $\alpha > \max\{1, \nu/\mu\}$;
(b) $v_1, v_2, h_1, h_2 \in C(\bar{\Omega})$ *and, for* $i = 1, 2$,

$$(\forall x \in \bar{\Omega}) \quad -1 \leqslant v_i(x) \leqslant 0,$$
$$0 \leqslant h_i(x) \leqslant \nu/\mu - 1 \quad \text{if } \mu < \nu,$$
$$-1 + \nu/\mu \leqslant h_i(x) \leqslant 0 \quad \text{if } \mu \geqslant \nu.$$

and
(c) $(v_1', h_1') = S(v_1, h_1), \quad (v_2', h_2') = S(v_2, h_2)$;
then

$$\|v_1' - v_2'\|_0 + \|h_2' - h_2'\|_0 \leqslant (1/\alpha)A\left[\|v_1 - v_2\|_0 + \|h_1 - h_2\|_0\right].$$

Proof. Now, from the definition of S we have

$$\begin{aligned}
\nabla^2(v_1' - v_2') - \lambda\alpha(v_1' - v_2')
&= \lambda\alpha\Big[e^{(1/\alpha)(h_1 - v_1)} - e^{(1/\alpha)(h_2 - v_2)} + v_1\big(e^{(1/\alpha)(h_1 - v_1)} - e^{(1/\alpha)(h_2 - v_2)}\big) \\
&\qquad + (v_1 - v_2)e^{(1/\alpha)(h_2 - v_2)} - (v_1 - v_2)\Big] \\
&= \lambda\alpha\Big[(1 + v_1)\big(e^{(1/\alpha)(h_1 - v_1)} - e^{(1/\alpha)(h_2 - v_2)}\big) \\
&\qquad + (v_1 - v_2)\big(e^{(1/\alpha)(h_2 - v_2)} - 1\big)\Big] \\
&= \lambda\alpha\Big[(1/\alpha)(1 + v_1)e^{\xi}(h_1 - h_2 + v_2 - v_1) \\
&\qquad + (v_1 - v_2)\big(e^{(1/\alpha)(h_2 - v_2)} - 1\big)\Big]
\end{aligned}$$

where ξ is a function with $\xi(x)$ lying between $(1/\alpha)(h_1(x) - v_1(x))$ and $(1/\alpha)(h_2(x) - v_2(x))$, for all $x \in \bar{\Omega}$. Since, for $i = 1, 2$, $(1/\alpha)(h_i - v_i) < \nu/\alpha\mu$ if $\mu < \nu$, and $(1/\alpha)(h_i - v_i) < 1/\alpha$ if $\mu \geqslant \nu$, we have

$$\begin{aligned}
&\Big|\lambda\alpha\Big[(1/\alpha)(1 + v_1)e^{\xi}(h_1 - h_2 + v_2 - v_1) + (v_1 - v_2)\big(e^{(1/\alpha)(h_2 - v_2)} - 1\big)\Big]\Big| \\
&\quad \leqslant \lambda\alpha\Big\{(1/\alpha)|1 + v_1|e^{\xi}\big(|h_1 - h_2| + |v_2 - v_1|\big) + |v_1 - v_2|\big|e^{(1/\alpha)(h_2 - v_2)} - 1\big|\Big\} \\
&\quad \leqslant \lambda\alpha\Big\{(1/\alpha)e^{\beta}\|h_1 - h_2\|_0 + \big((1/\alpha)e^{\beta} + |e^{\beta} - 1|\big)\|v_1 - v_2\|_0\Big\},
\end{aligned}$$

where $\beta = \max\{1/\alpha, \nu/\alpha\mu\}$.

Noting that $|e^{\beta} - 1| < \beta + (\beta^2/2)e^{\beta}$, and applying the maximum principle we see that

$$\begin{aligned}\|v_1' - v_2'\|_0 &\leqslant (1/\lambda\alpha)\big(\lambda\alpha\{(1/\alpha)e^{\beta}\|h_1 - h_2\|_0 \\ &\qquad + \big((1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta}\big)\|(v_1 - v_2)\|_0\}\big) \\ &= (1/\alpha)e^{\beta}\|h_1 - h_2\|_0 + \big((1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta}\big)\|v_1 - v_2\|_0.\end{aligned}$$

Also,

$$\nabla^2(h_1' - h_2') = 0 \quad \text{in } \Omega,$$

$$\frac{\partial(h_1' - h_2')}{\partial n} + \mu(h_1' - h_2') = (\mu - \nu)(v_1' - v_2') \quad \text{on } \partial\Omega,$$

where we note that if $\nu < \mu = \infty$ this boundary condition becomes

$$h_1' - h_2' = v_1' - v_2' \quad \text{on } \partial\Omega.$$

Hence it follows from the maximum principle that

$$\|h_1' - h_2'\|_0 \leqslant l(\mu, \nu)\|v_1' - v_2'\|_0$$

where $l(\mu, \nu) = |\mu - \nu|/\mu$ if $\mu < \infty$, and $l(\mu, \nu) = 1$ if $\mu = \infty$. Consequently

$$\begin{aligned}&\|v_1' - v_2'\|_0 + \|h_1' - h_2'\|_0 \\ &\quad \leqslant (1 + l(\mu, \nu))\{(1/\alpha)e^{\beta}\|h_1 - h_2\|_0 \\ &\qquad\qquad + \big((1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta}\big)\|v_1 - v_2\|_0\} \\ &\quad \leqslant (1 + l(\mu, \nu))\big((1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta}\big)\big[\|h_1 - h_2\|_0 + \|v_1 - v_2\|_0\big].\end{aligned}$$

Since $\alpha > \max\{1, \nu/\mu\}$, we have if $\mu \geqslant \nu$,

$$\begin{aligned}&(1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta} \\ &\quad = (1/\alpha)e^{1/\alpha} + (1/\alpha) + (1/2\alpha^2)e^{1/\alpha} < (1/\alpha)\big[e + 1 + \tfrac{1}{2}e\big];\end{aligned}$$

and if $\mu < \nu$,

$$\begin{aligned}(1/\alpha)e^{\beta} + \beta + (\beta^2/2)e^{\beta} &= (1/\alpha)e^{\nu/\alpha\mu} + (\nu/\alpha\mu) + \tfrac{1}{2}(\nu^2/\alpha^2\mu^2)e^{\nu/\alpha\mu} \\ &< (1/\alpha)\big[e + \nu/\mu + \tfrac{1}{2}(\nu/\mu)e\big].\end{aligned}$$

Therefore the result follows by taking

$$\begin{aligned}A &= (1 + l(\mu, \nu))(1 + \tfrac{3}{2}e) \quad \text{if } \mu \geqslant \nu, \\ &= (1 + l(\mu, \nu))\big(\nu/\mu + (1 + \tfrac{1}{2}\nu/\mu)e\big) \quad \text{if } \mu < \nu.\end{aligned}$$

Q.E.D.

THEOREM 2. *If α is sufficiently large then* (1), (2) *have a unique solution for all values of* λ.

PROOF. Let A be as given in Proposition 1; and suppose $\alpha > \max\{1, \nu/\mu, A\}$. If (u_1, v_1), (u_2, v_2) are solutions of (1), (2) for this value of α and any value of λ

https://doi.org/10.1017/S0334270000004781 Published online by Cambridge University Press

then (v_1, h_1), (v_2, h_2) are fixed points of S, where $h_1 = \alpha u_1 + v_1$, $h_2 = \alpha u_2 + v_2$. Therefore, by Proposition 1,

$$\|h_1 - h_2\|_0 + \|v_1 - v_2\|_0 \leqslant (1/\alpha)A\left[\|v_1 - v_2\|_0 + \|h_1 - h_2\|_0\right]$$
$$< \|v_1 - v_2\|_0 + \|h_1 - h_2\|_0.$$

Clearly this inequality can only be satisfied if $v_1 = v_2$ and $h_1 = h_2$, and hence $u_1 = u_2$. Consequently (1), (2) has a unique solution for any value of λ if $\alpha > \max\{1, \nu/\mu, A\}$. *Q.E.D.*

5. Bounds on the region of multiplicity

Implicit in all the proofs of these results on the uniqueness and nonuniqueness of solutions of (1), (2), given here and in Part II (Burnell, Lacey and Wake [2]), are estimates of the range of values of λ and α for which (1), (2) has multiple solutions. Here we shall state the results that follow from these proofs. Unfortunately, these estimates are rather crude as we made no effort in the proofs to find the best values of the parameters above (or below) which there is a unique (or multiple) solution. However we felt that it would be useful to record all these estimates. Once again we exclude the case $\mu < \nu = \infty$.

For any value of α let λ_b be the infimum of the set of values of λ for which (1), (2) has at least two solutions; and let λ^b denote the supremum of this set. Also, let K_μ, K_ν be the maps defined in Section 2, Theorem 1, and $\|K_\mu\|_0$, $\|K_\nu\|_0$ be their operator norms.

THEOREM 1. (a) $\lambda_b \geqslant e^{-1/\alpha}/(\|K_\mu\|_0 + \alpha\|K_\nu\|_0)$.

(b) *If* $\mu = \nu$ *then* $\lambda_b \geqslant e^{2-1/\alpha}/(\alpha\|K_\mu\|_0)$.

(c) *If* α *is sufficiently small then*

$$\lambda_b \leqslant \frac{\nu/\mu + \sqrt{\nu^2/\mu^2 - 4\alpha\nu/\mu}}{\nu/\mu - \sqrt{\nu^2/\mu^2 - 4\alpha\nu/\mu}} \exp\left\{-\left(\nu/\mu + \sqrt{\nu^2/\mu^2 - 4\alpha\nu/\mu}\right)/2\alpha\right\}\left\|\frac{f}{w}\right\|_0,$$

where $f\colon \bar{\Omega} \to \mathbf{R}$ *is a* C^1 *function with support in* Ω *and* w *is the unique solution of* $\nabla^2 w + f = 0$ *in* Ω, $\partial w/\partial n + \mu w = 0$ *on* $\partial\Omega$.

(d) *If* λ_0 *is the supremum of the set of values of* λ *for which the equation*

$$\left.\begin{aligned} &\nabla^2 u + \lambda e^u = 0 && \text{in } \Omega,\\ &\frac{\partial u}{\partial n} + \mu u = 0 && \text{on } \Omega, \end{aligned}\right\}$$

has a solution and α *is sufficiently small, then* $\lambda^b \geqslant \lambda_0$.

https://doi.org/10.1017/S0334270000004781 Published online by Cambridge University Press

(e) *If α is sufficiently small then*

$$\lambda^b \leqslant \max\left\{\frac{4(2-\nu/\mu)Ae^{(1-\nu/\mu)/\alpha}}{\alpha^3}, \frac{1}{k^2\alpha\left(1-(\|\nabla w_1\|_0)^2\right)^2}\right\},$$

where w_1 satisfies

$$\nabla^2 w_1 + 1/k = 0 \quad \text{in } \Omega,$$
$$w_1 = 0 \quad \text{on } \partial\Omega,$$

with $k > 1$ and $\|\nabla w_1\|_0 < 1$, $A = \nu/\min_{x\in\partial\Omega}\{-(\partial w_1/\partial n)(x)\}$.

References

[1] J. G. Burnell, A. A. Lacey and G. C. Wake, "Steady states of the reaction-diffusion equations. Part I: Questions of existence and continuity of solution branches", *J. Austral. Math. Soc. Ser. B* 24 (1983), 374–391.

[2] J. G. Burnell, A. A. Lacey and G. C. Wake, "Steady states of the reaction-diffusion equations. Part II: Uniqueness of solutions and some special cases", *J. Austral. Math. Soc. Ser. B* 24 (1983), 392–416.

[3] H. B. Keller and D. S. Cohen, "Some positone problems suggested by nonlinear heat generation", *J. Math. Mech.* 16 (1967), 1361–1376.