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ON THE ADDITIVITY OF UNBOUNDED SET FUNCTIONS

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The set functions associated with Schrödinger's equation are known to be unbounded on the algebra of cylinder sets. However, there do exist examples of scalar valued set functions which are unbounded, yet σ-additive on the underlying algebra of sets. The purpose of this note is to show that the set functions associated with Schrödinger's equation are not σ-additive on cylinder sets. In the course of the proof, general conditions implying the non-σ-additivity of unbounded set functions are given.

1. INTRODUCTION

It has been known for some time that there are substantial mathematical difficulties in implementing Feynman's program of interpreting quantum mechanics in terms of path integrals. One of these difficulties is described as follows.

Let λ be the Lebesgue measure on $\mathbb{R}$. Let Δ be the self-adjoint extension acting in $L^2(\mathbb{R})$ of the operator d^2/dx^2 defined on all smooth functions of compact support. Then for each $z \in \mathbb{C}$, $z \neq 0$, $\operatorname{Im} z \geqslant 0$, the semigroup of operators $S^z(t) = e^{i\Delta t/2z}$, $t \geqslant 0$ is defined by the functional calculus for self-adjoint operators on $L^2(\mathbb{R})$. For each $t > 0$, the operator $S^z(t)$ has a kernel

$$g_t^z(x-y) = \frac{e^{iz(x-y)^2/2t}}{[2\pi i t/z]^{1/2}}, \qquad \text{for all } x, y \in \mathbb{R}$$

in the sense that for all smooth functions ϕ with compact support in $\mathbb{R}$, $[S^z(t)\phi](x) = \int_{\mathbb{R}} g_t^z(x-y)\phi(y)dy$ for λ-almost all $x \in \mathbb{R}$. The branch of the square root is taken with a cut along the negative real axis.

Denote the operator of multiplication by the characteristic function χ_A of a Borel set $A \subseteq \mathbb{R}$, acting on $L^2(\mathbb{R})$, by $Q(A)$. The set functions we are interested in are constructed from the semigroups S^z and the spectral measure Q as follows.

Let Ω be the collection of all continuous functions ω: $[0, \infty) \to \mathbb{R}$. Let $t > 0$, $n = 1, 2, \ldots$, and let $0 \leqslant t_1 < t_2 < \ldots < t_n \leqslant t$. Suppose that $B_0, B_1, \ldots, B_{n+1}$ are Borel subsets of $\mathbb{R}$, and set

$$(1) \qquad E = \{\omega \in \Omega : \omega(0) \in B_0, \omega(t_1) \in B_1, \ldots, \omega(t_n) \in B_n, \omega(t) \in B_{n+1}\}.$$

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For each $z \in \mathbb{C}$, $z \neq 0$, $\operatorname{Im} z \geqslant 0$, define

$$M_t^z(E) = Q(B_{n+1})S^z(t-t_n)Q(B_n)\ldots S^z(t_2-t_1)Q(B_1)S^z(t_1)Q(B_0).$$

Then M_t^z is well-defined, and it extends to an additive set function on the algebra $\mathcal{S}_t$ generated by all such sets E, as n, $t_1, t_2, \ldots, t_n$ and $B_0, B_1, \ldots, B_n, B_{n+1}$ vary. We denote this unique extension, again, by M_t^z.

If $z = ai$, $a > 0$, then M_t^z is actually the restriction to $\mathcal{S}_t$ of an operator-valued measure, acting on $L^2(\mathbb{R})$, and defined on the σ-algebra $\sigma(\mathcal{S}_t)$ generated by $\mathcal{S}_t$. If $a = 1$, then M_t^i may be expressed in terms of Wiener measures P^x associated with the starting points $x \in \mathbb{R}$:

$$(M_t^i(A)\phi, \psi) = \int_{\mathbb{R}} P^x(\chi_A . \psi \circ X_t)\phi(x)\,d\lambda(x), \qquad A \in \mathcal{S}_t,\ \phi, \psi \in L^2(\mathbb{R}).$$

Here $X_s \colon \Omega \to \mathbb{R}$ denotes the random variable $X_s(\omega) = \omega(s)$, $s \geqslant 0$. There is an analogous formula for other cases of positive a in terms of scaled Wiener measures. The Feynman-Kac formula asserts that $e^{(\Delta/2+V)t} = \int_\Omega e^{\int_0^t V \circ X_s ds} dM_t^i$ for suitable functions $V \colon \mathbb{R} \to \mathbb{C}$. One attempt to attach meaning to the right hand side of the equation

$$e^{-i(-\Delta/2+V)t} = \int_\Omega e^{-i\int_0^t V \circ X_s ds} dM_t^1,$$

so representing a solution to Schrödinger's equation as a path integral, is given in [4]. The topic, in various guises, has a venerable history which is outlined in [2].

Now it follows from an observation of Cameron (see, for example [**2**, Theorem 5.1.1, p.217]) that if $\operatorname{Re} z \neq 0$ and $\phi, \psi \in L^2(\mathbb{R})$ are non-zero, then $(M_t^z\phi, \psi)$ is *not* the restriction to $\mathcal{S}_t$ of a scalar measure defined on the σ-algebra $\sigma(\mathcal{S}_t)$ generated by $\mathcal{S}_t$, because $(M_t^z\phi, \psi)$ is unbounded on $\mathcal{S}_t$, that is, the set $\{(M_t^z(A)\phi, \psi) \colon A \in \mathcal{S}_t\}$ is an unbounded subset of $\mathbb{C}$. Consequently, the standard theory of integration with respect to vector-valued measures does not apply to the operator-valued set functions $M_t^z \colon \mathcal{S}_t \to \mathcal{L}(L^2(\mathbb{R}))$ in the case that $\operatorname{Re} z \neq 0$. This is what is usually meant when it is stated that M_t^z is not σ-additive [**5**, p.11] (The operator valued Feynman integral $K_\lambda(\chi_E)$ considered in [5], for example, is the adjoint of $M_t^{i/\overline{\lambda}}(E)$, with E as defined above). However, it is well-known that there exist unbounded σ-additive set functions defined on an *algebra* of subsets of a set. Such a set function cannot be the restriction of a signed measure defined on a σ-algebra, otherwise it would be bounded.

Example. Let $\mathcal{A}$ be the algebra of subsets of $\mathbb{R}$, the set of real numbers, consisting of sets that are either finite, or have finite complements. Let $m \colon \mathcal{A} \to \mathbb{R}$ be the set

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function defined by

$$m(A) = \text{ cardinality of } A, \qquad \text{if } A \in \mathcal{A} \text{ is finite,}$$
$$m(A) = -\text{ cardinality of } \mathbb{R} \setminus A, \qquad \text{if } A \in \mathcal{A} \text{ is infinite.}$$

Then m is σ-additive on $\mathcal{A}$, but $\{m(A) : A \in \mathcal{A}\}$ is an unbounded subset of $\mathbb{R}$.

More interesting examples arise by taking the product of certain commuting spectral measures [6, 7].

The purpose of this note is to show that this phenomonon does not occur in the present situation — not only is the set function M_t^z unbounded on the algebra $\mathcal{S}_t$, it is not even σ-additive on $\mathcal{S}_t$ whenever $\operatorname{Re} z \neq 0$. Although this poor behaviour of the set functions M_t^z is not surprising, more analysis is required than in the proof of the unboundedness of M_t^z, which merely requires the calculation of the variation of the additive scalar set function $A \mapsto (M_t^z(A)\phi, \psi)$, $A \in \mathcal{S}_t$ for $\phi, \psi \in L^2(\mathbb{R})$.

In Section 2 we give a proof that $(M_t^z\phi, \psi)$ is not σ-additive on $\mathcal{S}_t$ for any non-zero $\phi, \psi \in L^2(\mathbb{R})$ whenever $\operatorname{Im} z > 0$, $\operatorname{Re} z \neq 0$ (Theorem 1). Also, a result which may be of independent interest is given formulating conditions under which an unbounded additive set function is not σ-additive on the underlying algebra of sets (Proposition 1).

That $(M_t^z\phi, \psi)$ is not σ-additive on $\mathcal{S}_t$ for non-zero $\phi, \psi \in L^2(\mathbb{R})$ whenever $\operatorname{Im} z = 0$, $z \neq 0$ is proved in Section 3 (Theorem 2). In this case, $(M_t^z\phi, \psi)$ is unbounded on the algebra generated by all sets E defined as in (1), with $n = 1, 2, \ldots$ and $t_1, \ldots, t_n$ fixed as $B_1, \ldots, B_n$ varies, so the argument used in Section 2 does not work. However, we give a simple proof that $(M_t^z\phi, \psi)$ is not σ-additive on the algebra generated by all sets E defined above with one fixed time t_1, after proving a general result concerning unbounded set functions (Proposition 2). This result is also relevant to the theory of Radon polymeasures [3]. A similar proof would work for set functions defined over a product of Hausdorff topological spaces endowed with Radon measures with values in $[0, \infty]$ — sufficiently general to deal with polymeasures associated with integral operators on manifolds, but we confine ourselves to $\mathbb{R}^3$.

Given an algebra of subsets $\mathcal{S}$ of a set Ω, a set function $m : \mathcal{S} \to \mathbb{C}$ is said to be *additive* if $m(A \cup B) = m(A) + m(B)$ for all $A, B \in \mathcal{S}$ such that $A \cap B = \emptyset$. It is *σ-additive* if for any pairwise disjoint family of sets $A_j \in \mathcal{S}$, $j = 1, 2, \ldots$ such that $\bigcup_{j=1}^{\infty} A_j \in \mathcal{S}$, the equality $m\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} m(A_j)$ holds.

The variation $|m| : \mathcal{S} \to [0, \infty]$ of an additive set function $m : \mathcal{S} \to \mathbb{C}$ is defined by

$$|m|(A) = \sup\left\{\sum_{B \in \pi} |m(B)| : \pi \in \Pi(A)\right\}, \qquad \text{for every } A \in \mathcal{S},$$

where $\Pi(A)$ is the collection of all finite partitions of $A \in \mathcal{S}$ by elements of $\mathcal{S}$. Then $|m|$ is an extended-real valued set function, and m is bounded and σ-additive on $\mathcal{S}$ if and only if $|m|$ has finite values and is σ-additive. Given a subalgebra $\mathcal{T}$ of $\mathcal{S}$, we denote the variation of the restriction $m\,|_{\mathcal{T}}$ of m to $\mathcal{T}$ by $|m|_{\mathcal{T}}$.

Note that if m is unbounded so that $|m|(\Omega) = \infty$, then the σ-additivity of $|m|$ on $\mathcal{S}$ does not guarantee the σ-additivity of m on $\mathcal{S}$. From the examples outlined in Section 3, we can produce an additive scalar-valued set function m whose variation is the restriction of a σ-finite measure, but m is not σ-additive on the algebra $\mathcal{S}$.

Given an operator valued set function $m: \mathcal{S} \to \mathcal{L}(L^2(\mathbb{R}))$, for all $\phi, \psi \in L^2(\mathbb{R})$, $m\phi: \mathcal{S} \to L^2(\mathbb{R})$ denotes the map defined by $m\phi(A) = m(A)\phi$ for all $A \in \mathcal{S}$, and $(m\phi, \psi): \mathcal{S} \to \mathbb{C}$ is defined by $(m\phi, \psi)(A) = (m(A)\phi, \psi)$ for all $A \in \mathcal{S}$, for the inner product $(f, g) = \int_{\mathbb{R}} f(x)g(x)dx$, $f, g \in L^2(\mathbb{R})$ of $L^2(\mathbb{R})$.

The Borel σ-algebra of a topological space T is denoted by $\mathcal{B}(T)$. By a Radon measure μ on T we mean a σ-additive set function $\mu: \mathcal{B}(T) \to [0, \infty]$ such that for every $x \in T$, there exists an open set U such that $x \in U$ and $\mu(U) < \infty$, and for every $A \in \mathcal{B}(T)$, $\mu(A) = \sup\{\mu(K): K \subseteq A, K \text{ a compact subset of } T\}$. A scalar-valued (that is, complex or real valued) measure μ is called a *Radon measure* if its variation $|\mu|$ is a Radon measure.

2. The case $\operatorname{Im} z > 0$, $\operatorname{Re} z \neq 0$

We start with a result concerning the σ-additivity of unbounded set functions defined on the unspecified algebra of subsets of a set.

Proposition 1. *Let $\mathcal{S}$ be an algebra of subsets of a set Ω. Let $m: \mathcal{S} \to \mathbb{C}$ be an additive set function. Suppose that $\mathcal{S}_n$, $n = 1, 2, \dots$ are subalgebras of $\mathcal{S}$ such that $\mathcal{S}_n \subseteq \mathcal{S}_{n+1}$ for every $n = 1, 2, \dots$, and the variation $|m|_{\mathcal{S}_n} : \mathcal{S}_n \to [0, \infty)$ is finite and σ-additive. Suppose that for all $n = 1, 2, \dots$*

$$a_n = \sup\left\{\frac{|m|_{\mathcal{S}_n}(S)}{|m|_{\mathcal{S}_k}(S)}: S \in \mathcal{S}_k,\ |m|_{\mathcal{S}_k}(S) > 0,\ 1 \leqslant k \leqslant n\right\} < \infty$$

Suppose also, that there exist sets $A_n \in \mathcal{S}_n$, $n = 1, 2, \dots$ such that $A_{n+1} \subseteq A_n$ for every $n = 1, 2, \dots$, $\bigcap_{n=1}^{\infty} A_n = \emptyset$, and $\lim_{n\to\infty} |m|_{\mathcal{S}_n}(A_n) = \infty$.

Then m is not σ-additive on $\mathcal{S}$. In particular, there exist sets $B_n \in \mathcal{S}_n$, $n = 1, 2, \dots$ such that $B_{n+1} \subseteq B_n$ for every $n = 1, 2, \dots$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and $\lim_{n\to\infty} |m(B_n)| = \infty$.

Proof: Let p, q be the real valued additive set functions on $\mathcal{S}$ such that $m = p + iq$. Because $|m|_{\mathcal{S}_n}(S) \leqslant |p|_{\mathcal{S}_n}(S) + |q|_{\mathcal{S}_n}(S)$ for all $S \in \mathcal{S}$ and $n = 1, 2, \dots$, it is not possible for both $\lim_{n\to\infty} \sup |p|_{\mathcal{S}_n}(A_n) < \infty$ and $\lim_{n\to\infty} \sup |q|_{\mathcal{S}_n}(A_n) < \infty$. We suppose that $\lim_{n\to\infty} \sup |p|_{\mathcal{S}_n}(A_n) = \infty$, otherwise we can replace m by im.

Let $p_n^+ = (|p|_{\mathcal{S}_n} + p \mid \mathcal{S}_n)/2$ be the non-negative part of $p \mid \mathcal{S}_n$. By choosing a subsequence of A_n, $n = 1, 2, \dots$, and replacing p_n by $-p_n$ if necessary, we may assume that $p_n^+(A_n) \to \infty$ as $n \to \infty$.

Let $n = 1, 2, \dots$. Now $|m|_{\mathcal{S}_n}(\Omega) < \infty$, and so $p \mid \mathcal{S}_n$ is σ-additive on $\mathcal{S}_n$ by virtue of the σ-additivity of $|m|_{\mathcal{S}_n}$. Denote the σ-additive extension of $p \mid \mathcal{S}_n$ to $\sigma(\mathcal{S}_n)$ by p_n. The σ-additive extensions to $\sigma(\mathcal{S}_n)$ of the set functions p_n^+ and $|m|_{\mathcal{S}_n}$ are denoted by the same symbols. An appeal to the Hahn decomposition theorem shows that there exists a set $X_n \in \sigma(\mathcal{S}_n)$ such that $p_n^+(S) = p_n(S \cap X_n)$ for all $S \in \mathcal{S}_n$.

Now for every $n = 1, 2, \dots$, choose $A_n' \in \mathcal{S}_n$ such that $|m|_{\mathcal{S}_n}((A_n \cap X_n)\Delta A_n') \leqslant 1/(2^n a_n + 2)$. Set $B_n = \bigcap_{k=1}^{n} (A_k' \cup (X_n \setminus X_k)) \cap A_n$. Then $B_n \downarrow \emptyset$ as $n \to \infty$ and

$$\begin{aligned}
|m|_{\mathcal{S}_n}((A_n \cap X_n)\Delta B_n) &\leqslant \sum_{k=1}^{n} |m|_{\mathcal{S}_n}((A_n \cap X_n) \setminus (A_k' \cup (X_n \setminus X_k))) \\
&\qquad + |m|_{\mathcal{S}_n}(A_n' \setminus (A_n \cap X_n)) \\
&\leqslant \sum_{k=1}^{n} |m|_{\mathcal{S}_n}((A_k \cap X_k) \setminus A_k') + \frac{1}{2} \\
&\leqslant \sum_{k=1}^{n} a_k \, |m|_{\mathcal{S}_k}((A_k \cap X_k) \setminus A_k') + \frac{1}{2} \\
&\leqslant 2
\end{aligned}$$

for all $n = 1, 2, \dots$. If one of the numbers $|m|_{\mathcal{S}_k}((A_k \cap X_k) \setminus A_k')$, $k = 1, \dots, n$ vanishes, then by the assumption it is clear that $|m|_{\mathcal{S}_n}((A_k \cap X_k) \setminus A_k') = 0$ too, so the inequality above still holds.

It follows that

$$\begin{aligned}
|p_n^+(A_n) - p(B_n)| = |p_n(A_n \cap X_n) - p(B_n)| &\leqslant |p|_{\mathcal{S}_n}((A_n \cap X_n)\Delta B_n) \\
&\leqslant |m|_{\mathcal{S}_n}((A_n \cap X_n)\Delta B_n) \leqslant 2,
\end{aligned}$$

so that $p(B_n) \to \infty$ as $n \to \infty$ too. The sequence $m(B_n)$, $n = 1, 2, \dots$ does not converge in $\mathbb{C}$, so m is not σ-additive on $\mathcal{S}$. $\square$

Fix $t > 0$. For each $n = 1, 2, \dots$, let $\mathcal{T}_n$ be the algebra of subsets of Ω generated by the random variables $X_{jt/n}$, $j = 0, 1, \dots, n$, that is, $\mathcal{T}_n$ is the smallest algebra containing all sets

$$\{\omega \in \Omega : X_0(\omega) \in B_0,\ X_{t/n}(\omega) \in B_1,\ X_{2t/n}(\omega) \in B_2, \dots, X_t(\Omega) \in B_n\},$$

with $B_0, B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R})$.

Let $z \in \mathbb{C}$, $\sigma = \operatorname{Im} z > 0$, $\operatorname{Re} z \neq 0$. If $\phi, \psi \in L^2(\mathbb{R})$, then $(M_t^z\phi, \psi)$ is bounded and σ-additive on $\mathcal{T}_n$ for each $n = 1, 2, \ldots$. Let $(M_t^z\phi, \psi)_n$ denote the restriction $(M_t^z\phi, \psi) \mid \mathcal{T}_n$ of $(M_t^z\phi, \psi)$ to $\mathcal{T}_n$. Then $|(M_t^z\phi, \psi)_n|$ is the variation of the σ-additive set function $(M_t^z\phi, \psi)_n$. In Lemma 1, $|(M_t^z\phi, \psi)_n|$ is given explicitly.

Let $\left(\Omega, \mathcal{U}, (P_\sigma^x)_{x\in\mathbb{R}}, (X_t)_{t>0}\right)$ be the Markov process such that for all $t > 0$,

$$P_\sigma^x(X_t \in B) = \int_B g_t^{i\sigma}(x-y)\,dy \text{ for all } x \in \mathbb{R}, B \in \mathcal{B}(\mathbb{R}),$$

This is just the diffusion process generated by $(1/2\sigma)\Delta$.

The following result allows us to provide a lower bound for the variation of $(M_t^z\phi, \phi)_n$.

Lemma 1. *Let $n = 1, 2, \ldots$, $A \in \mathcal{T}_n$, $\phi, \psi \in L^2(\mathbb{R})$. Then*

$$|(M_t^z\phi, \psi)_n|(A) = \left(\frac{|z|}{\sigma}\right)^{n/2} \int_{\mathbb{R}} P_\sigma^x(|\psi \circ X_t|\,\chi_A)\,|\phi(x)|\,dx.$$

Proof: Let E be the set defined in (1). Then

$$\begin{aligned}(M_t^z\phi, \psi)(E) &= (Q(B_{n+1})S^z(t-t_n)Q(B_n)\ldots S^z(t_2-t_1)Q(B_1)S^z(t_1)Q(B_0)\phi, \psi)\\ &= \int_{B_{n+1}}\int_{B_n}\cdots\int_{B_1}\int_{B_0} \overline{\psi}(x_{n+1})g_{t-t_n}^z(x_{n+1}-x_n)g_{t_n-t_{n-1}}^z(x_n-x_{n-1})\ldots\\ &\qquad \ldots g_{t_1}^z(x_1-x_0)\phi(x_0)\,dx_0\ldots dx_{n+1}.\end{aligned}$$

As a function of the product sets $B_0 \times B_1 \times \cdots \times B_{n+1}$, $B_k \in \mathcal{B}(\mathbb{R})$, $k = 0, 1, \ldots, n+1$, the right hand side is the restriction of a complex Borel measure m on $\mathbb{R}^{n+2}$ to the product of Borel subsets of $\mathbb{R}$. The measure m is the integral, with respect to Lebesgue measure λ_{n+2} on $\mathbb{R}^{n+2}$, of the function $f\colon \mathbb{R}^{n+2} \to \mathbb{R}$ defined by

$$\begin{aligned}f(x_0, x_1, \ldots, x_{n+1}) = \overline{\psi}(x_{n+1})g_{t-t_n}^z(x_{n+1}-x_n)g_{t_n-t_{n-1}}^z(x_n-x_{n-1})\ldots\\ \ldots g_{t_1}^z(x_1-x_0)\psi(x_0)\end{aligned}$$

for almost all $x_0, x_1, \ldots, x_{n+1} \in \mathbb{R}$.

Let $\mathcal{A}$ be the algebra of subsets of $\mathbb{R}^{n+2}$ generated by products of Borel subsets of $\mathbb{R}$. The variation of m with respect to $\mathcal{A}$ is equal the restriction, to $\mathcal{A}$, of the variation $|m|$ of m with respect to the Borel σ-algebra. Because $|m| = |f|\,.\lambda_{n+2}$ and

$|g_s^z| = (|z|/\sigma)g_s^{i\sigma}$ for all $s > 0$, we have

$$\begin{aligned}
&|(M_t^z\phi, \psi)_n|(A) \\
&= \int_{B_n} \cdots \int_{B_1} \int_{B_0} \left|\overline{\psi}(x_n) g_{t/n}^z(x_n - x_{n-1}) \ldots g_{t/n}^z(x_2 - x_1) g_{t/n}^z(x_1 - x_0)\phi(x_0)\right| \\
&\qquad\qquad dx_0\,dx_1 \ldots dx_n \\
&= \left(\frac{|z|}{\sigma}\right)^{n/2} \int_{B_n} \cdots \int_{B_1} \int_{B_0} |\overline{\psi}(x_n)|\, g_{t/n}^{i\sigma}(x_n - x_{n-1}) \ldots g_{t/n}^{i\sigma}(x_2 - x_1) g_{t/n}^{i\sigma}(x_1 - x_0) \\
&\qquad\qquad \times |\phi(x_0)|\, dx_0\,dx_1 \ldots dx_n \\
&= \left(\frac{|z|}{\sigma}\right)^{n/2} \int_{\mathbb{R}} P_\sigma^z(|\psi \circ X_t| \chi_A)\, |\psi(x)|\, dx,
\end{aligned}$$

for the set

$$A = \{\omega \in \Omega : X_0(\omega) \in B_0,\, X_{t/n}(\omega) \in B_1,\, X_{2t/n}(\omega) \in B_2, \ldots, X_t(\omega) \in B_n\} \in \mathcal{T}_n.$$

The validity of the equality follows for all $A \in \mathcal{T}_n$ by the additivity of the set functions on both sides of the equation. $\square$

Theorem 1. *For every non-zero $\phi, \psi \in L^2(\mathbb{R})$, there exist sets $B_n \in \mathcal{S}_t$, $n = 1, 2, \ldots$ such that $B_{n+1} \subseteq B_n$ for every $n = 1, 2, \ldots$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and $\lim_{n\to\infty} |(M_t^z(B_n)\phi, \psi)| = \infty$.*

Proof: For each $n = 1, 2, \ldots$, let $C_n = \{\omega \in \Omega : \omega(t/2) \geqslant n\}$. Then $C_{n+1} \subseteq C_n$ and $\bigcap_{n=1}^{\infty} C_n = \emptyset$. We show that for each non-zero $\phi, \psi \in L^2(\mathbb{R})$, there exist positive integers N_n, $n = 1, 2, \ldots$ such that $C_n \in \mathcal{T}_{N_n}$ for all $n = 1, 2, \ldots$, and $\lim_{n\to\infty} \left|(M_t^z\phi, \psi)_{N_n}\right|(C_n) = \infty$.

Denote the integer part of $x \in \mathbb{R}$ by $[x]$. For every $n = 1, 2, \ldots$, set

$$c_n = \left(e^{\Delta t/4\sigma} Q([n, \infty)) e^{\Delta t/4\sigma} |\phi|, |\psi|\right),\; N_n = 2.\max\left(\left[\frac{\ln n - \ln c_n}{\ln\left(\frac{|z|}{\sigma}\right)}\right] + 1, 1\right).$$

The assumption that ϕ and ψ are non-zero implies that $c_n > 0$. Then $C_n \in \mathcal{T}_{N_n}$ because N_n is even, and by Lemma 1,

$$\left|(M_t^z\phi, \psi)_{N_n}\right|(C_n) = \left(\frac{|z|}{\sigma}\right)^{N_n/2} \int_{\mathbb{R}} P_\sigma^z(|\psi \circ X_t| \chi_{C_n})\, |\phi(x)|\, dx = \left(\frac{|z|}{\sigma}\right)^{N_n/2} c_n > n,$$

showing that $\lim_{n\to\infty} \left|(M_t^z\phi, \psi)_{N_n}\right|(C_n) = \infty$.

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The conclusion then follows from Proposition 1 once we establish that

$$\sup\left\{\frac{|(M_t^z\phi,\,\psi)|_{\mathcal{T}_n}(S)}{|(M_t^z\phi,\,\psi)|_{\mathcal{T}_k}(S)} : S\in\mathcal{T}_k,\, |(M_t^z\phi,\,\psi)|_{\mathcal{T}_k}(S)>0,\, 1\leqslant k\leqslant n\right\}<\infty,$$

for every $n=1,2,\ldots$. According to Lemma 1,

$$\begin{aligned}\frac{|(M_t^z\phi,\,\psi)|_{\mathcal{T}_n}(S)}{|(M_t^z\phi,\,\psi)|_{\mathcal{T}_k}(S)} &= \frac{\left(\frac{|z|}{\sigma}\right)^{n/2}\int_{\mathbb{R}}P_\sigma^z(|\psi\circ X_t|\,\chi_S)\,|\phi(x)|\,dx}{\left(\frac{|z|}{\sigma}\right)^{k/2}\int_{\mathbb{R}}P_\sigma^z(|\psi\circ X_t|\,\chi_S)\,|\phi(x)|\,dx}\\ &= \left(\frac{|z|}{\sigma}\right)^{(n-k)/2}\leqslant\left(\frac{|z|}{\sigma}\right)^{(n-1)/2}\end{aligned}$$

for all $S\in\mathcal{T}_k$ such that $|(M_t^z\phi,\,\psi)|_{\mathcal{T}_k}(S)>0$, and all $1\leqslant k\leqslant n$, so the proof is complete. $\square$

Problem. If the sets B_k, $k=1,2,\ldots$ are of the form (1), with n and $t_1,\ldots,t_n$ not necessarily fixed, $B_{k+1}\subseteq B_k$ for every $k=1,2,\ldots$, and $\bigcap_{k=1}^{\infty}B_k=\emptyset$, then is it true that $\lim_{k\to\infty}|(M_t^z(B_k)\phi,\,\phi)|=0$? We know this to be true at least in the case that n and $t_1,\ldots,t_n$ are fixed [1, Theorem 1].

3. The case $\operatorname{Im} z=0$, $z\neq 0$

As in Section 2, we first state a general result which may be applied to the situation at hand. Let $\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})$ denote the algebra of subsets of $\mathbb{R}^3$ generated by all product sets $A\times B\times C$, $A,B,C\in\mathcal{B}(\mathbb{R})$.

Let $K_1\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$, $K_2\colon L^2(\mathbb{R})\to L^2(\mathbb{R})$ be bounded linear operators. For $\phi,\psi\in L^2(\mathbb{R})$, the additive set function $m\colon\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\to\mathbb{C}$ defined for each set $A\times B\times C$, $A,B,C\in\mathcal{B}(\mathbb{R})$ by

$$m_{\phi,\psi}(A\times B\times C)=(Q(C)K_2Q(B)K_1Q(A)\phi,\,\psi) \tag{2}$$

is separately σ-additive in each variable. We examine conditions under which this set function is not σ-additive on the algebra $\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})$.

Let $\operatorname{diag}=\{(x,x)\colon x\in\mathbb{R}\}$ and let $B_r(a)$ be the open ball of radius $r>0$ about $a\in\mathbb{R}$. Suppose that for $j=1,2$, K_j has a kernel $k_j(x,y)$, $x,y\in\mathbb{R}$, $x\neq y$ in the sense that $k_j\colon\mathbb{R}^2\setminus\operatorname{diag}\to\mathbb{C}$ is measurable, bounded on all compact subsets of $\mathbb{R}^2\setminus\operatorname{diag}$, and for all smooth functions f,g with compact support in $\mathbb{R}$,

$$(K_jf,\,g)=\lim_{\epsilon\to0+}\int_{\mathbb{R}}\int_{\mathbb{R}}\bar{g}(x)\chi_{B_\epsilon(x)^c}(y)k_j(x,y)f(y)\,dy\,dx.$$

Set $D=\mathbb{R}^3\setminus\{x\in\mathbb{R}^3 : x_1=x_2 \text{ or } x_2=x_3\}$.

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LEMMA 2. *Let $\phi, \psi \in L^2(\mathbb{R})$ and suppose that $m_{\phi,\psi}: \mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R}) \to \mathbb{C}$ is the additive set function defined by (2). Then,*

$$|m_{\phi,\psi}|(A) = \int_A |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3, \tag{3}$$

for all $A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$ such that $A \subseteq D$.

If k_1 and k_2 are locally bounded on $\mathbb{R}^3$, then (3) holds for all $A \in \mathcal{B}(\mathbb{R})\times_a\mathcal{B}(\mathbb{R})\times_a \mathcal{B}(\mathbb{R})$.

If, in either case, $|m_{\phi,\psi}|(A) < \infty$, then

$$m_{\phi,\psi}(A) = \int_A \psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1)dx_1dx_2dx_3,$$

PROOF: Let $\phi_n, \psi_n \in C_c^\infty(\mathbb{R})$, $n = 1, 2, \ldots$, $\lim_{n\to\infty} \psi_n = \psi$, $\lim_{n\to\infty} \phi_n = \phi$ in $L^2(\mathbb{R})$. Suppose that $f, g, h \in C_c^\infty\mathbb{R}$, $\operatorname{supp} f \cap \operatorname{supp} g = \emptyset$ and $\operatorname{supp} g \cap \operatorname{supp} h = \emptyset$. Then

$$\begin{aligned}&(Q(h)K_2Q(g)K_1Q(f)\phi_n, \psi_n)\\ &\quad= \int_{\mathbb{R}^3} \overline{\psi}_n(x_3)h(x_3)k_2(x_3, x_2)g(x_2)k_1(x_2, x_1)f(x_1)\phi_n(x_1)dx_1dx_2dx_3.\end{aligned} \tag{4}$$

This is seen as follows. If $\alpha \in C_c^\infty(\mathbb{R})$ then

$$\begin{aligned}(Q(g)K_1Q(f)\phi_n, \alpha) &= \lim_{\varepsilon\to 0+} \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\alpha}(x_2)g(x_2)\chi_{B_\varepsilon(x_2)^c}(x_1)k_1(x_2, x_1)f(x_1)\phi_n(x_1)dx_1dx_2\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\alpha}(x_2)g(x_2)k_1(x_2, x_1)f(x_1)\phi_n(x_1)dx_1dx_2,\end{aligned}$$

because for $\varepsilon < \inf\{|x-y| : x \in \operatorname{supp} f, y \in \operatorname{supp} g\}$, $g(x_2)\chi_{B_\varepsilon(x_2)^c}(x_1)f(x_1) = g(x_2)f(x_1)$ for all $x_1, x_2 \in \mathbb{R}$. Equality now follows for all $\alpha \in L^2(\mathbb{R})$ by dominated convergence, because k_1, k_2 are assumed to be locally bounded off the diagonal. Similarly, for every $\alpha \in L^2(\mathbb{R})$ which vanishes outside a compact set disjoint from $\operatorname{supp} h$, we have

$$(\alpha, K_2^*Q(\overline{h})\psi_n) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\psi}(x_3)h(x_3)k_2(x_3, x_2)\alpha(x_2)dx_2dx_3.$$

The integrand in (4) is a bounded measurable function with compact support in $\mathbb{R}^3$, so the equality (4) now follows from Fubini's theorem. By dominated convergence and the continuity of the operator $Q(h)K_2Q(g)K_1Q(f)$,

$$\begin{aligned}&(Q(h)K_2Q(t)K_1Q(f)\phi, \psi)\\ &\quad= \int_{\mathbb{R}^3} \overline{\psi}(x_3)h(x_3)k_2(x_3, x_2)g(x_2)k_1(x_2, x_1)f(x_1)\phi(x_1)dx_1dx_2dx_3.\end{aligned} \tag{5}$$

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Now let C_1, C_2, C_3 be compact subsets of $\mathbb{R}$ such that $C_1 \cap C_2 = \emptyset$ and $C_2 \cap C_3 = \emptyset$. Let

$$\mu_{C_1\times C_2\times C_3}(A) = \int_A \overline{\psi}(x_3)h(x_3)k_2(x_3, x_2)g(x_2)k_1(x_2, x_1)f(x_1)\phi(x_1)dx_1dx_2dx_3$$

for all $A \in \mathcal{B}(C_1 \times C_2 \times C_3)$. Then according to equation (5), $\mu_{C_1\times C_2\times C_3} : \mathcal{B}(C_1 \times C_2 \times C_3) \to \mathbb{C}$ is the unique Radon measure such that

$$(Q(h)K_2Q(g)K_1Q(f)\phi, \psi) = \mu_{C_1\times C_2\times C_3}(f \otimes g \otimes h)$$

for all functions $h, g, f \in C_c^\infty(\mathbb{R})$ such that $\operatorname{supp} f \subseteq C_1$, $\operatorname{supp} g \subseteq C_2$, $\sup h \subseteq C_3$. By virtue of the σ-additivity of the spectral measure Q, we have

$$(Q(C)K_2Q(B)K_1Q(A)\phi, \psi) = \mu_{C_1\times C_2\times C_3}(A \times B \times C)$$

whenever $A \subseteq C_1$, $B \subseteq C_2$, $C \subseteq C_3$ are Borel sets.

Let $\mathcal{T} = \mathcal{B}(C_1) \times_a \mathcal{B}(C_2) \times_a \mathcal{B}(C_3)$. It follows that

$$m_{\phi,\psi}(A) = \mu_{C_1\times C_2\times C_3}(A) = \int_A \psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1)dx_1dx_2dx_3,$$

$$|m_{\phi,\psi}|_{\mathcal{T}}(A) = |\mu_{C_1\times C_2\times C_3}|(A) = \int_A |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3$$

for all $A \in \mathcal{T}$. Now let $\mathcal{S}$ be the family of sets $A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$ such that there exists a compact subset C of D such that $A \subseteq C$. Extending the argument above to finite unions of compact product subsets of D, it follows that

$$|m_{\phi,\psi}|(A) = \int_A |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3 < \infty,$$

$$m_{\phi,\psi}(A) = \int_A \psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1)dx_1dx_2dx_3 \tag{6}$$

for every $A \in \mathcal{S}$.

Let $\mathcal{U} = \{A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \colon A \subseteq D\}$. An argument similar to that of [3, Proposition 1] shows that the restriction $|m_{\phi,\psi}|\,|_{\mathcal{U}}$ of $|m_{\phi,\psi}|$ to $\mathcal{U}$ extends to a Radon measure on D (the difference is that here D is not a product space, but we can still use the separate σ-additivity of $m_{\phi,\psi}$). Every subset of D belonging to the algebra $\mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$ can be written as the countable union of sets from $\mathcal{S}$, so

$$|m_{\phi,\psi}|(A) = \int_A |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3$$

for all $A \in \mathcal{U}$. Equality (6) is therefore true for all those sets $A \in \mathcal{U}$ such that $|m_{\phi,\psi}|(A) < \infty$.

If k_1 and k_2 are locally bounded on $\mathbb{R}^3$, then a similar proof works with D replaced by $\mathbb{R}^3$, except we do not have to worry about the diagonals. $\square$

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REMARK. For most operators of interest, such as singular integral operators, $K_\varepsilon \to K$ in the weak operator topology as $\varepsilon \to 0^+$, where K_ε is the continuous linear extension (which is assumed to exist) of the operator defined for all $f, g \in C_c^\infty(\mathbb{R})$ by

$$(K_\varepsilon f, g) = \int_{\mathbb{R}} \int_{\mathbb{R}} \bar{g}(x)\chi_{B_\varepsilon(x)^c}(y)k(x, y)f(y)dy\,dx.$$

Under such an assumption for the operators K_1 and K_2, it is not hard to show that equation (3) is valid for all $A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$.

PROPOSITION 2. *Let $\phi, \psi \in L^2(\mathbb{R})$ and suppose that the function*

$$x \to |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,,\ x \in \mathbb{R}^3,\ x_1 \neq x_2 \text{ and } x_2 \neq x_3$$

is not integrable on the set $\{x \in \mathbb{R}^3\colon x_1 \neq x_2 \text{ and } x_2 \neq x_3\}$. Then, for the set function $m_{\phi,\psi}$ defined by (2), there exist sets $B_j \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$, $j = 1, 2, \ldots$ such that $B_{j+1} \subseteq B_j$ for every $j = 1, 2, \ldots$, $\bigcap_{j=1}^{\infty} B_j \cap D = \emptyset$, and $\lim_{j\to\infty} |m_{\phi,\psi}(B_j)| = \infty$.

If k_1 and k_2 are locally bounded on $\mathbb{R}^3$, we can choose the sets B_j $j = 1, 2, \ldots$ so that $\bigcap_{j=1}^{\infty} B_j = \emptyset$.

PROOF: Let $C_j \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$, $j = 1, 2, \ldots$ be an increasing family of compact sets such that $D = \bigcup_{j=1}^{\infty} C_j$ in the case that both k_1 and k_2 are not locally bounded on $\mathbb{R}^3$, or $\mathbb{R}^3 = \bigcup_{j=1}^{\infty} C_j$ in the case that k_1 and k_2 are locally bounded. We prove the result for the former case; the proof for the latter case is similar. We denote scalar-valued representatives of the equivalence classes ψ, ϕ by the same symbols.

Assume that $\int_{\mathbb{R}^3} |\mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1))|\,dx_1dx_2dx_3 = \infty$, otherwise replace $m_{\phi,\psi}$ by $im_{\phi,\psi}$. Assume also that $\int_{\mathbb{R}^3} \mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1))^+$ $dx_1dx_2dx_3 = \infty$, otherwise replace $m_{\phi,\psi}$ by $-m_{\phi,\psi}$. For every $j = 1, 2, \ldots$, let

$$V_j = \{x \in C_j : \mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1)) \geqslant 0\}$$
$$W_j = \{x \in C_j : \mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1)) < 0\}.$$

Then $V_j \subseteq V_{j+1}$, $W_j \subseteq W_{j+1}$ for each $j = 1, 2, \ldots$, and $D = \bigcup_{j=1}^{\infty} V_j \cup W_j$. For each $j = 1, 2, \ldots$, let $\mathcal{T}_j$ be the algebra of sets $C_j \cap A$, $A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$. By Lemma 2, for each $j = 1, 2, \ldots$, the variation $|m_{\phi,\psi}|_{\mathcal{T}_j}$ of $m_{\phi,\psi}$ on $\mathcal{T}_j$ is the restriction of a Radon measure $\mu_j\colon \mathcal{B}(C_j) \to [0, \infty)$. Choose subsets $X_j, Y_j \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$ of C_j such that $\mu_j(V_j \Delta X_j) < 2^{-2j}$, $\mu_j(W_j \Delta Y_j) < 2^{-2j}$ for each $j = 1, 2, \ldots$ and set $S_n = \bigcup_{j=1}^{n} X_j$, $T_n = \bigcup_{j=1}^{n} Y_j$, $n = 1, 2, \ldots$.

Then for each $n = 1, 2, \ldots$

$$\begin{aligned}\mu_n(V_n \Delta S_n) &\leqslant \sum_{j=1}^{n} \mu_n(X_j \setminus V_n) + \mu_n\left(V_n \setminus \bigcup_{j=1}^{n} X_j\right) \\ &\leqslant \sum_{j=1}^{n} \mu_n(X_j \setminus V_j) + \mu_n(V_n \setminus X_n) = \sum_{j=1}^{n} \mu_j(X_j \setminus V_j) + \mu_n(V_n \setminus X_n) \\ &< 1.\end{aligned}$$

Similarly, $\mu_n(W_n \Delta T_n) < 1$. Since W_n and V_n are disjoint, $\mu_n(T_n \cap S_n) < 3$.

Let $\nu_n(A) = \int_A \mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1))dx_1dx_2dx_3$ for every $A \in \mathcal{B}(C_n)$. If $A \in \mathcal{T}_n$ then by Lemma 1, $\nu_n(A) = \mathrm{Re}\, m_{\phi,\psi}(A)$. In particular,

$$\begin{aligned}|\nu_n(V_n) - \mathrm{Re}\, m_{\phi,\psi}(S_n)| = |\nu_n(V_n) - \nu_n(S_n)| &= |\nu_n(V_n \setminus S_n) - \nu_n(S_n \setminus V_n)| \\ &\leqslant |\nu_n(V_n \setminus S_n)| + |\nu_n(S_n \setminus V_n)| \leqslant |\nu_n|\,(V_n \Delta S_n) \\ &\leqslant |\mu_n|\,(V_n \Delta S_n) < 1.\end{aligned}$$

Similarly, $|\nu_n(W_n) - \mathrm{Re}\, m_{\phi,\psi}(T_n)| < 1$.

Now by assumption,

$$\lim_{n\to\infty} \nu_n(V_n) = \lim_{n\to\infty} \int_{C_n} \mathrm{Re}\,(\psi(x_3)k_2(x_3, x_2)k_1(x_2, x_1)\phi(x_1))^+ dx_1dx_2dx_3 = \infty,$$

so $\lim\limits_{n\to\infty} \mathrm{Re}\, m_{\phi,\psi}(S_n) = \infty$. For each $j = 1, 2, \ldots$, choose $n_j = 1, 2, \ldots$ such that $n_j > j$ and

$$\mathrm{Re}\, m_{\phi,\psi}\left(S_{n_j}\right) > j - \mathrm{Re}\, m_{\phi,\psi}(T_j)$$

with $n_{j+1} > n_j$. Then

$$\begin{aligned}\mathrm{Re}\, m_{\phi,\psi}\left(S_{n_j} \cup T_j\right) &= \mathrm{Re}\, m_{\phi,\psi}\left(S_{n_j}\right) + \mathrm{Re}\, m_{\phi,\psi}(T_j) - \mathrm{Re}\, m_{\phi,\psi}\left(T_j \cap S_{n_j}\right) \\ &> j - \mu_j\left(T_j \cap S_{n_j}\right) \\ &\geqslant j - \mu_{n_j}\left(T_{n_j} \cap S_{n_j}\right) \\ &> j - 3.\end{aligned}$$

Set $D_0 = \bigcup\limits_{j=1}^{\infty} S_{n_j} \cup T_j$. Then $D_0 \subseteq D$ and

$$\int_{D \setminus D_0} |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3 = 0.$$

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The null set $D \setminus D_0$ belongs to the σ-algebra generated by $\mathcal{U} = \{A \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) : A \subseteq D\}$, so for every $\varepsilon > 0$, there exist sets $E_j \in \mathcal{U}$, $j = 1, 2, \ldots$ such that $D \setminus D_0 \subseteq \bigcup_{j=1}^{\infty} E_j$ and $\sum_{j=1}^{\infty} \int_{E_j} |\psi(x_3)|\,|k_2(x_3, x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|\,dx_1 dx_2 dx_3 < \varepsilon$. Choose $\varepsilon = 1/2$ and set $B_j = \mathbb{R}^3 \setminus \left(S_{n_j} \cup T_j \cup \bigcup_{k=1}^{j} E_k\right) \in \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$, for all $j = 1, 2, \ldots$. Then $\bigcap_{j=1}^{\infty} B_j \cap D = \emptyset$ and

$$\begin{aligned}
\operatorname{Re} m_{\phi,\psi}(B_j) &= \operatorname{Re} m_{\phi,\psi}(\mathbb{R}^3) - \operatorname{Re} m_{\phi,\psi}\left(S_{n_j} \cup T_j \cup \bigcup_{k=1}^{j} E_k\right) \\
&= \operatorname{Re} m_{\phi,\psi}(\mathbb{R}^3) - \operatorname{Re} m_{\phi,\psi}\left(S_{n_j} \cup T_j\right) \\
&\qquad - \operatorname{Re} m_{\phi,\psi}\left(\bigcup_{k=1}^{j} E_k\right) + \operatorname{Re} m_{\phi,\psi}\left(\left(S_{n_j} \cup T_j\right) \cap \bigcup_{k=1}^{j} E_k\right) \\
&\leqslant \operatorname{Re} m_{\phi,\psi}(\mathbb{R}^3) - j + 3 + 2\,|m_{\phi,\psi}|\left(\bigcup_{k=1}^{j} E_k\right) \\
&\leqslant \operatorname{Re} m_{\phi,\psi}(\mathbb{R}^3) - j + 4.
\end{aligned}$$

It follows that $\lim_{j\to\infty} |m_{\phi,\psi}(B_j)| = \infty$, as required. $\square$

REMARK. A similar proof shows that if the function F defined by

$$F(x) = |\psi(x_2)|\,|k_1(x_2, x_1)|\,|\phi(x_1)|, \qquad x \in \mathbb{R}^2,\ x_1 \neq x_2$$

is not integrable on $\mathbb{R}^2 \setminus \mathrm{diag}$, then the additive set function $A \times B \to (Q(B)K_1Q(A)\phi, \psi)$, $A, B \in \mathcal{B}(\mathbb{R})$ is not σ-additive on the algebra $\mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$. If k_1 is locally bounded on $\mathbb{R}^2$ and $\phi, \psi \in C_c^\infty(\mathbb{R})$, then F is integrable on $\mathbb{R}^2$.

If P is the spectral measure associated with the self-adjoint operator $(1/i)(d/dx)$ acting in $L^2(\mathbb{R})$, then the additive set function $A \times B \to (Q(A)P(B)\phi, \psi)$, $A, B \in \mathcal{B}(\mathbb{R})$ is not σ-additive on the algebra $\mathcal{B}(\mathbb{R}) \times_a \mathcal{B}(\mathbb{R})$ whenever $\phi, \psi \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$. Integration with respect to this set function is associated with the construction of pseudo-differential operators.

THEOREM 2. *Let $z \in \mathbb{C}$, $z \neq 0$, $\operatorname{Im} z = 0$. For every non-zero $\phi, \psi \in L^2(\mathbb{R})$, and every $0 < s < t$, there exist sets B_n, $n = 1, 2, \ldots$ belonging to the algebra generated by X_0, X_s and X_t, such that $B_{n+1} \subseteq B_n$ for every $n = 1, 2, \ldots$, $\bigcap_{n=1}^{\infty} B_n = \emptyset$, and $\lim_{n\to\infty} |(M_t^z(B_n)\phi, \psi)| = \infty$.*

PROOF: For all Borel subsets A, B, C of $\mathbb{R}$,

$$(M_t^z(\{X_0 \in A, X_s \in B, X_t \in C\})\phi, \psi) = (Q(C)S^z(t-s)Q(B)S^z(s)Q(A)\phi, \psi).$$

For each $u > 0$, the operator $S^z(u)$ has a kernel $g_u^z(x-y)$, $x, y \in \mathbb{R}$ such that $|g_u^z(x)| = |z|/\sqrt{2\pi u}$ for all $x \in \mathbb{R}$. Thus,

$$\int_{\mathbb{R}^3} |\psi(x_3)|\,\left|g_{t-s}^z(x_3-x_2)\right|\,|g_s^z(x_2-x_1)|\,|\phi(x_1)|\,dx_1dx_2dx_3$$
$$= \frac{|z|^2}{2\pi s(t-s)}\int_{\mathbb{R}^3} |\psi(x_3)|\,|\phi(x_1)|\,dx_1dx_2dx_3 = \infty.$$

The conditions of Proposition 2 hold, so the result follows. □

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