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GRAPHS WITH CONSTANT RATIO OF MEAN CURVATURES AND SPHERICAL CAPS

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Let f be a smooth nonconstant function defined on an n dimensional ball and zero on the boundary $n-1$ dimensional sphere. It is shown that the graph of f is a spherical cap (1) if f is positive and if the ratio H_k/H_r is a nonzero constant for $1 \leqslant k < r \leqslant n$ on the graph of f or (2) if the ratio H_n/H_k is a nonzero constant on the graph of f.

1. INTRODUCTION

Let M be an embedded hypersurface in the $n+1$ dimensional Euclidean space $\mathbb{R}^{n+1}$. Its rth mean curvature function H_r is the rth elementary symmetric function of principal curvature functions of M divided by $\binom{n}{r}$. Hence the Gauss–Kronecker curvature is denoted by H_n and the usual mean curvature function is denoted by H_1. It is well known that an embedded closed hypersurface in $\mathbb{R}^n$ with nonzero constant H_r is a round sphere [**1, 7**]. A closed embedded hypersurface M with nonzero constant ratio of mean curvature functions, $H_k/H_l = c$, is also a round sphere [**5, 6**].

On the other hand, it is also known that compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with nonzero constant $H_r, r \geqslant 2$ and spherical boundary are spherical caps [**2, 3**]. Hence it is natural to ask if compact embedded hypersurfaces in $\mathbb{R}^{n+1}$ with nonzero constant ratio H_r/H_k and spherical boundary are spherical caps. In this note, we give partial answers for graphs.

THEOREM 1. *Let f be a smooth positive function defined on an n-dimensional ball*

$$\mathbb{B}^n(\rho) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\| \leqslant \rho\}$$

and $f \equiv 0$ on the boundary $\mathbb{S}^{n-1}(\rho)$. If

$$H_r = cH_k, \quad 1 \leqslant k < r \leqslant n$$

for a nonzero constant c on the graph of f, then the graph of f is a spherical cap of radius $c^{1/(k-r)}$.

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Without the positivity assumption, we have as a corollary of the proof of Theorem 1 the following theorem:

THEOREM 2. *Let f be a smooth nonconstant function defined on an n-dimensional ball $\mathbb{B}^n(\rho)$ and $f \equiv 0$ on the boundary $\mathbb{S}^{n-1}(\rho)$. If*

$$H_n = cH_k$$

for a nonzero constant c on the graph of f, then the graph of f is a spherical cap of radius $c^{1/(k-n)}$.

The same results for a single nonzero constant H_r can be proved by using the maximum principle of [7], however, it seems not clear that the above theorems could be proved by using the same maximum principle even if the equations for the constant ratio of mean curvature functions are elliptic.

2. LEMMAS

Let M be a compact hypersurface in $\mathbb{R}^{n+1}$ with the boundary ∂M in the hyperplane

$$\mathbf{a}^{\perp} = \{\mathbf{x} \in \mathbb{R}^{n+1} : \langle \mathbf{x}, \mathbf{a} \rangle = 0\}, \quad \|\mathbf{a}\| = 1$$

and assume M, together with $\mathbf{a}^{\perp}$ encloses an $n+1$ dimensional region.

LEMMA 1. *Suppose that there is a point of M where all the principal curvatures are positive. If $H_r > 0$ on M, we have for $k < r$ that*

$$H_k \geqslant H_r^{k/r}$$

and the equality holds only at umbilic points. In particular, $H_k > 0$.

PROOF: See [**8**, p. 282] and [**4**, Section 12]. ∎

The boundary ∂M of M is a hypersurface in the n dimensional Euclidean space $\mathbf{a}^{\perp}$. Let h_{r-1} be the $(r-1)$th mean curvature function of ∂M as a hypersurface in $\mathbf{a}^{\perp}$. Let N be the unit normal vector field on M outward pointing to the region enclosed by M and $\mathbf{a}^{\perp}$ and let ν be the outward pointing unit conormal vector field along ∂M. Then the following integral formulas, given in [**2**, **3**], will also be used.

LEMMA 2.

$$\oint_{\partial M} \bigl(-\langle \nu, \mathbf{a} \rangle\bigr) ds = n \int_M H_1 \langle N, \mathbf{a} \rangle dM,$$

$$\oint_{\partial M} h_{r-1} \bigl(-\langle \nu, \mathbf{a} \rangle\bigr)^r ds = n \int_M H_r \langle N, \mathbf{a} \rangle dM, \quad r = 2, \dots, n.$$

3. Proofs

Let M be the graph of f in Theorem 1 and assume $\mathbb{B}^n(\rho)$ is contained in the hyperplane $\mathbf{a}^\perp$. We may assume

$$\langle N, \mathbf{a}\rangle \geqslant 0$$

on M and suppose the ratio of mean curvature functions H_r and H_k is a nonzero constant:

$$H_r = c\, H_k. \tag{1}$$

Since f is positive, M is not contained in the hyperplane $\mathbf{a}^\perp$. Then, since M is compact, one can see that there exists a point $x \in M$ at which all the principal curvature functions are positive by considering the family of n-dimensional spheres which contains $\mathbb{S}^{n-1}(\rho) = \partial\mathbb{B}^n(\rho)$. In particular, starting with a very large sphere in the family, having centre a long way above the hyperplane $\mathbf{a}^\perp$, there is then a first sphere in the family containing a point x of M. Clearly M must have positive principal curvatures at x. Then, as $H_r(x) > 0, H_k(x) > 0$, the constant c in (1) is positive. Now we claim that $H_r > 0$ on M. In order to prove the claim, let U be the connected component of the set $\{x \in M : H_r(x) > 0\}$, then U is an open set. On the other hand, for $x \in U$, we have from Lemma 1 and (1) that

$$\begin{aligned} H_k^{r/k}(x) &\geqslant H_r(x) = cH_k(x), \\ H_k(x) &\geqslant c^{k/(r-k)}, \\ H_r(x) &= cH_k(x) \geqslant c^{r/(r-k)}. \end{aligned}$$

That is,

$$U = \{x \in M : H_r(x) \geqslant c^{r/(r-k)}\},$$

hence U is also a closed set, which proves the claim. Moreover, the last line of the above computation and Lemma 1 give

$$H_1 \geqslant H_r^{1/r} \geqslant c^{1/(r-k)} \tag{2}$$

on M. Since M is a graph of a positive function f, we have $\langle \nu, \mathbf{a}\rangle \leqslant 0$ along the boundary $\mathbb{S}^{n-1}(\rho)$ and hence

$$-\langle \nu, \mathbf{a}\rangle = |\langle \nu, \mathbf{a}\rangle| \geqslant 0.$$

Since $h_{n-1} = 1/\rho^{n-1}, h_{k-1} = 1/\rho^{k-1}$ on $\mathbb{S}^{n-1}(\rho)$, we have from Lemma 2,

$$\begin{aligned} \frac{1}{\rho^{r-1}} \oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds &= n \int_M H_r \langle N, \mathbf{a}\rangle dM, \\ \frac{1}{\rho^{k-1}} \oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^k ds &= n \int_M H_k \langle N, \mathbf{a}\rangle dM. \end{aligned}$$

These identities and (1) then give

$$\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds = c\rho^{r-k} \oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^k ds.$$

Let us denote the $n-1$ dimensional volume of the sphere $\mathbb{S}^{n-1}(\rho)$ by A_ρ. Now, by the Hölder inequality we have

$$\begin{aligned}
\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^k ds &\leqslant \left(\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds\right)^{k/r} \left(\oint_{\mathbb{S}^{n-1}(\rho)} ds\right)^{(r-k)/r} \\
&= \left(\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds\right)^{k/r} A_\rho^{(r-k)/r}, \\
\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds &= c\rho^{r-k} \oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^k ds \\
&\leqslant c\rho^{r-k} \left(\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds\right)^{k/r} A_\rho^{(r-k)/r}, \\
\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds &\leqslant c^{r/(r-k)} \rho^r A_\rho.
\end{aligned}$$

Since

$$\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle| ds \leqslant \left(\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle|^r ds\right)^{1/r} A_\rho^{(r-1)/r}$$

the last inequality gives

$$\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle| ds \leqslant c^{1/(r-k)} \rho A_\rho. \tag{3}$$

On the other hand, since $\langle N, \mathbf{a}\rangle \geqslant 0$ on M, (2) gives

$$H_1 \langle N, \mathbf{a}\rangle \geqslant c^{1/(r-k)} \langle N, \mathbf{a}\rangle. \tag{4}$$

Then, since the n dimensional volume of the n dimensional ball $\mathbb{B}^n(\rho)$ is $\rho A_\rho / n$, we have by Lemma 2 and the Divergence theorem

$$\begin{aligned}
\oint_{\mathbb{S}^{n-1}(\rho)} |\langle \nu, \mathbf{a}\rangle| ds &= n \int_M H_1 \langle N, \mathbf{a}\rangle dM \\
&\geqslant nc^{1/(r-k)} \int_M \langle N, \mathbf{a}\rangle dM \\
&= nc^{1/(r-k)} \mathrm{vol}(\mathbb{B}^n(\rho)) \\
&= c^{1/(r-k)} \rho A_\rho.
\end{aligned}$$

Comparing this computation with (3), one can see that the equality must hold in (4) at every point of M, which implies that the inequality in Lemma 1 should be an equality, which shows that every point of M is an umbilic point. Since M in not contained in

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the hyperplane, M is a part of a round sphere, that is, a spherical cap. Let σ be the radius of the spherical cap, then since $H_r = 1/\sigma^r, H_k = 1/\sigma^k$, we can see from (1) that $\sigma = c^{1/(k-r)}$. This completes the proof of Theorem 1. $\square$

If one assumes $H_n = cH_k$, one can show that $H_n > 0$ on M in the same way as in the above proof, then M is convex and f is a positive function, hence one can see that

$$\langle \nu, \mathbf{a} \rangle \leqslant 0$$

along the boundary without the assumption of the positiveness of f in the case of Theorem 2. Now the same proof gives a proof of Theorem 2 as well. $\square$

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