THE GENUS OF THE n-CUBE

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The definition of the genus $\gamma(G)$ of a graph G is very well known (König **2**): it is the minimum genus among all orientable surfaces in which G can be drawn without intersections of its edges. But there are very few graphs whose genus is known. The purpose of this note is to answer this question for one family of graphs by determining the genus of the n-cube.

The graph Q_n called the *n-cube* has 2^n vertices each of which is a binary sequence $a_1 a_2 \ldots a_n$ of length n, where $a_i = 0$ or 1. Two of its vertices are adjacent (joined by an edge) whenever their sequences differ in exactly one place. Thus each vertex of Q_n has degree n, i.e. it is adjacent with n other vertices. Hence the number of edges of Q_n is $n2^{n-1}$.

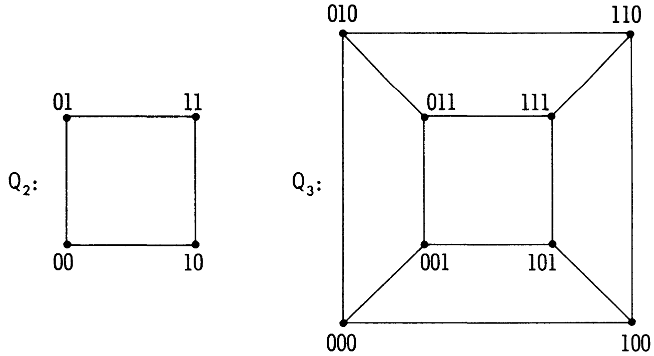

FIGURE 1

THEOREM. *Whenever $n \geqslant 2$, the genus of the n-cube is given by*

$$\gamma(Q_n) = (n-4)2^{n-3} + 1. \tag{1}$$

Let $\gamma_n = (n-4)2^{n-3} + 1$. The proof that $\gamma(Q_n) \geqslant \gamma_n$ will be seen to follow readily from Euler's classical polyhedron formula. To show that $\gamma(Q_n) \leqslant \gamma_n$, an inductive construction will be provided by which Q_n can be embedded in an orientable surface of genus γ_n.

We first show that $\gamma(Q_n) \geqslant \gamma_n$. As mentioned in **(1)**, it follows from Euler's formula that the genus $\gamma(G)$ of any even graph G (having no cycles of odd length) with p vertices and q edges satisfies the inequality

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(2) $$\gamma(G) \geqslant \tfrac{1}{4}q - \tfrac{1}{2}(p-2).$$

Since each n-cube, Q_n, is obviously an even graph, we have

$$\gamma(Q_n) \geqslant \tfrac{1}{4}n2^{n-1} - \tfrac{1}{2}(2^n - 2)$$
$$= (n-4)2^{n-3} + 1.$$

To demonstrate the converse inequality we prove by induction on n that the following slightly stronger result holds:

PROPOSITION. *The n-cube Q_n can be embedded on an orientable surface of genus γ_n in such a way that every quadrilateral in Q_n whose vertices differ only in the first and last places of their sequences is a face.*

The proposition clearly holds for Q_2. Assume that it is true for Q_{n-1}. Take two orientable surfaces S_0 and S_1 both of genus γ_{n-1} which have Q_{n-1} embedded on each of them in such a way that (i) the resulting embeddings are "mirror images" of each other, and (ii) each of the embeddings satisfies the proposition. In both S_0 and S_1 each vertex v lies on exactly one quadrilateral face $F(v)$ whose vertices differ only in their first and last $(n-1)$-st places.

We now construct a surface S of genus γ_n. To the sequence of each vertex in S_0, suffix a 0; to each vertex in S_1, suffix a 1. A "handle" can be placed from the face $F(00\ldots00)$ of S_0 to the face $F(00\ldots01)$ of S_1, thereby forming a surface of genus $2\gamma_{n-1}$. There are $\frac{1}{4}(2^{n-1}-4)$ other faces on S_0 (and of course also on S_1) which are faces $F(v)$ for four vertices v. If "handles" are added joining corresponding faces from S_0 and S_1, we obtain a surface S of genus $2\gamma_{n-1} + (2^{n-3}-1)$, which is readily seen to equal γ_n. On this surface S we can embed Q_n, because each point of S_0 can be joined to the corresponding point of S_1 (i.e. points differing only in the nth place can be joined) without intersections using the new handles, as shown in Figure 2. Moreover, each quadrilateral of Q_n whose edges join vertices differing in the first and last (the nth) places is a face.

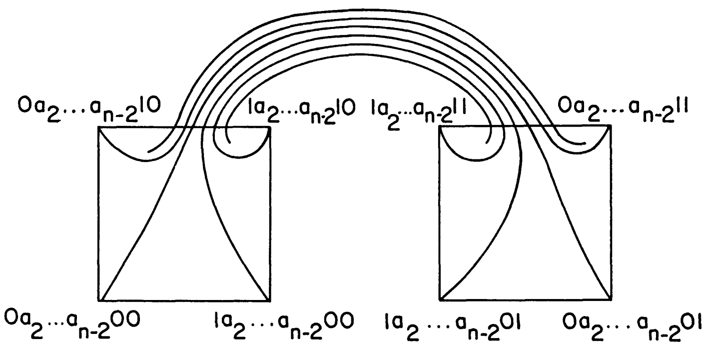

FIGURE 2

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Thus Q_n is embedded in this surface S of genus γ_n in accordance with the proposition showing that $\gamma(Q_n) \leqslant \gamma_n$.

Both of the inequalities having been established, the theorem is proved.

Figure 3 illustrates this construction in detail by showing the embedding of Q_4 into a surface of genus $\gamma_4 = 1$, i.e. a torus, using two mirror-image copies of Q_3.

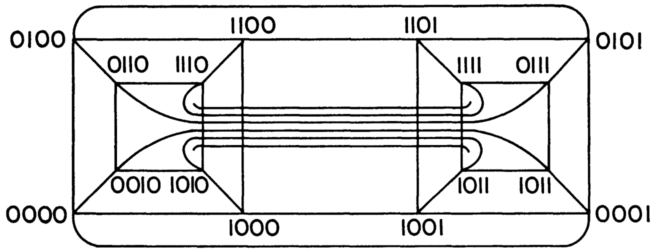

FIGURE 3

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