ON BESSEL POLYNOMIALS

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1. Introduction. Recently a number of papers have been written on Bessel polynomials which arise as the solutions of the classical wave equation in spherical coordinates. Krall and Frink **(5)** studied in some detail the properties of these polynomials $y_n(x, a, b)$ defined as

$$(1) \qquad y_n(x, a, b) = {}_2F_0(-n, a + n - 1; -x/b).$$

In particular, they gave the differential equation, nth differential formula, orthogonal property and recurrence relations. The special case $y_n(x)$ of these polynomials, obtained by taking $a = b = 2$, was studied in greater detail than the generalised polynomials. Besides the above properties, a generating function was also derived for $y_n(x)$. It was also shown that $y_n(x)$ are closely connected with Bessel functions of half-integral order.

Later, Burchnall **(2)** identified these polynomials by certain other polynomials studied by him many years back **(3)**. By the help of certain differential operators he filled in certain gaps in Krall and Frink's work and also gave simple proofs of some of their formulae. He derived a generating function for $y_n(x, a, b)$ and gave some properties regarding the zeros of the polynomials $y_n(x)$, which were not given by Krall and Frink.

Very recently Rainville **(6)** and Brafman **(1)** have given some generating functions for these polynomials. The generating functions given by Rainville were very general and one of them includes Burchnall's result as a particular case.

But it seems rather surprising that none of the above authors seem to have noticed that these polynomials $y_n(x, a, b)$ are merely a special limiting case of the classical Jacobi polynomials $P_n^{(\alpha, \beta)}(x)$. In this paper I use this definition and derive most of the results given by Krall and Frink and Burchnall as simple limiting cases of known results for these polynomials.

Since Jacobi polynomials have been extensively studied, one can find many more interesting properties for $y_n(x, a, b)$ simply as limiting cases.

In §9 I give certain simple expansions involving these polynomials to indicate the possibility of getting more results of this type.

I conclude the paper by giving a very simple relationship between $y_n(x, a, b)$ and the Whittaker's function $W_{k,m}(x)$. I am grateful to Professor Bailey for pointing out this interesting relationship.

2. Definition. We know that **(7,** 4.21.2**)**

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\,\Gamma(\alpha+1)}\ {}_2F_1\left(-n,\ n+\alpha+\beta+1;\ \alpha+1;\ \frac{1-x}{2}\right).$$

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Thus from the definition (1) of $y_n(x, a, b)$ it follows easily that

$$(2) \qquad y_n(x, a, b) = \lim_{\epsilon\to\infty} \frac{\Gamma(n+1)\,\Gamma(\epsilon)}{\Gamma(\epsilon+n)}\, P_n^{(\epsilon-1, a-\epsilon-1)}\left(1 + \frac{2\epsilon x}{b}\right).$$

3. The differential equation. The differential equation satisfied by $P_n^{(\alpha,\beta)}(x)$ is known to be (**7**, 4.2.1)

$$(1 - x^2)\frac{d^2y}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)\,x]\frac{dy}{dx} + n(n + \alpha + \beta + 1)\,y = 0.$$

Changing the variables by putting

$$1 + \frac{2\epsilon x}{b} \text{ for } x, \quad \alpha = \epsilon - 1 \text{ and } \beta = a - \epsilon - 1$$

and using (2) we get on taking the limit that the differential equation satisfied by $y_n(x, a, b)$ is

$$(3) \qquad x^2 \frac{d^2y}{dx^2} + (ax + b)\frac{dy}{dx} = n(n + a - 1)\,y.$$

This is the equation given by Krall and Frink (**5**, equation 2).

4. The nth differential formula. It is known that (**7**, 4.3.1)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{2^n\,n!}(-)^n \frac{d^n}{dx^n}\{(1-x)^{n+\alpha}\,(1+x)^{n+\beta}\}.$$

Using the definition (2) we get

$$y_n(x, a, b) = \frac{(-)^n\,\Gamma(n+1)}{2^n\,n!}\lim_{\epsilon\to\infty}\frac{\Gamma(\epsilon)}{\Gamma(\epsilon+n)}\left(-\frac{2\epsilon x}{b}\right)^{1-\epsilon}\left(2 + \frac{2\epsilon x}{b}\right)^{1+\epsilon-a}\left(\frac{b}{2\epsilon}\right)^n$$
$$\times \frac{d^n}{dx^n}\left[\left(-\frac{2\epsilon x}{b}\right)^{n+\epsilon-1}\left(2 + \frac{2\epsilon x}{b}\right)^{n+a-\epsilon-1}\right].$$

This on simplification gives

$$(4) \qquad y_n(x, a, b) = b^{-n}\,x^{2-a}\,e^{b/x}\frac{d^n}{dx^n}(x^{2n+a-2}\,e^{-b/x}),$$

which gives Krall and Frink's result (**5**, equation 47).

5. Recurrence formulae. We can easily deduce from the known results (**7**; 4.5.1, 4.5.7 (i) and (ii) and 4.5.4 (ii)) the following four recurrence formulae for $y_n(x, a, b)$:

$$(5) \quad (n + a - 1)(2n + a - 2)\,y_{n+1} = [(2n + a)(2n + a - 2)\,x/b + a - 2](2n + a - 1)\,y_n + n(2n + a)\,y_{n-1},$$

$$(6) \qquad x^2(2n + a - 2)\frac{dy_n}{dx} = [n(2n + a - 2)\,x - nb]\,y_n + nby_{n-1},$$

$$(7) \qquad (2n + a)\,x^2\frac{dy_n}{dx} = (1 - n - a)[(2n + a)\,x + b]\,y_n + b(n + a - 1)\,y_{n+1},$$

$$(8) \qquad (2n + a)\,x\,y_n(x, a + 1, b) = b[y_{n+1} - y_n].$$

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The first two recurrence formulae were given by Krall and Frink (**5**, 51) but the last two are new. It may be remarked that the proof of (6) above as given by Krall and Frink was especially long.

6. The generating function. The generating function for the Jacobi polynomials is (**7**, 4.4.5)

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\, w^n$$
$$= 2^{\alpha+\beta}(1 - 2xw + w^2)^{-\frac{1}{2}}\{1 - w + (1 - 2xw + w^2)^{\frac{1}{2}}\}^{-\alpha}$$
$$\times \{1 + w + (1 - 2xw + w^2)^{\frac{1}{2}}\}^{-\beta},$$

valid for sufficiently small values of $|w|$. Putting

$$w = \tfrac{1}{2}tb/\epsilon,\ \alpha = \epsilon - 1,\ \beta = a - \epsilon - 1,\ x \equiv 1 + 2\epsilon x/b,$$

and using the definition (2) we get, on taking the limit,

$$(9)\quad [\tfrac{1}{2} - \tfrac{1}{2}(1 - 2xt)^{\frac{1}{2}}]^{2-a}(1 - 2xt)^{-\frac{1}{2}} \exp\,[\{1 - (1 - 2xt)^{\frac{1}{2}}\}\, b/2x]$$
$$= \sum_{n=0}^{\infty} (\tfrac{1}{2}b)^n y_n(x, a, b)\, t^n/n!.$$

This gives the generating function for $y_n(x, a, b)$ given by Burchnall (**2**, §6). From this by obvious substitutions we can get the pseudo-generating function for $y_n(x, a, b)$ given by Burchnall [**2**, §6 (24)] viz,

$$\frac{(1 - u)^{2-a}e^{bxu}}{1 - 2u} = \sum_{n=0}^{\infty} \frac{[bu(1 - u)]^n}{n!} x^n\, y_n(x^{-1}, a, b).$$

For $a = b = 2$, (9) gives the generating function for $y_n(x)$ given by Krall and Frink (**5**, 25).

7. A contour integral representation. An immediate consequence of the generating function (9) not noted by the above authors is the following contour integral representation for the polynomials $y_n(x, a, b)$. If a is an integer (to make the integrand single-valued), we have

$$\frac{(\frac{1}{2}b)^n}{n!} y_n(x, a, b)$$
$$= \frac{1}{2\pi i}\int_C t^{-n-1}[\tfrac{1}{2} - \tfrac{1}{2}(1 - 2xt)^{\frac{1}{2}}]^{2-a}(1 - 2xt)^{-\frac{1}{2}} \exp\,[\{1 - (1 - 2xt)^{\frac{1}{2}}\}\, b/2x]dt,$$

or

$$y_n(x, a, b) = \frac{1}{2\pi i}\frac{n!}{b^n} x^n \int_C \frac{u^{2-a}}{[u(1 - u)]^{n+1}} e^{bu/x}\, du,$$

where C is a simple closed curve round the origin, small enough not to pass through the point $u = 1$, and a is an integer.

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8. Orthogonal property and properties of zeros. Since the polynomials $y_n(x, a, b)$ are orthogonal over a unit circle and the Jacobi polynomials over the real interval $(-1, 1)$, the limiting process does not help and we may have to resort to other methods (e.g. **7**, 11.5). But it seems to be more complicated than the classical method given by Krall and Frink and hence there seems to be no interest in going into the details. Once knowing the differential equation satisfied by $y_n(x, a, b)$ we can easily apply Krall and Frink's method.

Further, we can also deduce some of the properties of the zeros of these polynomials given by Burchnall, e.g., it is known that if $x_\nu (\nu = 1, 2, \ldots, n)$ are the zeros of $P_n^{(\alpha,\beta)}(x)$ (**7**, ex. 14, p. 370) then

$$\sum_{\nu=1}^{n} x_\nu = \frac{n(\beta - \alpha)}{2n + \alpha + \beta}.$$

Let the zeros of

$$P_n^{(\epsilon-1, a-\epsilon-1)}(1 + 2\epsilon x/b)$$

be denoted by $X_\nu (\nu = 1, 2, \ldots, n)$; then it is obvious that

$$\sum_{\nu=1}^{n} X_\nu = \sum_{\nu=1}^{n} \frac{b}{2\epsilon}(x_\nu - 1) = \frac{nb}{2\epsilon}\left[\frac{a - 2\epsilon - 1}{2n + a - 2} - 1\right].$$

Now let $\{\xi_\nu\}$ be the zeros of $y_n(x, a, b)$; then obviously

$$\sum_{\nu=1}^{n} \xi_\nu = \lim_{\epsilon\to\infty} \sum_{\nu=1}^{n} X_\nu = -\tfrac{1}{2}b.$$

Hence, we have

$$\sum_{\nu=1}^{n} \xi_\nu = -\tfrac{1}{2}b, \tag{10}$$

where $\xi_\nu (\nu = 1, 2, \ldots, n)$ are the zeros of $y_n(x, a, b)$.

This generalizes the result of Burchnall, who deduced that the sum of zeros of $y_n(x)$ is to equal to -1.

It is probable that one might find other properties of the zeros of $y_n(x, a, b)$ as limiting properties of the zeros of $P_n^{(\alpha,\beta)}(x)$, but I have not gone into the detail of that aspect.

9. Some expansions involving $y_n(x, a, b)$. We can get some interesting expansions for these polynomials from known expansions of Burchnall and Chaundy (**4**). I mention some of them, omitting the very simple proofs.

From (**4**, 30) we get

$$y_n(x, b + b' + 1, c) = \sum_{r=0}^{n} \frac{(-n)_r(b + n)_r(b')_r}{r!} (x/c)^{2r} y_{n-r}(x, b+2r+1, c) \times y_{n-r}(x, b' - n + 2r + 1, c), \tag{11}$$

which may be taken as a quasi-addition theorem for the parameters b and b'.

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Similarly, from **(4**, 38**)**, we get

$$(12)\ y_{2n}(2x, b+1, c) = \sum_{r=0}^{2n} \frac{(-2n)_{2r}(b+2n)_r}{r!}\left(\frac{x}{c}\right)^{2r} \times y_{2n-2r}(x, 2b+2n+4r+1, c);$$

this may be taken as a duplication-theorem for these polynomials.

In the same way one can write down many other expansions from Burchnall and Chaundy, and other known expansions involving ordinary hypergeometric functions.

10. The relationship of $y_n(x, a, b)$ with Whittaker's function $W_{k,m}(x)$. It is known that **(8**, §16.3**)**

$$W_{k,m}(x) \sim e^{-\frac{1}{2}x}x^k\left\{1 + \sum_{p=1}^{\infty} \frac{(-)^p(m-k+\frac{1}{2})_p(-m-k+\frac{1}{2})_p}{p!\,x^p}\right\},$$

for large values of $|x|$ when $|\arg x| \leqslant \pi - \alpha$. Putting $m = -k + \frac{1}{2} + n$, where n is a positive integer, we get

$$W_{k,-k+\frac{1}{2}+n}(x) = e^{-\frac{1}{2}x}x^k \sum_{p=0}^{n} \frac{(-)^p(-n)_p(1-2k+n)_p}{p!\,x^p}.$$

Put $k = 1 - \frac{1}{2}a$. Then

$$W_{1-\frac{1}{2}a,\frac{1}{2}a-\frac{1}{2}+n}(x) = e^{-\frac{1}{2}x}x^{1-\frac{1}{2}a}\,{}_2F_0(-n, a+n-1, -1/x).$$

Hence

$$(13)\qquad y_n(x, a, b) = e^{-\frac{1}{2}bx}(b/x)^{1-\frac{1}{2}a}W_{1-\frac{1}{2}a,\frac{1}{2}a-\frac{1}{2}+n}(b/x).$$

From this definition and from the known properties of Whittaker's function which has been widely studied one can deduce all the important properties of $y_n(x, a, b)$ together with many new ones. This definition has the advantage that it avoids the limiting process given in §2.

It easily follows from (13) and the integral **(8**, §16.12**)**

$$W_{k,m}(x) = \frac{e^{-\frac{1}{2}x}x^k}{\Gamma(\frac{1}{2}-k+m)}\int_0^\infty t^{-k-\frac{1}{2}+m}(1+t/x)^{k-\frac{1}{2}+m}e^{-t}dt \quad (\Re(k-\tfrac{1}{2}-m) \leqslant 0),$$

that

$$(14)\qquad y_n(x, a, b) = \frac{1}{\Gamma(a+n-1)}\int_0^\infty t^{a-2+n}(1+tx/b)^n e^{-t}dt,$$

valid for $\Re(a+n-1) \geqslant 0$.

In a similar manner we can find other new properties for $y_n(x, a, b)$.

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