Compositio Mathematica **112:** 147–182, 1998.

(Shifted) Macdonald polynomials: q-Integral representation and combinatorial formula

ANDREI OKOUNKOV
Department of Mathematics, University of Chicago, 5734 South University Avenue, Chicago, IL 60637-1546, U.S.A.
e-mail: okounkov@math.uchicago.edu

Received 27 August 1996; accepted in final form 14 April 1997

Abstract. We extend some results about *shifted Schur functions* to the general context of *shifted Macdonald polynomials*. We strengthen some theorems of F. Knop and S. Sahi and give two explicit formulas for these polynomials: a q-integral representation and a combinatorial formula. Our main tool is a q-integral representation for ordinary Macdonald polynomial. We also discuss duality for shifted Macdonald polynomials and Jack degeneration of these polynomials.

Mathematics Subject Classification (1991): 05E05.

Key words: Macdonald polynomials, shifted Macdonald polynomials, combinatorial formula, q-integral representation

1. Introduction

The orthogonality of Schur functions

$$(s_\mu, s_\lambda) = \delta_{\mu\lambda},$$

is the orthogonality relation for characters of the unitary group $U(n)$. The orthogonality of characters means that a character (as a function on the group) vanishes in all but one irreducible representation.

There is a very remarkable basis in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}(n))$, which is similar to the basis of characters in the algebra of central functions on $U(n)$. Many properties of this basis can be found in [OO, Ok]. The elements of this basis are indexed by partitions μ with at most n parts; they are denoted by $\mathbb{S}_\mu$ and called *quantum immanants*. The element $\mathbb{S}_\mu$ has degree $|\mu|$ and vanishes in as many irreducible representations, as possible, namely

$$s^*_\mu(\lambda) = 0, \quad \text{unless } \mu \subseteq \lambda,$$

where $s^*_\mu(\lambda)$ is the eigenvalue of $\mathbb{S}_\mu$ in the representation with highest weight λ. The function $s^*_\mu(\lambda)$ is called the *shifted Schur function* for it is a polynomial in λ with highest term $s_\mu(\lambda)$. This polynomial is symmetric in variables

$$\lambda_i - i, \quad i = 1, 2, \ldots,$$

JEFF. INTERPRINT: PIPS Nr.:139217 MATHKAP
comp4127.tex; 27/04/1998; 8:29; v.7; p.1

and is also stable, which means it does not change if we add some zeroes to λ. The number $s^*_\mu(\lambda)$ has a remarkable combinatorial interpretation: up to some simple factors, it counts standard tableaux on λ/μ.

There is the following formula for $\mathbb{S}_\mu$ in terms of the generators E_{ij} of $\mathcal{U}(\mathfrak{gl}(n))$. Fix any standard tableau T on μ. Let $c_T(i)$ denote the content of the ith square in T. Then we have (see [Ok1], and also [N, Ok2])

$$\mathbb{S}_\mu = \mathrm{tr}((E - c_T(1)) \otimes \cdots \otimes (E - c_T(|\mu|)) \cdot P_T), \tag{1.1}$$

where E is the following matrix with entries in $\mathcal{U}(\mathfrak{gl}(n))$

$$E = (E_{ij})_{ij} \in \mathrm{Mat}(n) \otimes \mathcal{U}(\mathfrak{gl}(n)),$$

P_T is the orthogonal projection on the Young basis vector indexed by T

$$P_T \in \mathbb{Q}S(|\mu|),$$

and we use the standard representation

$$\mathbb{Q}S(|\mu|) \to \mathrm{Mat}(n)^{\otimes|\mu|}.$$

The formula (1.1) corresponds to the following *combinatorial formula* for the shifted Schur functions (see [OO], Section 11, and also [Ok1], Section 3.7). We call a tableau T on a diagram μ a *reverse tableau* if its entries strictly decrease down the columns and weakly decrease in the rows. Denote by $T(s)$ the entry of T in the square s and by $c(s)$ the content of the square s. Then we have

$$s^*_\mu(x_1, x_2, \ldots) = \sum_T \prod_{s\in\mu} (x_{T(s)} - c(s)), \tag{1.2}$$

where T ranges over all reverse tableau on μ.

The shifted Schur functions have numerous applications to the finite- and especially infinite-dimensional representation theory. Their (q, t)-analogs, with which we deal in this paper, inherit most of their power.

In the general (q, t)-situation the center of $\mathcal{U}(\mathfrak{gl}(n))$ gets replaced by the commutative algebra generated by Macdonald q-difference operators [M]. The eigenfunctions of this commutative algebra are Macdonald polynomials $P_\lambda(q, t)$, which replace Schur functions. The eigenvalue of a central element in the representation with highest weight λ becomes the eigenvalue of a q-difference operator on the Macdonald polynomial $P_\lambda(q, t)$. It is known (see, for example [EK]) that the algebra generated by Macdonald operators can be naturally identified with the center of the q-deformed $\mathcal{U}(\mathfrak{gl}(n))$.

The eigenvalue of a Macdonald operator on $P_\lambda(q, t)$ is known to be a polynomial in q^{λ_i} which is symmetric in variables

$$q^{\lambda_i} t^{-i}.$$

Therefore the natural (q,t)-analog of the shifted Schur function should be a polynomial

$$P^*_\mu(x),$$

of degree

$$\deg P^*_\mu(x) = |\mu|,$$

which is symmetric in variables

$$x_i t^{-i}, \quad i = 1, 2, \dots,$$

and satisfies the following vanishing condition

$$P^*_\mu(q^\lambda) = 0, \quad \text{unless } \mu \subseteq \lambda, \tag{1.3}$$

and $P^*_\mu(q^\mu) \neq 0$. It is an overdetermined system of linear conditions on $P^*_\mu(x)$. It is easy to see that these polynomials are unique within a scalar factor provided they exist. We shall specify the normalization in section 4. We call these polynomials *shifted Macdonald polynomials*.

It is well known that the number $q^{j-1}t^{1-i}$ is the (q,t)-analog of the content of the a square $s = (i,j)$. Together with G. Olshanski we have conjectured (unpublished) the following analog of the formula (1.2). By

$$a'(s) = j - 1, \qquad l'(s) = i - 1$$

denote the *arm-colength* and the *leg-colength* of the square $s = (i,j)$. Then

$$P^*_\mu(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\mu} t^{1-T(s)}(x_{T(s)} - q^{a'(s)}t^{-l'(s)}), \tag{1.4}$$

where the sum is over all reverse tableau on μ with entries in $\{1,2,\dots\}$ and $\psi_T(q,t)$ is the same (q,t)-weight of a tableau which enters the combinatorial formula for ordinary Macdonald polynomials (see [M], Section VI.7)

$$P_\mu(x;q,t) = \sum_T \psi_T(q,t) \prod_{s\in\mu} x_{T(s)}. \tag{1.5}$$

The coefficients $\psi_T(q,t)$ are rational functions of q and t.

However, only in the present paper we obtain a proof of this formula.

First theorems about the polynomials $P^*_\mu(x)$ were obtained by F. Knop and S. Sahi [KS, K, S]. In particular, they identified the highest degree term of the inhomogeneous polynomial $P^*_\mu(x)$. Other fundamental properties (such as *integrality*) were also established. Their approach was based on q-difference equations for polynomials $P^*_\mu(x)$.

The combinatorial formula (1.4) implies both the highest degree term identification [K, S] and the *extra vanishing* property (1.3) proved in [K]. It is easy to see that all conditions from the definition of $P^*_\mu(x)$ are obvious in (1.4) except for the symmetry in $x_i t^{-i}$. We were not able to find any direct proof of this symmetry and shall give a quite indirect proof.

We shall use an approach based on a *q-integral representation* for the polynomial $P^*_\mu(x)$. It is independent of the difference equations approach and generalizes the *coherence* property of quantum immanants (see [OO], Section 10 and also [Ok1], Section 5.1).

Our main technical tool is a q-integral representation for Macdonald polynomials (see Theorem I below). Since Macdonald polynomials satisfy a q-difference equation it is natural to expect that

$$P_\mu(x_1, \ldots, x_n)$$

can be written as a multiple q-integral where x_i occur as limits of integration. Indeed, $P_\mu(x_1, \ldots, x_n)$ can be written as such a multiple integral of the Macdonald polynomial

$$P_\mu(y_1, \ldots, y_{n-1})$$

in a smaller set of variables with respect to a multivariate analog of the symmetric beta measure. In the simplest case $n = 2$ this integral reduces to a particular case of the q-analog of the beta integral studied in [AA, AV]. In the Schur function case this integral can be evaluated explicitly and gives the determinant ratio formula. The integral representation allows to characterize $P_\mu(x)$ (and also $P^*_\mu(x)$) as eigenfunctions of commuting integral operators, see Remark 4.7.

There are important q-integral formulas involving Macdonald polynomials due to Kadell (see [M], example VI.9.3 and also [Ka]) as well as integral representation of Macdonald polynomials via ordinary contour integrals (see [AOS] and references therein).

The crucial property of our integral is that the domain of integration is of the form

$$\int_{x_2}^{x_1} d_q y_1 \cdots \int_{x_n}^{x_{n-1}} d_q y_{n-1} (\cdots).$$

This allows to obtain a q-integral representation for the polynomial P^*_μ just by a minor modification of the integral, see Section 4. In other words, the orthogonal polynomials P_μ and the Newton interpolation polynomials P^*_μ have essentially the same integral representation.

In a sense the relationship between P_μ and P^*_μ is even closer than between s_μ and s^*_μ. One explanation for this is that the finite difference calculus (which is suitable to s^*_μ) unifies with the ordinary calculus (which is suitable to s_μ) in the q-difference calculus.

The lower degree terms of the inhomogeneous polynomial P^*_μ carry some important additional information and make some properties of P^*_μ look even more natural than the corresponding properties of the ordinary Macdonald polynomials. For example, the *duality* for shifted Macdonald polynomials has a clear combinatorial interpretation (see Section 6) which gives a new interpretation of the duality for Macdonald polynomials.

In Section 7 we consider shifted Jack polynomials. This degeneration was also considered by F. Knop and S. Sahi in [KS]. Even if one is interested in Jack polynomials only it proves to be easier to work with general Macdonald polynomials and then let $q \to 1$ in very final formulas.

It would be very interesting to find a (q,t)-analog of the formula (1.1).

I am grateful to I. Cherednik, S. Sahi and especially to G. Olshanski for many helpful discussions. I would like to thank R. Askey, A. N. Kirillov, K. Mimachi, M. Noumi, M. Wakayma for their remarks on the preliminary version of this paper (q-alg 9605013). In particular, K. Mimachi gave me a copy of the preprint [MN].

Since the completion of this paper binomial type formulas for P^*_μ and also for ordinary Macdonald polynomials were obtained (see [Ok3] and also [OO3]). In particular, they provide new and, perhaps, more natural ways to prove the q-integral representations obtained here. Among other application let us mention the papers [KOO] and [Ok5].

The analogs of P^*_μ for the root system of type BC_n were considered in [Ok4]. Those polynomials have essentially all properties of P^*_μ except for the difference equations.

2. Notations

All q-shifted factorials in this paper will be with the same base q, for example

$$(a)_\infty = (1-a)(1-qa)(1-q^2a)\ldots.$$

This product converges if $|q| < 1$. Put

$$(a)_\theta = \frac{(a)_\infty}{(q^\theta a)_\infty}.$$

The numbers q and $t = q^\theta$ will be the two parameters of the Macdonald polynomials P_μ. With this notation

$$(a)_\theta = \frac{(a)_\infty}{(ta)_\infty}.$$

We shall need also another q-shifted power

$$\langle a\rangle_r = (a-1)(a-q)\cdots(a-q^{r-1}), \quad r = 0,1,2,\ldots,$$

which is a q-analog of the falling factorial power of a.

Recall the definition of the q-integral

$$\int_b^a f(y)d_qy = \int_0^a f(y)d_qy - \int_0^b f(y)d_qy,$$

where

$$\int_0^a f(y)d_qy = a(1-q)\sum_{i=0}^{\infty} f(aq^i)q^i.$$

We have

$$\int_0^a y^r d_qy = \frac{1}{[r+1]}a^{r+1}, \tag{2.1}$$

where

$$[r] = \frac{1-q^r}{1-q}$$

is the q-analog of the number r. Recall also the following q-analog of the beta function integral [GR, 1.11.7]

$$\int_0^1 y^{a-1}(qy)_{b-1}d_qy = B_q(a,b), \tag{2.2}$$

where $\Re a > 0, b \neq 0, -1, -2, \ldots$,

$$B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)},$$

and, finally,

$$\Gamma_q(a) = (1-q)^{1-a}(q)_{a-1}.$$

Given two vectors

$$x = (x_1, \ldots, x_n), \quad y = (y_1, \ldots, y_{n-1}),$$

where $n = 2, 3, \ldots$, write

$$y \prec x$$

if

$$y_i \in [x_i, x_{i+1}], \quad i = 1, \ldots, n-1.$$

Denote by

$$V(x)=\prod_{i<j}(x_i-x_j)$$

the Vandermonde determinant in variables $x_1,\dots,x_n$. Put

$$V^\theta(x)=V(x)\prod_{i\neq j}(qx_i/x_j)_{\theta-1}.$$

We will integrate symmetric polynomials in $y_1,\dots,y_{n-1}$ over the domain $y\prec x$ with respect to the following measure

$$d\beta(y|x)=V(y)\prod_{i,j}(qy_i/x_j)_{\theta-1}d_qy, \tag{2.3}$$

which is a multivariate analog of the beta measure in (2.2). Here

$$d_qy=d_qy_1\dots d_qy_{n-1}.$$

With the Macdonald notation

$$\Pi(x,y;q,t)=\prod\frac{(tx_iy_j)_\infty}{(x_iy_j)_\infty}$$

we have

$$d\beta(y|x)=V(y)\Pi(1/x,ty;q,q/t)d_qy, \tag{2.4}$$

By Λ denote the algebra of symmetric functions with coefficients in rational functions in q and t. By $\Lambda(n)$ denote the algebra on symmetric functions in n variables with the same coefficients.

3. q-Integral representation of Macdonald polynomials

Given a partition μ with the number of parts $\ell(\mu)\leqslant n$ put

$$C(\mu,n)=\prod B_q(\mu_i+(n-i)\theta,\theta). \tag{3.1}$$

In this section we will prove the following theorem

THEOREM I. *Suppose* $\ell(\mu)<n$, *then*

$$\frac{1}{V^\theta(x)}\int_{y\prec x}P_\mu(y)d\beta(y|x)=C(\mu,n)P_\mu(x). \tag{3.2}$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

On the one hand this formula expresses the Macdonald polynomial $P_\mu(x)$ in terms of the Macdonald polynomial $P_\mu(y)$ in the smaller set of variables. By stability of Macdonald polynomials it suffices to know P_μ in $|\mu|$ variables. Therefore (3.2) gives a q-integral representation of P_μ. This formula is a generalization of the determinant ratio formula for Schur functions (see below).

On the other hand, since P_μ form a basis in the space of symmetric polynomials, (3.2) tells how to integrate symmetric polynomials over the domain $y \prec x$ with respect to the measure (2.3).

In the limit $q \to 1$ the formula (3.2) becomes the integral representation for Jack polynomials found by G. Olshanski [Ol].

Both sides of (3.2) are analytic functions of q and t in the polydisc

$$|q| < 1, |t| < 1.$$

More precisely, if

$$|q|, |t| < \delta < 1,$$

then we have to assume, for example, that

$$\delta^{1/2} < |x_i| < \delta^{-1/2}, \quad i = 1, \ldots, n$$

and that

$$x_i \neq x_j, \quad i \neq j$$

in order to avoid zero factors.

Therefore it suffices to prove the equality (3.2) for

$$t = q^\theta, \quad \theta = 2, 3, \ldots. \tag{3.3}$$

Indeed, an analytic function, which vanishes on (3.3) should vanish on all lines

$$q = \text{const} \neq 0$$

by the 1-dimensional uniqueness theorem.

LEMMA 3.1. *Suppose $f(y_1)$ is a polynomial. Then*

$$\frac{1}{x_1 - x_2} \int_{x_2}^{x_1} f(y_1) d_q y_1 \tag{3.4}$$

is a symmetric polynomial in x_1 and x_2.

Proof. Follows from (2.1). □

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

The following computation is from [OO], Section 10.

LEMMA 3.2. *Suppose* $f(y)$ *is a symmetric polynomial. Then*

$$\int_{y \prec x} f(y)V(y)d_q y \tag{3.5}$$

is a skew-symmetric polynomial in x.

Proof. Since $f(y)V(y)$ is skew-symmetric, it is a linear combination of the following determinants

$$\det \begin{pmatrix} y_1^{\nu_1} & \cdots & y_1^{\nu_{n-1}} \\ \vdots & & \vdots \\ y_{n-1}^{\nu_1} & \cdots & y_{n-1}^{\nu_{n-1}} \end{pmatrix}$$

for some numbers $\nu_1 > \nu_2 > \cdots$. Integrating we get up to a constant factor

$$\det \begin{pmatrix} x_1^{\nu_1+1} - x_2^{\nu_1+1} & \cdots & x_1^{\nu_{n-1}+1} - x_2^{\nu_{n-1}+1} \\ \vdots & & \vdots \\ x_{n-1}^{\nu_1+1} - x_n^{\nu_1+1} & \cdots & x_{n-1}^{\nu_{n-1}+1} - x_n^{\nu_{n-1}+1} \end{pmatrix}.$$

Which equals

$$\det \begin{pmatrix} x_1^{\nu_1+1} & \cdots & x_1^{\nu_{n-1}+1} & 1 \\ x_2^{\nu_1+1} & \cdots & x_2^{\nu_{n-1}+1} & 1 \\ \vdots & & \vdots & \vdots \\ x_n^{\nu_1+1} & \cdots & x_n^{\nu_{n-1}+1} & 1 \end{pmatrix}. \tag{3.6}$$

To see this subtract in (3.6) the second line from the first one, then the third line from the second one and so on. Clearly, (3.6) is skew-symmetric in x. □

REMARK 3.3. In the Schur functions case $q = t$ the computation from the previous lemma gives in fact an explicit computation of the integral in the left-hand side of (3.2) which shows that (3.2) is a generalization of the determinant ratio formula for Schur functions.

PROPOSITION 3.4. *Suppose* $f(y)$ *is symmetric polynomial. Then*

$$\frac{1}{V^\theta(x)} \int_{y \prec x} f(y) d\beta(y|x) \tag{3.7}$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

is a symmetric polynomials in x of degree $\deg f$.

Proof. Multiply both the integrand and denominator in (3.7) by

$$\left(\prod x_i\right)^{(n-1)(\theta-1)}$$

so that they become polynomials. The integrand becomes a skew-symmetric polynomial in y with coefficients in symmetric polynomials in x, therefore by Lemma 3.2 the integral is a skew-symmetric polynomial in x. Denote this polynomial by J.

By Lemma 3.1 the polynomial J is divisible by $(x_1 - x_2)$.

Since the integrand vanishes at the points

$$y_1 = x_1/q, \ldots, x_1/q^{\theta-1},$$

as well as at the points

$$y_1 = x_2/q, \ldots, x_2/q^{\theta-1},$$

we can replace the integration

$$\int_{x_2}^{x_1} d_q y_1$$

by integration

$$\int_{x_2}^{x_1/q^s} d_q y_1, \quad s = 1, \ldots, \theta - 1,$$

or by

$$\int_{x_2/q^s}^{x_1} d_q y_1, \quad s = 1, \ldots, \theta - 1.$$

Therefore J is divisible also by

$$x_1 - q^s x_2, \quad s = 1, \ldots, \theta - 1,$$

and

$$q^s x_1 - x_2, \quad s = 1, \ldots, \theta - 1.$$

Since J is skew-symmetric in x it is divisible by

$$V(x) \prod_{i \neq j} \prod_{s=1}^{\theta-1} (x_i - q^s x_j).$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

Therefore (3.7) is a symmetric polynomial in x. It is clear that its degree equals

$$\deg f + (n-1)(n-2)/2 + (n-1) - n(n-1)/2 = \deg f,$$

where the summands come from $f(y)$, $V(y)$, integration, and $V^\theta(x)$ respectively. This concludes the proof. □

Denote by $I(\mu, n)$ the left-hand side of (3.2). We want to apply the Macdonald operator

$$D = \sum_{i=1}^{n} \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} T_{q,x_i} \tag{3.8}$$

to $I(\mu, n)$. Here

$$[T_{q,x_i} f](x_1, \dots, x_i, \dots, x_n) = f(x_1, \dots, qx_i, \dots, x_n).$$

The operator (3.8) will be also denoted by $D(q,t)$ and by $D_x(q,t)$ when it should be stressed that it acts on variables x. The Macdonald polynomials are eigenfunctions of this operator [M]

$$DP_\mu = \left(\sum q^{\mu_i} t^{n-i} \right) P_\mu. \tag{3.9}$$

We shall need the three following general lemmas about the operator D. For these lemmas we do not need any special assumptions about q and t.

LEMMA 3.5. *For all q and t we have the following commutation relation*

$$D(1/q, 1/t) \frac{1}{V^\theta(x)} = \frac{(q/t)^{(n-1)}}{V^\theta(x)} D(1/q, t/q). \tag{3.10}$$

Proof. Direct computation. □

LEMMA 3.6. *Put*

$$D_{1/x}(q,t) = \sum_i \prod_{j \neq i} \frac{t/x_i - 1/x_j}{1/x_i - 1/x_j} T_{q,1/x_i}.$$

then

$$D_{1/x}(q,t) = t^{n-1} D_x(1/q, 1/t). \tag{3.11}$$

Proof. Direct computation. □

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

By definition (3.8) the operator D acts on symmetric polynomials in n variables. This action indeed depends on n; in other words it is not compatible with the restriction homomorphisms

$$\Lambda(n) \to \Lambda(n-1). \tag{3.12}$$

In the next lemma we shall deal with two finite sets of variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ and we suppose for simplicity that $n \geqslant m$. Denote by D_x and D_y the operators D in variables x and y respectively. Put

$$\Pi_{n,m} = \Pi(x, y; q, t).$$

LEMMA 3.7.

$$D_x \Pi_{n,m} = (t^{n-m} D_y + [n-m]_t) \Pi_{n,m}, \tag{3.14}$$

where $[n-m]_t = (1 - t^{n-m})/(1-t)$.

Proof. As explained in [M, VI. 4] (this is clear from (3.9)) the following modification of the operator (3.8)

$$E = t^{-n} D - \sum_1^n t^{-i} \tag{3.15}$$

is compatible with homomorphisms (3.12) and therefore defines an operator

$$\Lambda \to \Lambda,$$

which is self-adjoint. By [M, VI.2.13]

$$E_x \Pi(x, y; q, t) = E_y \Pi(x, y; q, t).$$

Therefore

$$\begin{aligned} D_x \Pi_{n,m} &= t^n \left(E_x + \sum_1^n t^{-i} \right) \Pi_{n,m} \\ &= t^n \left(E_y + \sum_1^n t^{-i} \right) \Pi_{n,m} \\ &= t^n \left(t^{-m} D_y - \sum_1^m t^{-i} + \sum_1^n t^{-i} \right) \Pi_{n,m} \\ &= \left(t^{n-m} D_y + [n-m]_t \right) \Pi_{n,m}. \end{aligned}$$ □

In particular,

$$D_x \Pi_{n,n-1} = (t D_y + 1)\, \Pi_{n,n-1}. \tag{3.16}$$

PROPOSITION 3.8.

$$D(1/q,1/t)I(\mu,n) = \left(\sum_i q^{-\mu_i}t^{i-n}\right) I(\mu,n). \tag{3.17}$$

Proof. Recall that we consider the case $\theta = 2, 3, \ldots$. Put

$$I_0 = \int_{y\prec x} P_\mu(y)d\beta(y|x).$$

By (3.10) we have to prove that

$$(q/t)^{n-1}D(1/q,t/q)I_0 = \left(\sum_i q^{-\mu_i}t^{i-n}\right) I_0.$$

By (3.11) it is equivalent to

$$D_{1/x}(q,q/t)I_0 = \left(\sum_i q^{-\mu_i}t^{i-n}\right) I_0.$$

We have

$$T_{q,1/x_i}I_0 = \int_{x_2}^{x_1}\cdots\int_{x_i/q}^{x_{i-1}}\int_{x_{i+1}}^{x_i/q}\cdots\int_{x_n}^{x_{n-1}} T_{q,1/x_i}P_\mu(y)d\beta(y|x). \tag{3.18}$$

Since the integrand in (3.18) vanishes if

$$y_j, y_{j-1} = x_j/q, \quad j \neq i,$$

we can rewrite (3.18) as follows

$$T_{q,1/x_i}I_0 = \int_{y\prec x/q} T_{q,1/x_i}P_\mu(y)d\beta(y|x).$$

Therefore

$$D_{1/x}(q,q/t)I_0 = \int_{y\prec x/q} D_{1/x}(q,q/t)P_\mu(y)d\beta(y|x). \tag{3.19}$$

Now by (3.16)

$$D_{1/x}(q,q/t)\Pi(1/x,ty;q,q/t) = \left(\frac{q}{t}D_y(q,q/t)+1\right)\Pi(1/x,ty;q;q/t). \tag{3.20}$$

By (3.20) and (2.4), the integral (3.18) equals

$$\int_{y\prec x/q} P_\mu(y)V(y)\left[\left(\frac{q}{t}D_y(q,q/t)+1\right)\Pi(1/x,ty;q;q/t)\right]d_qy. \tag{3.21}$$

Consider one summand in (3.21)

$$\frac{q}{t}\int_{y\prec x/q} P_\mu(y)V(y)\left[\prod_{j\neq i}\frac{qy_i/t-y_j}{y_i-y_j}\,T_{q,y_i}\,\Pi(1/x,ty;q;q/t)\right]d_qy, \tag{3.22}$$

where $i=1,\dots,n-1$. Replace qy_i by a new variable. To simplify notation denote this new variable by y_i. Then (3.22) becomes

$$\frac{1}{t}\int_{x_2/q}^{x_1/q}\cdots\int_{x_{i+1}}^{x_i}\cdots\int_{x_n/q}^{x_{n-1}/q}\left[\prod_{j\neq i}\frac{y_i/t-y_j}{y_i-y_j}\,T_{1/q,y_i}\,P_\mu(y)\right]d\beta(y|x). \tag{3.23}$$

Since the beta measure vanishes if

$$y_j = x_j/q, x_{j+1}/q, \tag{3.24}$$

the integral (3.23) equals

$$\frac{1}{t}\int_{y\prec x}\left[\prod_{j\neq i}\frac{y_i/t-y_j}{y_i-y_j}T_{1/q,y_i}P_\mu(y)\right]d\beta(y|x). \tag{3.25}$$

By the same vanishing of the beta measure for (3.24)

$$\begin{aligned}\int_{y\prec x/q} & P_\mu(y)V(y)\Pi(1/x,ty;q;q/t)d_qy\\ &=\int_{y\prec x}P_\mu(y)V(y)\Pi(1/x,ty;q;q/t)d_qy.\end{aligned} \tag{3.26}$$

Therefore the integral (3.21) equals

$$I_0+\frac{1}{t}\int_{y\prec x}\left[D_y(1/q,1/t)P_\mu(y)\right]d\beta(y|x), \tag{3.27}$$

It is well known (and follows, for example, from the formula for the scalar product) that

$$P_\mu(x;q,t)=P_\mu(x;1/q,1/t). \tag{3.28}$$

By (3.9) and (3.28)

$$D_y(1/q,1/t)P_\mu(y)=\left(\sum_1^{n-1}q^{-\mu_i}t^{i-n+1}\right)P_\mu(y).$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

Hence (3.27) equals

$$\left(\sum_1^n q^{-\mu_i} t^{i-n}\right) I_0,$$

as desired. □

COROLLARY 3.9. *$I(\mu,n)$ equals the Macdonald polynomial $P_\mu(x)$ up to a scalar factor.*

We calculate this factor in the next proposition. It will conclude the proof of the theorem.

Consider the highest monomial in of $I(\mu,n)$ with respect to the lexicographic ordering of monomials

PROPOSITION 3.10. *The highest monomial in $I(\mu,n)$ equals*

$$C(\mu,n)\prod x_i^{\mu_i}. \tag{3.29}$$

Proof. Multiply both the integrand and denominator in $I(\mu,n)$ by

$$\prod(-1)^{(\theta-1)(n-i)} q^{-(n-i)\theta(\theta-1)/2} x_i^{(n-1)\theta}.$$

Then the highest monomial in the denominator equals

$$\prod_i x_i^{(2\theta-1)(n-i)}.$$

Calculate the highest term of the integrand. We have to give x_i and y_i the same priority because we obtain x_i integrating y_i. Therefore the highest term of the integrand is

$$\prod_i x_i^{(n-i-1)(\theta-1)} y_i^{\mu_i+(n-i-1)+(n-i)(\theta-1)}(x_i-qy_i)\cdots(x_i-q^{\theta-1}y_i)$$
$$=\prod_i x_i^{(n-i-1)(\theta-1)} y_i^{\mu_i+(n-i)\theta-1}(x_i-qy_i)\cdots(x_i-q^{\theta-1}y_i).$$

Now calculate the highest term of the integral. The terms which come from the lower limits are negligible, therefore the highest term of the integral equals

$$\prod_i x_i^{(n-i-1)(\theta-1)}\int_0^{x_i} y_i^{\mu_i+(n-i)\theta-1}(x_i-qy_i)\cdots(x_i-q^{\theta-1}y_i)d_qy_i$$
$$=\prod_i x_i^{(n-i-1)(\theta-1)+\mu_i+(n-i)\theta+\theta-1}$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

$$\begin{aligned}&\times \int_0^1 z^{\mu_i+(n-i)\theta-1}(1-qz)\cdots(1-q^{\theta-1}z)d_q z\\
&= \prod_i B_q(\mu_i+(n-i)\theta,\theta)x_i^{\mu_i+(n-i)(2\theta-1)}\\
&= C(\mu,n)\prod_i x_i^{\mu_i+(n-i)(2\theta-1)},\end{aligned}$$

where we used change of variables $y_i = zx_i$, beta function integral (2.2) and definition (3.1). Therefore the highest term of the ratio equals

$$C(\mu,n)\prod_i x_i^{\mu_i}. \qquad \square$$

EXAMPLE 3.11. The integral representation (3.2) gives one more way to calculate the special value (see [M], VI.6.17)

$$P_\mu(1,t,t^2,\dots,t^{n-1}). \tag{3.30}$$

What is special about this value is that for

$$\theta = 1,2,\dots$$

only one summand, corresponding to the point

$$y_i = t^{i-1}$$

does not vanish in (3.2).

4. q-Integral representation for shifted Macdonald polynomials

Denote by $\Lambda^*(n)$ the algebra of polynomials $f(x_1,\dots,x_n)$ which are symmetric in new variables

$$x_i' = x_i t^{1-i}.$$

We shall call such polynomials *shifted symmetric*. The algebra $\Lambda^*(n)$ is filtered by degree of polynomials. Denote by Λ^* the projective limit of filtered algebras $\Lambda^*(n)$ with respect to the homomorphisms

$$\Lambda^*(n) \to \Lambda^*(n-1),$$

$$f(x_1,\dots,x_n) \mapsto f(x_1,\dots,x_{n-1},1).$$

The algebra Λ^* can be naturally identified with the algebra of those polynomials in Macdonald commuting difference operators which are stable (that is compatible with homomorphisms (3.12)). An example of such operator is the operator E from (3.15).

The graded algebra corresponding to the filtered algebra Λ^* can be naturally identified with the algebra of symmetric functions in variables x_i'. The algebra Λ^* is generated, for example, by the following analogs of the power sums

$$p_k^*(x) = \sum_i (x_i^k - 1)t^{k(1-i)}, \quad k = 1, 2, \ldots. \tag{4.1}$$

Given a partition μ put

$$n(\mu) = \sum_i (i-1)\mu_i = \sum_j \mu_j'(\mu_j' - 1)/2.$$

Recall that for each square $s = (i, j) \in \mu$ the numbers

$$a(s) = \mu_i - j, \qquad a'(s) = j - 1,$$
$$l(s) = \mu_j' - i, \qquad l'(s) = i - 1,$$

are called arm-length, arm-colength, leg-length, and leg-colength respectively. By definition, put

$$H(\mu) = t^{-2n(\mu)} q^{n(\mu')} \prod_{s\in\mu} (q^{a(s)+1} t^{l(s)} - 1).$$

This number will play the same role as the hook-length product played in [OO].

Suppose $\ell(\mu) \leqslant n$. By P_μ^* denote the element of $\Lambda^*(n)$ which satisfies the two following conditions

$$\deg P_\mu^* = |\mu|, \tag{4.2}$$

$$P_\mu^*(q^\lambda) = H(\mu)\delta_{\lambda,\mu}, \quad |\lambda| \leqslant |\mu|, \ell(\lambda) \leqslant n, \tag{4.3}$$

where $q^\lambda = (q^{\lambda_1}, \ldots, q^{\lambda_n})$. Here $H(\mu)$ is just normalization constant and is introduced for convenience. Note that the condition (4.3) is weaker than the condition (1.3) in the introduction. Here we have a square system of linear equations on P_μ^*. We shall prove (1.3) below in (4.11).

It is clear that if P_μ^* exists then it is unique. The existence of P_μ^* will follow from an explicit formula for it. The existence of this polynomial was proved by different methods in [K, S].

In the same way as in [OO], Example 3.5, we have

PROPOSITION 4.1. *The sequence*

$$\{P_\mu^* \in \Lambda^*(n)\}_{n \geqslant \ell(\mu)}$$

defines an element of Λ^*.

Proof. The polynomial

$$P^*_\mu(x_1, \ldots, x_{n-1}, 1)$$

satisfies all conditions for the shifted Macdonald polynomial in $(n-1)$ variables provided $n > \ell(\mu)$. $\square$

PROPOSITION 4.2. *Suppose* $\mu_n > 0$ *and put*

$$\mu^- = (\mu_1 - 1, \ldots, \mu_n - 1).$$

Then

$$P^*_\mu(x_1, \ldots, x_n) = q^{|\mu^-|} \prod_i (x_i^* t^{1-i} - t^{1-n}) P^*_{\mu^-}(x_1/q, \ldots, x_n/q). \tag{4.4}$$

Proof. We have to verify that the right-hand side of (4.4) satisfies all conditions for P^*_μ. It is clear, that its degree equals $|\mu|$. Evaluate it at q^μ. We have

$$q^{|\mu^-|} t^{-n(n-1)} \prod_i (q^{\lambda_i} t^{n-i} - 1) H(\mu^-) = H(\mu).$$

Finally, it is easy to see that it vanishes at all points q^λ, where $|\lambda| \leqslant |\mu|$ and $\ell(\lambda) \leqslant n$. $\square$

In this section we shall obtain a q-integral formula for polynomials P^*_μ, which is a minor modification of the formula (3.2).

We shall consider the following integration

$$\int_{y \prec^* x} d_q y = \int_{x_2}^{qx_1} d_q y_1 \cdots \int_{x_n}^{qx_{n-1}} d_q y_{n-1}.$$

Introduce new variables

$$x_i^* = x_i t^{n-i}, \qquad y_i^* = y_i t^{n-1-i},$$

and put

$$V^*(x) = V(x^*),$$

$$V^{*\theta}(x) = V^\theta(x^*),$$

$$d\beta^*(y|x) = d\beta(y^*|x^*).$$

We have

THEOREM II. *Suppose* $\ell(\mu) < n$ *then*

$$\frac{1}{V^{*\theta}(x)} \int_{y \prec^* x} P^*_\mu(y) d\beta^*(y|x) = t^{|\mu|} C(\mu, n) P^*_\mu(x). \tag{4.5}$$

Together with (4.4) the theorem gives a formula for polynomials P^*_μ. In the proof we shall assume that

$$\theta = 1, 2, \ldots . \tag{4.6}$$

Indeed, the left-hand side of (4.5) is analytic in the polydisc $|q|, |t| < 1$ and being a polynomial vanishing at certain points provided (4.6) must be such a polynomial for the other values of q and t as well.

We need the two following elementary lemmas.

LEMMA 4.3. *For any* A

$$\frac{(qA/t)_{\theta-1}}{(q/A)_{\theta-1}} = A^{\theta-1} q^{-\theta(\theta-1)/2}. \tag{4.7}$$

LEMMA 4.4. *For any partition* μ *and* $\theta = 1, 2, \ldots$

$$V^{*\theta}(q^\mu) \neq 0. \tag{4.8}$$

Proof of the theorem. Denote by I^* the integral in the left-hand side of (4.5). First show that it is an element of $\Lambda^*(n)$. In variables x^*, y^* it can be rewritten as

$$\frac{1}{V^\theta(x^*)} \int_{x_2^*}^{q^{1-\theta} x_1^*} \cdots \int_{x_n^*}^{q^{1-\theta} x_{n-1}^*} P^*_\mu(y) d\beta(y^*|x^*). \tag{4.9}$$

The polynomial

$$P^*_\mu(y)$$

is symmetric in y_i^*. Because of the vanishing of the beta measure we can replace integration

$$\int_{x_{i+1}^*}^{q^{1-\theta} x_i^*}$$

by integration

$$\int_{x_{i+1}^*}^{x_i^*}.$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

Thus, by Proposition 3.4 it is a symmetric polynomial in x_i^*, that is an element of $\Lambda^*(n)$.

The main point is the verification of the vanishing condition. Let us evaluate $I^*(\mu, n)$ at

$$x = q^\lambda,$$

where λ is a partition. Then the summands which enter the q-integral correspond to points

$$y_i = q^{\gamma_i}, \quad \lambda_i \geqslant \gamma_i \geqslant \lambda_{i+1}, \quad i = 1, \dots, n-1.$$

(Recall that $q < 1$ and therefore $q^{\lambda_{i+1}}$ is in fact the upper limit of integration for y_i and q^{λ_i+1} is the lower one. This explains also the minus sign below in (4.10).) Note that γ is a partition

$$\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_{n-1}$$

and $|\gamma| \leqslant |\lambda|$.

Now suppose that $|\lambda| \leqslant |\mu|$ and $\lambda \neq \mu$. Then always

$$\gamma \neq \mu.$$

By definition, $P_\mu^*(y)$ vanishes at all such points! Since by (4.8) the denominator does not vanish, $I^*(\mu, n)$ vanishes.

The only calculation left is evaluation of the integral at

$$x = q^\mu,$$

which means

$$x_i^* = q^{\mu_i} t^{n-i}.$$

In particular $x_n = x_n^* = 1$. This evaluation is elementary but quite messy. To the end of the proof the indices i and j will range from 1 to $n-1$. Only one summand in the integral does not vanish. This summand equals the product of the following factors:

$$H(\mu)$$

from the value of $P_\mu^*(y)$ at this point,

$$t^{-(n-1)(n-2)/2} \prod_{i<j} (x_i^* - x_j^*)$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

from the Vandermonde determinant,

$$((q/t)_{\theta-1})^{n-1}\prod(qx_i^*/t)_{\theta-1}\prod_{i\neq j}((q/t)x_i^*/x_j^*)_{\theta-1}$$

from the other factors in the beta density, and finally

$$(-1)^{n-1}(1-q)^{n-1}t^{-(n-1)}\prod x_i^* \tag{4.10}$$

from the q-Lebesgue measure d_qy^*.

The denominator equals

$$\prod_{i<j}(x_i^*-x_j^*)\prod_{i\neq j}(qx_i^*/x_j^*)_{\theta-1}\prod_i((x_i^*-1)(qx_i^*)_{\theta-1}(q/x_i^*)_{\theta-1}).$$

By (4.7)

$$\prod_{i\neq j}\frac{((q/t)x_i^*/x_j^*)_{\theta-1}}{(qx_i^*/x_j^*)_{\theta-1}}=q^{-\theta(\theta-1)(n-1)(n-2)/2}.$$

Again, by (4.7)

$$\frac{(-1)x_i^*(1-q)(q/t)_{\theta-1}(qx_i^*/t)_{\theta-1}}{(x_i^*-1)(qx_i^*)_{\theta-1}(q/x_i^*)_{\theta-1}}=\frac{(x_i^*)^\theta q^{\theta(\theta-1)}(1-q)(q)_{\theta-1}}{(x_i^*)_\theta},$$

which equals

$$(x_i^*)^\theta q^{\theta(\theta-1)}B_q(\mu_i+(n-i)\theta,\theta).$$

Therefore the result equals

$$C(\mu,n)H(\mu)$$

times the following power of q

$$\begin{aligned}&\theta|\mu|+\theta^2n(n-1)/2-\theta(\theta-1)(n-1)\\&\quad-\theta(\theta-1)(n-1)(n-2)/2-\theta(n-1)(n-2)/2-\theta(n-1),\end{aligned}$$

which equals $\theta|\mu|$. □

Write

$$\mu\subseteq\lambda$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

if $\lambda_i \geqslant \mu_i$ for all i, and

$$\mu \subset \lambda$$

if $\mu \subseteq \lambda$ and $\mu \neq \lambda$. In the sequel we shall deal with some sums like

$$\sum_{\nu} c_\nu P_\nu$$

where the only thing that matters is which P_ν can enter the sum, no matter with which specific constant factors c_ν. Constant factor like c_ν will always denote such unspecific factors.

The following proposition is a strengthening of a result by S. Sahi and F. Knop.In particular, it gives the highest degree term of P^*_μ, which was found in [K, S] by different methods.

PROPOSITION 4.5.

$$\begin{aligned} P^*_\mu(x_1, x_2, \dots, x_n) &= P_\mu(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}) \\ &\quad + \sum_{\nu \subset \mu} c_\nu P_\nu(x_1, x_2 t^{-1}, \dots, x_n t^{1-n}), \end{aligned} \tag{4.11}$$

where c_ν are some constants[*] *which depend on μ and n.*

Proof. Induct on n and $|\mu|$. Suppose $\mu_n > 0$. By (4.4) we have

$$\begin{aligned} &P^*_\mu(x_1, \dots, x_n) \\ &\quad = q^{|\mu^-|} \left(\sum_{k=0}^{n} (-t^{-n})^k e_{n-k}(\dots, x_i t^{1-i}, \dots) \right) P^*_{\mu^-}(x_1/q, \dots, x_n/q). \end{aligned}$$

By inductive assumption (4.11) is true for μ^-. It is well known that the product

$$e_r P_\nu$$

is a linear combination of such Macdonald polynomials P_η that

$$\eta/\nu$$

is a vertical r-strip. Therefore (4.11) is true also for μ.

Suppose $\mu_n = 0$. By inductive assumption

$$\begin{aligned} P^*_\mu(y_1, \dots, y_{n-1}) &= P_\mu(y_1, \dots, y_{n-1} t^{2-n}) \\ &\quad + \sum_{\nu \subset \mu} c_\nu P_\nu(y_1, \dots, y_{n-1} t^{2-n}). \end{aligned}$$

[*] Explicit formulas for these coefficients are given by a particular case of the *binomial theorem* for Macdonald polynomials, see the recent paper [Ok3]

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

Therefore

$$P^*_\mu(y_1,\dots,y_{n-1}) = t^{(2-n)|\mu|}P_\mu(y^*_1,\dots,y^*_{n-1}) + \sum_{\nu\subset\mu} c'_\nu P_\nu(y^*_1,\dots,y^*_{n-1}),$$

for some constants c'_ν. Integrating as in (3.2) and (4.5) we obtain

$$P^*_\mu(x_1,\dots,x_n) = t^{(1-n)|\mu|}P_\mu(x^*_1,\dots,x^*_n) + \sum_{\nu\subset\mu} c''_\nu P_\nu(x^*_1,\dots,x^*_n),$$

which is equivalent to (4.11). □

Using (4.5) we can reprove one more result from [K], which generalizes the corresponding result for the shifted Schur functions [OO], Theorem 3.1.

PROPOSITION (VANISHING PROPERTY) 4.6.

$$P^*_\mu(q^\lambda) = 0, \quad \text{unless } \mu \subseteq \lambda. \tag{4.12}$$

Proof. Induct on n and $|\mu|$. If $\mu_n > 0$ apply Proposition 4.2.

If $\mu_n = 0$ then apply the very same argument which proved vanishing in the proof of the theorem. □

REMARK 4.7. The formulas (3.2) and (4.5) define certain q-integral operators

$$\Lambda(n-1) \to \Lambda(n)$$

and

$$\Lambda^*(n-1) \to \Lambda^*(n).$$

Using projection (3.12) (and its analog for Λ^*) in the inverse direction we can characterize Macdonald polynomials and shifted Macdonald polynomials as eigenfunctions (with distinct eigenvalues) of some integral operators (in the same way as for Schur functions [OO], Section 10).

In fact, there are countably many commuting integral operators which correspond to iterated integration (3.2) and (4.5)

$$\Lambda(n) \to \Lambda(N), \quad n < N,$$

and projection back

$$\Lambda(N) \to \Lambda(n).$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

5. Combinatorial formula for shifted Macdonald polynomials

In this section we shall establish branching rule for shifted Macdonald polynomials, or (what is the same) a combinatorial formula for P^*_μ in terms of semistandard tableaux on μ.

First, we need some qualitative results about the branching for P^*_μ.

PROPOSITION 5.1. *Let d be a variable, then*

$$P^*_\mu(dx_1,\dots,dx_n)=\sum_{\nu\subseteq\mu}c_\nu(d)P^*_\nu(x_1,\dots,x_n), \tag{5.1}$$

where $c_\nu(d)$ are some polynomials in d which depend also on μ and n.

Proof. It is clear that an expansion

$$P^*_\mu(dx_1,\dots,dx_n)=\sum_{\nu}c_\nu(d)P^*_\nu(x_1,\dots,x_n)$$

exists. We have to show that

$$c_\nu(d)=0, \quad \text{unless } \nu\subseteq\mu.$$

By (4.11) the bases

$$\{P^*_\mu(x_1,\dots,x_n)\} \tag{5.2}$$

and

$$\{P_\mu(\dots,x_it^{1-i},\dots)\} \tag{5.3}$$

are mutually triangular with respect to the partial ordering by inclusion of diagrams. Since dilatation by d in diagonal in the basis (5.3) it is triangular in the basis (5.2). □

Put

$$\mu'=(\mu_1,\dots,\mu_{n-1}).$$

We have

PROPOSITION 5.2.

$$P^*_\mu(x_1,\dots,x_{n-1},d)=\sum_{\nu\subseteq\mu'}c_\nu(d)P^*_\nu(x_1,\dots,x_{n-1}), \tag{5.4}$$

where $c_\nu(d)$ are some polynomials in d which depend also on μ and n.

Proof. Again we have to show that certain summands cannot occur.

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

If $d=1$ then by stability of P^*_μ

$$P^*_\mu(x_1,\dots,x_{n-1},1)=\begin{cases}P^*_\mu(x_1,\dots,x_{n-1}), & \mu_n=0\\ 0, & \mu_n>0.\end{cases}$$

For general d introduce new variables ξ_i

$$x_i=d\xi_i,\quad i=1,\dots,n-1,$$

and use (5.1). □

Put

$$`\mu=(\mu_2,\dots,\mu_n).$$

Recall that

$$\nu\prec\mu$$

means that

$$\mu_1\geqslant\nu_1\geqslant\mu_2\geqslant\cdots\geqslant\nu_{n-1}\geqslant\mu_n,$$

which is equivalent to

$$`\mu\subseteq\nu\subseteq\mu'.$$

PROPOSITION 5.3.

$$P^*_\mu(d,x_2,\dots,x_n)=\sum_{\nu\prec\mu}f_{\mu,\nu}(d)P^*_\nu(x_2,\dots,x_n), \tag{5.5}$$

where $f_{\mu,\nu}(d)$ are some polynomials in d.

Note that because of stability of shifted Macdonald polynomials the polynomial $f_{\mu,\nu}(d)$ does *not* depend on n.

Proof. By the shifted symmetry

$$P^*_\mu(d,x_2,\dots,x_n)=P^*_\mu(x_2/t,\dots,x_n/t,t^{n-1}d).$$

Hence by (5.4)

$$P^*_\mu(d,x_2,\dots,x_n)=\sum_{\nu\subseteq\mu'}c_\nu(t^{n-1}d)P^*_\nu(x_2/t,\dots,x_n/t).$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

By (5.1)

$$P^*_\nu(x_2/t,\dots,x_n/t)=\sum_{\eta\subseteq\nu} c_\eta(1/t)P^*_\eta(x_2,\dots,x_n).$$

Therefore only summands with

$$\nu\subseteq\mu'$$

can occur in (5.5).

Now assume that there is a summand in (5.5) with

$$\mu\not\subseteq\nu.$$

We can choose ν minimal, that is in such a way that P^*_η with $\eta\subset\nu$ do not enter the sum (5.5). Then for all sufficiently large d

$$P^*_\mu(d,\nu_1,\dots,\nu_{n-1})\neq 0$$

because

$$P^*_\nu(\nu_1,\dots,\nu_{n-1})\neq 0$$

and all other summands in (5.5) vanish by the vanishing property (4.12). Clearly, this contradicts (4.12) and makes our assumption impossible. □

Now recall the corresponding result for Macdonald polynomials

$$P_\mu(d,x_2,\dots,x_n)=\sum_{\nu\prec\mu}\psi_{\mu,\nu}d^{|\mu/\nu|}P_\nu(x_2,\dots,x_n), \tag{5.6}$$

where $\psi_{\mu,\nu}$ are certain nonzero rational functions of q and t which can be found in [M], Section VI.7. Introduce the following notation

$$\langle d\rangle_{\mu/\nu}=\prod_{s\in\mu/\nu}(d-q^{a'(s)}t^{-l'(s)}),$$

where the numbers $a'(s)$ and $l'(s)$ for a square $s\in\mu$ were defined in Section 4. The number $q^{a'(s)}t^{-l'(s)}$ is the q-analog of the *content* of the square s. If

$$\nu=\emptyset,\qquad \mu=(r),$$

then

$$\langle d\rangle_{\mu/\nu}=\langle d\rangle_r.$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

Therefore the number $\langle d\rangle_{\mu/\nu}$ is a generalization of the factorial power $\langle d\rangle_r$. The main result of this section is the following

THEOREM III.

$$P_\mu^*(d, x_2, \dots, x_n) = \sum_{\nu \prec \mu} \psi_{\mu,\nu} t^{-|\nu|} \langle d\rangle_{\mu/\nu} P_\nu^*(x_2, \dots, x_n), \tag{5.7}$$

It is clear that iterating this formula we obtain the semistandard tableaux sum formula (1.4) for polynomials P_μ^*. We shall use induction on n. In fact we shall need only the following corollary of this theorem

COROLLARY 5.4. *For all* $r = 1, \dots, n$.

$$P_\mu^*(x_1, \dots, x_r, q^{\mu_{r+1}}, \dots, q^{\mu_n}) = c x_1^{\mu_1} \dots x_r^{\mu_r} + \cdots, \tag{5.8}$$

where c *is a nonzero factor and dots stand for lower monomials in lexicographic order.*

Proof. Given a partition

$$\eta = (\eta_1, \dots, \eta_{n-1})$$

set

$$^{(i)}\eta = (\eta_i, \dots, \eta_{n-1}).$$

Then

$$c = t^{-\sum_1^r |^{(i+1)}\mu|} \left(\prod \psi_{^{(i)}\mu, ^{(i+1)}\mu}\right) H(^{(r+1)}\mu). \qquad \square$$

Proof of the theorem. Induction on n. The case $n = 1$ is clear. Suppose $n > 1$. We shall find

$$|\mu/\nu|$$

distinct zeros of the polynomial $f_{\mu,\nu}(d)$. Fix some i and show that

$$f_{\mu,\nu}(d) = 0, \qquad d = q^{\mu_i - 1} t^{1-i}, \dots, q^{\nu_i} t^{1-i}. \tag{5.9}$$

We shall prove (5.9) by induction on $^{(i)}\nu$, in other words we shall deduce (5.9) from the assumption that

$$f_{\mu,\eta}(d) = 0, \qquad d = q^{\mu_i - 1} t^{1-i}, \dots, q^{\eta_i} t^{1-i} \tag{5.10}$$

for all η such that

$$^{(i)}\eta \subset {}^{(i)}\nu.$$

Suppose d is as (5.9). We have

$$\begin{aligned} &P^*_\mu(d, x_2, \dots, x_i, q^{\nu_i}, \dots, q^{\nu_{n-1}}) \\ &\quad = \sum_{\eta \prec \mu} f_{\mu,\eta}(d) P^*_\eta(x_2, \dots, x_i, q^{\nu_i}, \dots, q^{\nu_{n-1}}). \end{aligned} \tag{5.11}$$

By the vanishing property (4.12) only summands satisfying

$$^{(i)}\eta \subseteq^{(i)} \nu$$

are nonzero. On the other hand, if

$$^{(i)}\eta \subset^{(i)} \nu,$$

then in particular

$$\eta_i \leqslant \nu_i,$$

and by our Assumption (5.10) the corresponding summand vanishes. Therefore only summands with

$$^{(i)}\eta =^{(i)} \nu$$

enter the sum. By (5.8) each summand has the following form

$$c f_{\mu,\eta}(d)(x_2^{\eta_1} \dots x_i^{\eta_{i-1}} + \cdots), \tag{5.12}$$

where c is a nonzero factor and dots stand for lower monomials in lexicographic order.

On the other hand, by shifted symmetry

$$\begin{aligned} &P^*_\mu(d, x_2, \dots, x_i, q^{\nu_i}, \dots, q^{\nu_{n-1}}) \\ &\quad = P^*_\mu(x_2/t, \dots, x_i/t, t^{i-1}d, q^{\nu_i}, \dots, q^{\nu_{n-1}}). \end{aligned}$$

By the vanishing property (4.12) this should vanish if d is as in (5.9) and

$$x_j/t = q^{\lambda_j}, \quad j = 2, \dots, i$$

for all sufficiently large integers

$$\lambda_2 \geqslant \cdots \geqslant \lambda_i.$$

Therefore for such d the polynomial (5.11) should be identically zero. By virtue of (5.12) it is impossible unless

$$f_{\mu,\eta}(d) = 0$$

for all η such that ${}^{(i)}\eta = {}^{(i)}\nu$. This proves (5.9).

Since

$$\deg f_{\mu,\nu} \leqslant |\mu/\nu|$$

the polynomial $f_{\mu,\nu}$ equals

$$\langle d \rangle_{\mu/\nu}$$

up to a factor. This factor is clear from (4.11) and (5.6). □

EXAMPLE 5.5. The shifted analogs of the elementary symmetric functions are

$$P^*_{(1^k)}(x) = \sum_{i_1<\cdots<i_k} t^{k-\sum i_s} \prod_s (x_{i_s} - t^{s-k}). \tag{5.13}$$

REMARK 5.6. The shifted symmetry of P^*_μ results in some complicated identities for coefficients $\psi_{\mu,\nu}$ which are not clear from the explicit formula for these coefficients.

6. Duality

The duality we shall discuss in this section relates shifted Macdonald polynomials with parameters q and t to shifted Macdonald polynomials with parameters $1/t$ and $1/q$. Denote by

$$\Lambda^*_t$$

the algebra Λ^* constructed in Section 4. Denote by

$$p^*_k(x;t) = \sum_i t^{k(1-i)}(x_i^k - 1) \tag{6.1}$$

the power-sum generators of Λ^*_t. Consider the following isomorphism $\omega^*_{q,t}$

$$\Lambda^*_t \xrightarrow{\omega^*_{q,t}} \Lambda^*_{1/q},$$

$$p^*_k(t) \mapsto \frac{1-q^k}{1-(1/t)^k} p^*_k(1/q). \tag{6.2}$$

Note a slight difference with the Macdonald duality automorphism. The isomorphism (6.2) has the following clear combinatorial interpretation which generalizes the corresponding result for shifted Schur function [OO], Theorem 4.1.

PROPOSITION 6.1. *For all* $f \in \Lambda_t^*$

$$[\omega_{q,t}^*(f)](t^{-\lambda}) = f(q^{\lambda'}). \tag{6.3}$$

Proof. It suffices to check (6.3) on generators (6.1). For any partition λ we have

$$\frac{1}{q-1}\sum_i t^{1-i}(q^{\lambda_i}-1) = \frac{1}{1/t-1}\sum_j q^{j-1}(t^{-\lambda'_j}-1), \tag{6.4}$$

which is Example VI.5.1(a) in [M]. To see (6.4) one can use the same trick as in [OO], Theorem 4.1, and sum

$$\sum_{(i,j)\in\lambda} q^{j-1}t^{1-i}$$

first along rows and then along columns. It is clear that (6.4) means

$$\frac{1-q}{1-1/t}p_1^*(t^{-\lambda};1/q) = p_1^*(q^{\lambda'};t).$$

Now replace q by q^k and t by t^k in (6.4) to obtain

$$\frac{1-q^k}{1-(1/t)^k}p_k^*(t^{-\lambda};1/q) = p_k^*(q^{\lambda'};t),$$

as desired. □

COROLLARY 6.2.

$$\omega_{q,t}^*(P_\mu^*(q,t)) = cP_{\mu'}^*(1/t,1/q), \tag{6.5}$$

where c *is a constant which depends on* μ.

Proof. Follows from (6.3) and definition of P_μ^*. □

Recall the following notation of Macdonald

$$b_\lambda(q,t) = \prod_{s\in\lambda}\frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}.$$

In particular,

$$b_{\lambda'}(t,q) = \prod_{s\in\lambda}\frac{1-q^{a(s)+1}t^{l(s)}}{1-q^{a(s)}t^{l(s)+1}}.$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

THEOREM IV.

$$\omega^*_{q,t}(P^*_\mu(q,t)) = (-t)^{|\mu|} b_{\mu'}(t,q) P^*_{\mu'}(1/t, 1/q).$$

Proof. It is clear that in order to calculate the constant in (6.5) we have to evaluate the equality (6.5) at $t^{-\mu'}$. By (6.3) and definition of P^*_μ we have

$$\begin{aligned} c &= \frac{H(\mu;q,t)}{H(\mu';1/t,1/q)} \\ &= \frac{t^{-2n(\mu)} q^{n(\mu')} \prod_{s\in\mu}(q^{a(s)+1}t^{l(s)}-1)}{q^{2n(\mu')}t^{-n(\mu)}\prod_{s\in\mu}(q^{-a(s)}t^{-l(s)-1}-1)} \\ &= (-t)^{|\lambda|} b_{\lambda'}(t,q), \end{aligned}$$

because

$$\sum_{s\in\mu} l(s) = n(\mu), \qquad \sum_{s\in\mu} a(s) = n(\mu'). \qquad \square$$

EXAMPLE 6.3. Since the automorphism (6.2) preserves the degree we can look at the corresponding isomorphism of the graded algebras

$$\Lambda \to \Lambda,$$

which differs from Macdonald automorphism

$$p_k \to (-1)^{k-1} \frac{1-q^k}{1-t^k} p_k$$

by factor

$$(-t)^{\deg},$$

where deg is the degree of a polynomial. Since the highest term of P^*_μ is the Macdonald polynomial P_μ and

$$P_\mu(q,t) = P_\mu(1/q, 1/t),$$

the Theorem IV gives a new proof of the duality theorem of Macdonald together with computation of the constant factor.

7. Degeneration

In this section we consider *shifted Jack polynomials*. These polynomials were considered by F. Knop and S. Sahi in [KS]. These polynomials are indexed by partitions μ, depend on a parameter θ and equal, by definition, to

$$P^*_\mu(x;\theta) = \lim_{q\to 1}(q-1)^{-|\mu|} P^*_\mu(q^x; q, q^\theta). \tag{7.1}$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

In this section let us change notations as follows. The dependence on q will be denoted explicitly in all prelimit expressions like in the right-hand side of (7.1). The expressions without q will denote the corresponding degenerations. For example, we write

$$\langle z;q\rangle_r = (z-1)(z-q)\cdots(z-q^{r-1}). \tag{7.2}$$

This is the shifted Macdonald polynomial in one variable

$$P_r^*(z;q,t) = \langle z;q\rangle_r.$$

The corresponding shifted Jack polynomial equals

$$P_r^*(z;\theta) = \langle z\rangle_r,$$

where

$$\langle z\rangle_r = z(z-1)\cdots(z-r+1). \tag{7.3}$$

In the same way write

$$\langle z\rangle_{\mu/\nu} = \prod_{s\in\mu/\nu}(z-a'(s)+\theta l'(s)),$$

this is a generalization of (7.3).

The easiest way to see that (7.1) is a polynomial of degree $|\mu|$ is to look at the combinatorial formula for $P_\mu^*(q,t)$. The branching rule (5.7) has the following limit

$$\begin{aligned} &P_\mu^*(x_1,x_2,\ldots,x_n;\theta)\\ &\quad=\sum_{\nu\prec\mu}\psi_{\mu,\nu}(\theta)\langle x_1\rangle_{\mu/\nu}P_\nu^*(x_2,\ldots,x_n;\theta), \end{aligned} \tag{7.4}$$

where $\psi_{\mu,\nu}(\theta)$ are the branching coefficients for ordinary Jack polynomials, which are rational functions in θ and can be found in [St] and [M], Section VI.10. Note that

$$\theta = 1/\alpha,$$

where α is the traditional parameter for Jack polynomials.

Because of the symmetry and vanishing properties of the shifted Macdonald polynomials we have

$$P_\mu^*(x;\theta) \quad \text{is symmetric in variables } x_i-\theta i, \tag{7.5}$$

$$P^*_\mu(\lambda;\theta)=0 \quad \text{unless } \mu\subseteq\lambda, \tag{7.6}$$

$$P^*_\mu(\mu;\theta)=\prod_{s\in\mu}(a(s)+\theta l(s)+1). \tag{7.7}$$

The product in (7.7) is a θ-analog of the hook length product; denote it by $H(\mu;\theta)$.

Before discussing the limit of integral formula (4.5) for $P^*_\mu(x;q,t)$ consider some elementary properties of finite sums. We have

$$\sum_{y=x_2}^{x_1}\langle y\rangle_r=\frac{\langle x_1+1\rangle_{r+1}-\langle x_2\rangle_{r+1}}{r+1}, \tag{7.8}$$

provided x_1-x_2 is a nonnegative integer. Using (7.8) one can evaluate sums

$$\sum_{y=x_2}^{x_1} f(y) \tag{7.9}$$

for arbitrary polynomials $f(y)$. If x_1-x_2 is not a nonnegative integer then *define* (7.9) using (7.8). In particular, we have

$$\sum_{y=x_2}^{x_1}=-\sum_{y=x_1+1}^{x_2-1}.$$

Now suppose that f is a polynomial and

$$\lim_{q\to1}\frac{1}{(q-1)^s}f(q^y)=\bar f(y)$$

for some s and $\bar f$. Then

$$\lim_{q\to1}\frac{1}{(q-1)^{s+1}}\int_{q^{x_2}}^{q^{x_1+1}} f(y)d_qy=\sum_{y=x_2}^{x_1}\bar f(y), \tag{7.10}$$

which follows directly from definitions if x_1-x_2 is an integer.

Now we calculate the limit of (4.5). Recall that [GR, 1.10]

$$\lim_{q\to1}\Gamma_q(z)=\Gamma(z),$$

where q tends to 1 from below. Therefore

$$\lim_{q\to1}\frac{(q^z)_{\theta-1}}{(1-q)^{\theta-1}}=\frac{\Gamma(z+\theta-1)}{\Gamma(z)}.$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

In the sequel we assume that

$$\theta = 1, 2, \dots .$$

In this case

$$\frac{\Gamma(z+\theta-1)}{\Gamma(z)} = z(z+1)\dots(z+\theta-1).$$

Recall that we denote by $V^{*\theta}(x;q)$ the denominator in (4.5)

$$V^{*\theta}(x;q) = \prod_{i<j}(q^{\theta(n-i)}x_i - q^{\theta(n-j)}x_j)\prod_{i\neq j}(q^{\theta(j-i)+1}x_i/x_j)_{\theta-1}.$$

We have

$$\lim_{q\to 1}\frac{V^{*\theta}(q^x;q)}{(1-q)^{n(n-1)(\theta+1/2)}} = (-1)^{n(n-1)/2}V^{*\theta}(x), \tag{7.11}$$

where

$$V^{*\theta}(x) = \prod_{i<j}(x_i - x_j + \theta(j-i))\prod_{i\neq j}\langle x_i - x_j + \theta(j-i) + \theta - 1\rangle_{\theta-1}. \tag{7.12}$$

Denote by

$$\beta^*(y,x;q,q^\theta) = \prod_{i<j}(q^{\theta(n-1-i)}y_i - q^{\theta(n-1-j)}y_j)\prod_{i,j}(q^{\theta(j-i)+1-\theta}y_i/x_j)_{\theta-1}$$

the density of the beta measure in (4.5). In the same way we have

$$\lim_{q\to 1}\frac{\beta^*(q^y,q^x;q,q^\theta)}{(1-q)^{n(n-1)\theta+(n-1)(n-2)/2}} = (-1)^{(n-1)(n-2)/2}\beta^*(y,x;\theta), \tag{7.13}$$

where, by definition,

$$\beta^*(y,x;\theta) = \prod_{i<j}(y_i - y_j + \theta(j-i))\prod_{i,j}\langle y_i - x_j + \theta(j-i) - 1\rangle_{\theta-1}. \tag{7.14}$$

Using (7.10), (7.11), and (7.14) we obtain the following degeneration of (4.5)

$$\frac{1}{V^{*\theta}(x)}\sum_{y\prec x}P^*_\mu(y;\theta)\beta(y,x;\theta) = C(\mu,n)P^*_\mu(x;\theta), \tag{7.15}$$

where

$$\sum_{y\prec x} = \sum_{y_1=x_2}^{x_1}\cdots\sum_{y_{n-1}=x_n}^{x_{n-1}},$$

https://doi.org/10.1023/A:1000436921311 Published online by Cambridge University Press

and

$$C(\mu, n) = \prod B(\mu_i + (n-i)\theta, \theta).$$

For $\theta = 1$ we obtain the coherence property of the shifted symmetric functions [OO], Section 10.

Acknowledgements

The author is grateful to the Institute for Advanced Study for hospitality and to NSF for financial support under grant DMS 9304580.

References

[AA] Andrews, G. and Askey, R.: Another q-extension of the beta function, *Proc. Amer. Math. Soc.* 81(1) (1981).

[AOS] Awata, H., Odake, S. and Shirashi, J.: *Integral Representations of the Macdonald Symmetric Functions*, q-alg/9506006.

[AV] Al-Salam, W. A. and Verma, A: Some remarks on q-beta integral *Proc. Amer. Math. Soc.* 85(3) (1982).

[EK] Etingof, P. and Kirillov, A. Jr.: Macdonald's polynomials and representations of quantum groups, *Math. Res. Lett.* 1 (1994) 279–296.

[GR] Gasper, G. and Rahman, M.: *Basic Hypergeometric Series*, Cambridge University Press, 1990.

[Ka] Kadell, K. W. J.: The q-Selberg polynomials for $n = 2$, *Trans. AMS* 310 (1988) 535–553.

[K] Knop, F.: *Symmetric and Non-Symmetric Quantum Capelli Polynomials*, q-alg 9603028 (to appear).

[KOO] Kerov, S., Okounkov, A. and Olshanski, G.: *The Boundary of Young Graph with Jack Edge multiplicities*, q-alg/9703037 (to appear).

[KS] Knop, F. and Sahi, S.: Difference equations and symmetric polynomials defined by their zeros, *Internat. Math. Res. Notices* 10 (1996) 473–486.

[M] Macdonald, I. G.: *Symmetric Functions and Hall Polynomials, second edition*, Oxford University Press, 1995.

[MN] Mimachi, K. and Noumi, M.: *Notes on Eigenfunctions for Macdonald's q-Difference Operators*, preprint.

[N] Nazarov, M.: *Yangians and Capelli identities*, to appear in A. A. Kirillov representation theory seminar, G. Olshanski (ed.), Adv. in Math. Sciences, Amer. Math. Soc., q-alg 9601027.

[Ok1] Okounkov, A.: Quantum immanants and higher Capelli identities, *Transformation Groups* 1(1) (1996) 99–126.

[Ok2] Okounkov, A.: Young Basis, Wick Formula, and Higher Capelli identities, *Internat. Math. Res. Not.* 17 (1996) 817–839.

[Ok3] Okounkov, A.: *Binomial Formula for Macdonald Polynomials*, q-alg 9608021 (to appear).

[Ok4] Okounkov, A.: *Shifted Macdonald Polynomials with 3 Parameters and Binomial Formula for Koornwinder Polynomials*, to appear, q-alg/9611011.

[Ok5] Okounkov, A.: *On n-Point Correlations in the Log-Gas at Rational Temperature*, to appear, hep-th/9702001.

[Ol] Olshanski, G.: *private communication*.

[OO] Okounkov, A. and Olshanski, G.: Shifted Schur functions, *Algebra i Analiz* 9(2) (1997) (Russian); English version to appear in St. Petersburg Math. J. 9(2) (1998).

[OO2] Okounkov, A. and Olshanski, G.: Shifted Schur functions II, to appear in A. A. Kirillov representation theory seminar, G. Olshanski ed., Adv. in Math. Sciences, Amer. Math. Soc.

[OO3] Okounkov, A. and Olshanski, G.: Shifted Jack polynomials, binomial formula, and applications, *Math. Res. Letters* 4 (1997) 69–78.

[S] Sahi, S.: *Interpolation, Integrality, and a Generalization of Macdonald's Polynomials*, Internat. Math. Res. Notices (1996) 10 457–471.

[St] Stanley, R. P.: Some Combinatorial Properties of Jack Symmetric Functions, *Adv. in Math.* 77 (1989) 76–115.