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DISTANCE FUNCTIONS AND ORLICZ-SOBOLEV SPACES

D. E. EDMUNDS AND R. M. EDMUNDS

1. Introduction. Let Λ be a bounded, non-empty, open subset of $\mathbf{R}^n$ and given any x in $\mathbf{R}^n$, let

$$d(x) = \operatorname{dist}(x, \mathbf{R}^n \backslash \Lambda);$$

let $k \in \mathbf{N}$ and suppose that $p \in (1, \infty)$. It is known (c.f. e.g. [**4**]) that if u belongs to the Sobolev space $W^{k,p}(\Lambda)$ and $u/d^k \in L^p(\Lambda)$, then $u \in W_0^{k,p}(\Lambda)$. Further results in this direction are given in [**5**] and [**9**]. Moreover, if m is the mean distance function in the sense of [**2**], then it turns out that

$$v/m \in L^2(\Lambda) \quad \text{if } v \in W_0^{1,2}(\Lambda).$$

Under appropriate smoothness conditions on the boundary of Λ, m and d are equivalent, and thus $W_0^{1,2}(\Lambda)$ may in this case be characterized as the subspace of $W^{1,2}(\Lambda)$ consisting of all functions $u \in W^{1,2}(\Lambda)$ such that $u/d \in L^2(\Lambda)$.

The object of this paper is to give various extensions of these results, and, in particular, to provide an analogous characterization for Orlicz-Sobolev spaces. For the sake of definiteness we deal with spaces modelled upon the particular Orlicz function ϕ defined by

$$\phi(t) = \exp(t^\nu) - 1 \quad (\nu \in (1, \infty));$$

this function occurs naturally in Sobolev embedding theory [**1**] and is typical of these required in the study of strongly non-linear elliptic equations. Corresponding results for other specific Orlicz functions are possible.

2. Preliminaries. Throughout Λ will stand for a non-empty, open, bounded subset of $\mathbf{R}^n$ with boundary $\partial\Lambda$ and closure $\bar{\Lambda}$; points of $\mathbf{R}^n$ will be represented by $x = (x_1, \dots, x_n)$; k will stand for a natural number. Let $p \in [1, \infty)$ and let $L^p(\Lambda)$ be the Banach space of (equivalence classes of) real- or complex-valued functions u such that $|u|^p$ is Lebesgue-integrable over Λ, with norm $\|\cdot\|_{p,\Lambda}$ defined by

$$\|u\|_{p,\Lambda} = \left(\int_\Lambda |u(x)|^p dx \right)^{1/p}.$$

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Given any $\alpha \in \mathbf{N}_0^n$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, we write

$$|\alpha| = \sum_{i=1}^{n} \alpha_i, \quad D_i = \partial/\partial x_i \quad \text{and} \quad D^\alpha = \prod_{i=1}^{n} D_i^{\alpha_i}.$$

Let $k \in \mathbf{N}$; by $W^{k,p}(\Lambda)$ is meant the linear space

$$\{u \in L^p(\Lambda): D^\alpha u \in L^p(\Lambda) \text{ for all } \alpha \in \mathbf{N}_0^n \text{ with } |\alpha| \leqq k\},$$

endowed with the norm $\|\cdot\|_{k,p,\Lambda}$, where

$$\|u\|_{k,p,\Lambda} = \sum_{|\alpha| \leqq k} \|D^\alpha u\|_{p,\Lambda};$$

the closure in $W^{k,p}(\Lambda)$ of the space $C_0^\infty(\Lambda)$ of all infinitely differentiable functions with compact support in Λ is denoted by $W_0^{k,p}(\Lambda)$.

An Orlicz function is any map $\phi:[0, \infty) \to [0, \infty)$ which is continuous, convex and such that

$$\lim_{t \to 0} \phi(t)/t = 0, \quad \lim_{t \to \infty} \phi(t)/t = \infty.$$

The Orlicz class $\tilde{L}_\phi(\Lambda)$ is the set of all (equivalence classes of) functions $u:\Lambda \to \mathbf{R}$ such that

$$\int_\Lambda \phi[u(x)]dx < \infty;$$

the Orlicz space $L_\phi(\Lambda)$ is the linear hull of $\tilde{L}_\phi(\Lambda)$, provided with the Luxemburg norm $\|\cdot\|_{(\phi),\Lambda}$ given by

$$\|u\|_{(\phi),\Lambda} = \inf\left\{\lambda > 0: \int_\Lambda \phi(|u(x)|/\lambda)dx \leqq 1\right\};$$

$L_\phi(\Lambda)$ is a Banach space which is, in general, neither reflexive nor separable. The closure $E_\phi(\Lambda)$ in $L_\phi(\Lambda)$ of the family of all bounded, measurable functions on $\bar{\Lambda}$ is, however, both separable and contained in $\tilde{L}_\phi(\Lambda)$. Let $k \in \mathbf{N}$; the Orlicz-Sobolev space $W^kE_\phi(\Lambda)$ is defined by

$$W^kE_\phi(\Lambda) = \{u \in E_\phi(\Lambda): D^\alpha u \in E_\phi(\Lambda) \quad \text{for all } \alpha \in \mathbf{N}_0^n \text{ with } |\alpha| \leqq k\},$$

together with the norm $\|\cdot\|_{k,(\phi),\Lambda}$, where

$$\|u\|_{k,(\phi),\Lambda} = \sum_{|\alpha| \leqq k} \|D^\alpha u\|_{(\phi),\Lambda}.$$

The closure of $C_0^\infty(\Lambda)$ in $W^kE_\phi(\Lambda)$ is denoted by $W_0^kE_\phi(\Lambda)$.

Henceforth ϕ will stand for the Orlicz function defined by

$$\phi(t) = \exp(t^\nu) - 1 \quad (t \in [0, \infty));$$

here ν is a given number in the open interval $(1, \infty)$.

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3. The distance function d. We recall that d is defined by

$$d(x) = \operatorname{dist}(x, \mathbf{R}^n \backslash \Lambda).$$

To establish the characterization of $W_0^k E_\phi(\Lambda)$ which is the main object of this paper, we shall use the result mentioned in Section 1 and proved in [4], that if $p \in (1, \infty)$, then $u \in W_0^{k,p}(\Lambda)$ if $u \in W^{k,p}(\Lambda)$ and $u/d^k \in L^p(\Lambda)$. The proof of this in [4] uses the fact that the map which takes each $f \in L^p(\Lambda)$ into the corresponding maximal function is a bounded map of $L^p(\Lambda)$ into itself, and so the hypothesis that $p > 1$ is essential. However, Professor C. Kenig has pointed out to us that a Whitney decomposition of Λ into cubes may be used to establish this result even when $p = 1$, and for the reader's convenience we indicate briefly below the main lines of this argument for the case $k = 1$, the proof for $k > 1$ being similar. We are grateful to Professor Kenig for supplying the essential idea of this proof.

By the Whitney decomposition theorem [7],

$$\Lambda = \bigcup_{j=1}^{\infty} Q_j,$$

where each Q_j is a closed cube with sides parallel to the coordinate axes, $\mathring{Q}_j \cap \mathring{Q}_l = \emptyset$ if $j \neq l$, and for each $j \in \mathbf{N}$,

$$\operatorname{diam} Q_j \leqq \operatorname{dist}(Q_j, \mathbf{R}^n \backslash \Lambda) \leqq 4 \operatorname{diam} Q_j.$$

Let $\epsilon_1 \in (0, 1/4)$, let $x^{(j)}$ be the centre of Q_j, let l_j be the length of each side of Q_j and put

$$Q_j^* = (1 + \epsilon_1)(Q_j - x^{(j)}) + x^{(j)} : Q_j \subset Q_j^*$$

and the Q_j^* need not pairwise disjoint. Let $\phi \in C_0^\infty(\mathbf{R}^n)$ be such that $0 \leqq \phi \leqq 1$, $\phi(x) = 1$ for all

$$x \in Q_0 := \left[-\frac{1}{2}, \frac{1}{2}\right]^n, \ \phi(x) = 0 \text{ for all } x \notin (1 + \epsilon)Q_0;$$

for each $j \in \mathbf{N}$ put

$$\phi_j(x) = \phi\left(\frac{x - x^{(j)}}{l_j}\right) \quad (x \in \mathbf{R}^n).$$

Then $\phi_j(x) = 1$ for all $x \in Q_j$, $\phi_j(x) = 0$ for all $x \notin Q_j^*$, and there is a constant A such that for all $x \in \mathbf{R}^n$, all $i \in \{1, 2, \ldots, n\}$ and all $j \in \mathbf{N}$,

$$|D_i \phi_j(x)| \leqq A(\operatorname{diam} Q_j)^{-1}.$$

Put

$$\phi_j^*(x) = \phi_j(x)/\Phi(x),$$

where

$$\Phi(x) = \sum_{i=1}^{\infty} \phi_i(x) \quad (x \in \Lambda);$$

then for all $x \in \Lambda$,

$$\sum_{j=1}^{\infty} \phi_j^*(x) = 1.$$

It is also shown in [7] that each point of Λ is contained in at most 12^n of the Q_j^*. Finally, put

$$d_j = \operatorname{dist}(Q_j, \mathbf{R}^n \setminus \Lambda)$$

and note that

$$\sqrt{n} l_j \leqq d_j \leqq 4\sqrt{n} l_j \quad (j \in \mathbf{N}).$$

Now let $u \in W^{1,p}(\Lambda)$ and suppose that $u/d \in L^p(\Lambda)$, where $p \in [1, \infty)$. For each $x \in \Lambda$,

$$u(x) = \sum_{j=1}^{\infty} u_j(x),$$

where $u_j(x) = u(x)\phi_j^*(x)$; then given $\epsilon > 0$,

$$u(x) = \sum_{d_j > \epsilon} u_j(x) + \sum_{d_j \leqq \epsilon} u_j(x) := v(x) + w(x).$$

As there are points $y^{(j)} \in Q_j^*$ and $z^{(j)} \in Q_j$ such that

$$\operatorname{dist}(y^{(j)}, \mathbf{R}^n \setminus \Lambda) = \operatorname{dist}(Q_j^*, \mathbf{R}^n \setminus \Lambda) \quad \text{and}$$

$$|x^{(j)} - y^{(j)}| \leqq \frac{1}{2}\epsilon_1 \sqrt{n} l_j,$$

it follows that

$$d_j \leqq \operatorname{dist}(Q_j^*, \mathbf{R}^n \setminus \Lambda) + \frac{1}{2}\epsilon_1 \sqrt{n} l_j,$$

and so

$$\operatorname{dist}(Q_j^*, \mathbf{R}^n \setminus \Lambda) \geqq d_j - \frac{1}{2}\epsilon_1 \sqrt{n} l_j \geqq \left(1 - \frac{1}{2}\epsilon_1\right) d_j \geqq \frac{1}{2} d_j,$$

for all $j \in \mathbf{N}$. Since the series for v has a finite number of terms only, each term being a function with compact support in Λ, $v \in C_0^{\infty}(\Lambda)$. To show that $u \in W_0^{1,p}(\Lambda)$, it is thus sufficient to prove that

$$||u - v||_{1,p,\Lambda} = ||w||_{1,p,\Lambda} \leqq C(\epsilon),$$

where $C(\epsilon) \to 0$ as $\epsilon \to 0$.

Let $x \in \operatorname{supp} u_j$ for some $j \in \mathbf{N}$ with $d_j \leqq \epsilon$. Then $x \in Q_j^*$; thus

$$d(x) \leqq d_j + (1 + \epsilon_1)\sqrt{n}l_j \leqq d_j(2 + \epsilon_1) \leqq \epsilon(2 + \epsilon_1) < 3\epsilon.$$

Put $\Lambda(\epsilon) = \{x \in \Lambda : d(x) < 3\epsilon\}$. Then

$$||w||_{p,\Lambda}^p = \int_{\Lambda(\epsilon)} \left|\sum_{d_j \leqq \epsilon} u_j\right|^p \leqq \int_{\Lambda(\epsilon)} \left|\sum_{j=1}^{\infty} |u|\phi_j^*\right|^p = \int_{\Lambda(\epsilon)} |u|^p \to 0$$

as $\epsilon \to 0$. To estimate $||D_i w||_{p,\Lambda}$ note that

$$D_i w = (D_i u)\left(\sum_{d_j \leqq \epsilon} \phi_j^*\right) + u \sum_{d_j \leqq \epsilon} D_i \phi_j^*$$

and that

$$D_i\phi_j^* = (1/\Phi)D_i\phi_j - \phi_j(D_i\Phi)/\Phi^2.$$

If $x \in \operatorname{supp} \phi_j$,

$$d(x) \leqq d_j(2 + \epsilon_1) \leqq 3d_j \quad \text{and}$$
$$|D_i\phi_j(x)| \leqq A(\sqrt{n}l_j)^{-1} \leqq 12A/d(x);$$

also $\Phi(x)$ is a finite sum,

$$D_i\Phi(x) = \sum_{m=1}^{\infty} D_i\phi_m(x) \quad \text{and}$$

$$d(x) \leqq 3d_m, \ |D_i\phi_m(x)| \leqq 12A/d(x)$$

if in addition $x \in \operatorname{supp} \phi_m$. Hence for all $x \in \Omega$,

$$|D_i w(x)| \leqq |D_i u(x)| \left|\sum_{d_j \leqq \epsilon} \phi_j^*(x)\right|$$
$$+ 12A|u(x)| \sum_{d_j \leqq \epsilon} \chi_{Q_j^*}(x)\left\{1 + \sum_{m=1}^{\infty} \chi_{Q_m^*}(x)\right\}/d(x),$$

from which it follows that

$$||D_i w||_{p,\Lambda} \leqq \left(\int_{\Lambda(\epsilon)} |D_i u|^p\right)^{1/p} + 12A(1 + 12^n)\left(\int_{\Lambda(\epsilon)} (|u|/d)^p\right)^{1/p}$$
$$\to 0 \text{ as } \epsilon \to 0.$$

It is now clear that $u \in W_0^{1,p}(\Lambda)$, as required.

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THEOREM 1. *Suppose that* $\partial\Lambda$ *is of class* C^{∞}, *let* $u \in W^k E_{\phi}(\Lambda)$ *and assume that* $u/d^k \in L_{\phi}(\Lambda)$. *Then* $u \in W_0^k E_{\phi}(\Lambda)$.

Proof. Evidently $u \in W^{k,p}(\Lambda)$ and $u/d^k \in L^p(\Lambda)$ for all $p \in [\nu, \infty)$. Thus, by the result discussed above, $u \in W_0^{k,p}(\Lambda)$ for all $p \in [\nu, \infty)$, and so $u \in C^{k-1}(\bar{\Lambda})$ and $D^{\alpha}u|_{\partial\Lambda} = 0$ for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq k - 1$ (cf. e.g. [**6**]). Extend u to all of $\mathbf{R}^n$ by setting it equal to zero on $\mathbf{R}^n \backslash \Lambda$, and denote this extension again by u.

For the moment suppose that $u \in C^{\infty}(\bar{\Lambda})$. Since $\partial\Lambda$ is of class C^{∞}, there is a covering of $\partial\Lambda$ by open sets $U_1, \dots, U_N$ such that each U_i is homeomorphic, via a C^{∞} diffeomorphism Φ_i with C^{∞} inverse, to the open unit ball B in $\mathbf{R}^n$, and with

$$\Phi_i(U_i \cap \Lambda) = \{y \in \mathbf{R}^n : |y| < 1, y_n > 0\},$$

$$\Phi_i(U_i \cap \partial\Lambda) = \{y \in \mathbf{R}^n : |y| < 1, y_n = 0\}.$$

Let U_0 be an open set in $\mathbf{R}^n$ with $\bar{U}_0 \subset \Lambda$ and such that $U_0, U_1, \dots, U_N$ is an open covering of $\bar{\Lambda}$; let $\phi_0, \phi_1, \dots, \phi_N$ be a C^{∞} partition of unity subordinate to this covering. For each $\epsilon > 0$, let $\psi_{\epsilon}:[0, \infty) \to [0, 1]$ be a C^{∞} function such that

$$\psi_{\epsilon}(t) = \begin{cases} 0, & 0 \leqq t \leqq \epsilon, \\ 1, & 2\epsilon \leqq t, \end{cases}$$

and

$$|\psi_{\epsilon}^{(j)}(t)| \leqq c\epsilon^{-j}$$

for all $t \geqq 0$ and all $j \in \{1, 2, \dots, k\}$, where c is an absolute constant. Put $u_i = u\phi_i$; then

$$u = \sum_{i=0}^{N} u_i \quad \text{and}$$

$$\operatorname{supp} u_i \subset U_i \cap \bar{\Lambda} \quad \text{for each } i.$$

Since $D^{\alpha}u|_{\partial\Lambda} = 0$ for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq k - 1$, it follows that

$$D^{\alpha}u_i(\Phi_i^{-1}(y)) = O(y_n^{k-j})$$

as $y_n \downarrow 0$ for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq k$ and $\alpha_n = j$, and all $i \in \{1, 2, \dots, N\}$. It is now routine to verify that there is a constant C such that for all $p \in (1, \infty)$, all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq k$, all $\epsilon > 0$, and all $i \in \{1, 2, \dots, N\}$,

$$\|D^{\alpha}\{u_i - (\psi_{\epsilon} \circ (\Phi_i)_n)u_i\}\|_{p,\Lambda \cap U_i}^p \leqq C^p\epsilon.$$

Put

$$v^{(\epsilon)} = u_0 + \sum_{i=1}^{N} u_i(\psi_\epsilon \circ (\Phi_i)_n),$$

let $\mu > 0$ and denote by M the smallest integer such that $N \leqq 2^M$; put $u_i = 0$ for $i \in \{N + 1, \dots, 2^M\}$, if $N < 2^M$. Then by the convexity of ϕ,

$$\begin{aligned}
&\int_\Lambda \phi\left(\frac{D^\alpha u - D^\alpha v^{(\epsilon)}}{\mu}\right)dx \\
&= \int_\Lambda \phi\left(\frac{1}{\mu}\sum_{i=1}^{2^M} D^\alpha\{u_i - u_i(\psi_\epsilon \circ (\Phi_i)_n)\}\right)dx \\
&\leqq 2^{-M}\int_\Lambda \sum_{i=1}^{N} \phi\left(\frac{2^M}{\mu} D^\alpha\{u_i - u_i(\psi_\epsilon \circ (\Phi_i)_n)\}\right)dx \\
&\leqq 2^{-M}\epsilon N \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{2^M C}{\mu}\right)^{j\nu} \\
&\to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

Since $v^{(\epsilon)} \in C_0^\infty(\Lambda)$, it follows that $u \in W_0^k E_\phi(\Lambda)$.

All that is left is to remove the assumption that $u \in C^\infty(\bar{\Lambda})$. Suppose that we merely know that

$$u \in C^{k-1}(\bar{\Lambda}) \cap W^k E_\phi(\Lambda),$$

with $D^\alpha u|_{\partial\Lambda} = 0$ for all $\alpha \in \mathbf{N}_0^n$ with $|\alpha| \leqq k - 1$. With the notation used above, put

$$v_i = u_i \circ \Phi_i^{-1}:\{y = (y', y_n):|y| < 1, y_n \geqq 0\} \to \mathbf{R}$$

and let $\tilde{v}_i$ be the extension of v_i to B by oddness:

$$\tilde{v}_i(y', y_n) = -v_i(y', -y_n)$$

for all $y \in B$ with $y_n < 0$ ($i \in \{1, 2, \dots, N\}$). Each $\tilde{v}_i$ belongs to $W^k E_\phi(B)$. To check that this is so, first note that for all $\alpha \in \mathbf{N}_0^n$ with $|a| \leqq k$, and all $\psi \in C_0^\infty(B)$, integration by parts shows that with $B_\pm = \{y \in B:y_n \gtrless 0\}$,

$$\begin{aligned}
\int_B \tilde{v}_i D^\alpha \psi dy &= \int_{B_+} v_i(y)D^\alpha\psi(y)dy + \int_{B_-} \tilde{v}_i(y)D^\alpha\psi(y)dy \\
&= \int_{B_+} v_i(y)D^\alpha\psi(y)dy + \int_{B_+} v_i(y', y_n)D^\alpha\psi(y', -y_n)dy \\
&= \int_{B_+} (-1)^{|\alpha|}\psi(y)D^\alpha v_i(y)dy
\end{aligned}$$

$$+ \int_{B_+} (-1)^{|\alpha|}\psi(y', -y_n)D^\alpha v_i(y)dy,$$

the integrated terms vanishing on $\{y \in B: y_n = 0\}$ since $D^\beta v_i(y) = 0$ on this set if $|\beta| \leqq k - 1$. Hence

$$\begin{aligned}\int_B \tilde{v}_i D^\alpha \psi dy &= \int_{B_+} (-1)^{|\alpha|}\psi(y)D^\alpha v_i(y)dy \\ &\quad - \int_{B_-} (-1)^{|\alpha|}\psi(y)D^\alpha v_i(y', -y_n)dy \\ &= \int_B (-1)^{|\alpha|}\psi(y)w(y)dy,\end{aligned}$$

where

$$w(y) = \begin{cases} D^\alpha v_i(y) \text{ if } y \in B_+, \\ -D^\alpha v_i(y', -y_n) \text{ if } y \in B_-. \end{cases}$$

This shows that $\tilde{v}_i$ has the appropriate generalized derivatives in B, and the anti-symmetry then leads to the conclusion that $\tilde{v}_i \in W^k E_\phi(B)$, for $i \in \{1, 2, \ldots, N\}$. Moreover, supp $\tilde{v}_i \subset B$ as

$$\text{supp } v_i \subset \{y \in B: y_n \geqq 0\}.$$

Extend each such $\tilde{v}_i$ by zero to all of $\mathbf{R}^n$, and denote this extension again by $\tilde{v}_i$, for notational simplicity. Given any $\epsilon > 0$, write $\tilde{v}_{i,\epsilon} = \rho_\epsilon * \tilde{v}_i$, where ρ is the usual mollifier and

$$\rho_\epsilon(y) = \epsilon^{-n}\rho(y/\epsilon).$$

Application of Lemma 2.1 of [**3**] now shows that as $\epsilon \to 0$, $\tilde{v}_{i,\epsilon} \to \tilde{v}_i$ in $W^k E_\phi(B)$. Moreover, a simple calculation shows that for all $\alpha \in \mathbf{N}_0^n$, with $|\alpha| \leqq k - 1$, $D^\alpha \tilde{v}_{i,\epsilon}(y) = 0$ if $|y| < 1$ and $y_n = 0$. Also $\tilde{v}_{i,\epsilon} \in C_0^\infty(B)$ for $i \in \{1, 2, \ldots, N\}$. Finally, put

$$u_{i,\epsilon} = \tilde{v}_{i,\epsilon} \circ \Phi_i \quad \text{for } i \in \{1, 2, \ldots, N\}, \quad \text{and}$$

$$u_{(\epsilon)} = \sum_{i=1}^N \phi_i u_{i,\epsilon} + \phi_0(\rho_\epsilon * u_0).$$

Then

$$u_{(\epsilon)} \in C^\infty(\bar{\Lambda}),$$

$$D^\alpha u_{(\epsilon)}|_{\partial\Lambda} = 0 \text{ for all } \alpha \in \mathbf{N}_0^n \text{ with } |\alpha| \leqq k - 1, \quad \text{and}$$

$$u_{(\epsilon)} \to u \quad \text{in } W^k E_\phi(\Lambda).$$

Use of the first part of the proof now finishes the argument.

4. Mean distance functions. In this section we give results in the opposite direction to that of Section 3.

Definition 1. Let $p \in (1, \infty)$, *let* $\xi \in \mathbf{R}^n$ be such that $|\xi| = 1$, and define

$$\rho_\xi(x) = \min\{\,|t|:x + t\xi \notin \Lambda\} \quad (x \in \mathbf{R}^n).$$

The mean p-distance function m_p (with respect to Λ) is defined by

$$(m_p(x))^{-p} = \omega_n^{-1} \int_{|\xi|=1} (\rho_\xi(x))^{-p} d\sigma(\xi) \quad (x \in \Lambda),$$

where ω_n is the surface area of the unit ball in $\mathbf{R}^n$ and $d\sigma$ is the surface measure on the unit sphere.

This is an obvious generalization of the mean distance function of E. B. Davies [**2**], which is the function m_2, in our notation.

LEMMA 1. *Suppose that* $1 < p < q < \infty$. *Then for all* $x \in \Lambda$,

$$d(x) \leqq m_q(x) \leqq m_p(x).$$

Proof. Since $d(x) \leqq \rho_\xi(x)$ for all $\xi \in \mathbf{R}^n$ with $|\xi| = 1$, it follows that

$$(m_q(x))^{-q} \leqq \omega_n^{-1} \int_{|\xi|=1} (d(x))^{-q} d\sigma(\xi) = (d(x))^{-q},$$

and so $d(x) \leqq m_q(x)$. By Hölder's inequality,

$$(m_p(x))^{-p} \leqq \left(\int_{|\xi|=1} (\rho_\xi(x))^{-q} d\sigma(\xi) \right)^{p/q} \omega_n^{-p/q} = (m_q(x))^{-p},$$

which shows that $m_q(x) \leqq m_p(x)$.

The next theorem is a simple extension of a result of Davies which deals with the case $p = 2$.

THEOREM 2. *For all* $f \in C_0^\infty(\Lambda)$, *and any* $p \in (1, \infty)$,

$$\int_\Lambda |f(x)|^p/(m_p(x))^p dx \leqq \left(\frac{p}{p-1}\right)^p \int_\Lambda |\operatorname{grad} f(x)|^p dx. \tag{1}$$

Proof. We begin by establishing a one-dimensional version of (1), namely, that if $a, b \in \mathbf{R}$,

$$\rho(t) = \min(|t - a|, |t - b|) \quad \text{and} \quad g \in C_0^\infty((a, b)),$$

then

$$\int_a^b |g(t)|^p/(\rho(t))^p dt \leqq \left(\frac{p}{p-1}\right)^p \int_a^b |g'(t)|^p dt. \tag{2}$$

To prove this, let

$$O_+ = \{t \in (a, b):g(t) > 0\}, \quad O_- = \{t \in (a, b):g(t) < 0\}.$$

Since O_+ is open,

$$O_+ = \bigcup_{m=1}^{\infty} I_m,$$

where the I_m are pairwise disjoint, open invervals. Let $I_m = (a_m, b_m)$ and set

$$c_m = \frac{1}{2}(a_m + b_m).$$

Then since $g(a_m) = 0$,

$$\int_{a_m}^{c_m} |g(t)|^p/(t - a_m)^p dt$$
$$= \int_{a_m}^{c_m} (t - a_m)^{-p}\left(\int_{a_m}^{t} (|g(s)|^p)' ds\right)dt$$
$$= \int_{a_m}^{c_m} (|g(s)|^p)'\left(\int_{s}^{c_m} (t - a_m)^{-p} dt\right)ds$$
$$= \frac{p}{p-1}\int_{a_m}^{c_m} (g(s))^{p-1} g'(s)(s - a_m)^{1-p} ds$$
$$- \frac{1}{p-1}(c_m - a_m)^{1-p}(g(c_m))^p.$$

Thus

$$\int_{a_m}^{c_m} |g(t)|^p/t - a_m)^p dt$$
$$\leqq \frac{p}{p-1}\left(\int_{a_m}^{c_m} (g(s))^p (s - a_m)^{-p} ds\right)^{(p-1)/p}\left(\int_{a_m}^{c_m} |g'(s)|^p ds\right)^{1/p},$$

and so

$$\int_{a_m}^{c_m} |g(t)|^p/(t - a_m)^p dt \leqq \left(\frac{p}{p-1}\right)^p \int_{a_m}^{c_m} |g'(t)|^p dt. \tag{3}$$

It follows similarly that

$$\int_{a_m}^{c_m} |g(t)|^p (b_m - t)^p dt \leqq \left(\frac{p}{p-1}\right)^p \int_{c_m}^{b_m} |g'(t)|^p dt, \tag{4}$$

and (3) and (4) together show that

$$\int_{I_M} |g(t)|^p/(\rho_m(t))^p dt \leqq \left(\frac{p}{p-1}\right)^p \int_{I_M} |g'(t)|^p dt,$$

where

$$\rho_m(t) = \min(|t - a_m|, |t - b_m|).$$

Evidently a similar inequality holds for each interval into which O_- may be decomposed; thus as $\rho(t) \geqq \rho_m(t)$ for all $t \in I_m$,

$$\begin{aligned}
&\int_a^b |g(t)|^p/(\rho(t))^p dt \\
&= \int_{O_+} |g(t)|^p/(\rho(t))^p + \int_{O_-} |g(t)|^p/(\rho(t))^p dt \\
&\leqq \left(\frac{p}{p-1}\right)^p \left\{ \int_{O_+} |g'(t)|^p dt + \int_{O_-} |g'(t)|^p dt \right\} \\
&\leqq \left(\frac{p}{p-1}\right)^p \int_a^b |g'(t)|^p dt,
\end{aligned}$$

and (2) is proved.

Now let $\{e_1, e_2, \ldots, e_n\}$ be any orthonormal basis of $\mathbf{R}^n$ and let $x \in \Lambda$ have coordinates $(x_1, x_2, \ldots, x_n)$ with respect to this basis; put

$$\rho_i(x) = \rho_{e_i}(x).$$

Then

$$\int (|f(x)|/\rho_i(x))^p dx_i \leqq \left(\frac{p}{p-1}\right)^p \int |D_i f(x)|^p dx_i,$$

where the integrals are over a line through an arbitrary point in Λ and parallel to the x_i-axis. Hence

$$\begin{aligned}
&\int_\Lambda (|f(x)|/\rho_i(x))^p dx \\
&\leqq \left(\frac{p}{p-1}\right)^p \int_\Lambda |D_i f(x)|^p dx \\
&\leqq \left(\frac{p}{p-1}\right)^p \int_\Lambda |\operatorname{grad} f(x)|^p dx.
\end{aligned}$$

It follows that for any $\xi \in \mathbf{R}^n$ with $|\xi| = 1$,

$$\int_\Lambda (|f(x)|/\rho_\xi(x))^p dx \leqq \left(\frac{p}{p-1}\right)^p \int_\Lambda |\operatorname{grad} f(x)|^p dx,$$

and so

$$\begin{aligned}
&\int_{|\xi|=1} \int_\Lambda (|f(x)|/\rho_\xi(x))^p dx d\sigma(\xi) \\
&\leqq \omega_n \left(\frac{p}{p-1}\right)^p \int_\Lambda |\operatorname{grad} f(x)|^p dx.
\end{aligned}$$

Thus

$$\int_\Lambda |f(x)|^p \int_{|\xi|=1} (\rho_\xi(x))^{-p} d\sigma(\xi) dx$$

$$\leqq \omega_n \left(\frac{p}{p-1}\right)^p \int_\Lambda |\text{grad}\, f(x)|^p dx,$$

which amounts to (1).

COROLLARY 1. *Let* $f \in W_0^{1,p}(\Lambda)$ *for some* $p \in (1, \infty)$. *Then*

$$f/m_p \in L^p(\Lambda).$$

Proof. Since (1) plainly holds for all elements of $W_0^{1,p}(\Lambda)$, the result is immediate.

In [**2**] Davies has given conditions on Λ sufficient to ensure that m_2 is equivalent to d. We give these below: they also ensure the equivalence of m_p and d, for all $p \in [1, \infty)$.

Given any $\theta \in \left(0, \frac{\pi}{2}\right)$, the boundary of Λ is said to satisfy a *θ-cone condition* if there is a right-circular cone $\mathscr{C}$ of semi-angle θ and fixed height h such that given any $x \in \partial\Lambda$, there is a cone $\mathscr{C}_x \subset \mathbf{R}^n \setminus \Lambda$ congruent to $\mathscr{C}$ and with vertex x. Let $\omega(a)$ denote the solid angle subtended at the origin by a ball of radius $a < 1$ and with centre a distance 1 from the origin:

$$\omega(a) = \int_0^{\sin^{-1} a} \sin^{n-2} t dt \Big/ \left(2 \int_0^{\pi/2} \sin^{n-2} t dt\right).$$

PROPOSITION 1. *Suppose that* $\theta \in (0, \pi/2)$ *and that* $\partial\Lambda$ *satisfies a* θ*-cone condition. Then there is a constant* $b \in (0, 1/4]$ *such that for all* $x \in \Lambda$ *and all* $p \in [1, \infty)$,

$$m_p(x) \leqq 2(\omega(b \sin \theta))^{-1/p} d(x).$$

As the proof is the same as that given by Davies for $p = 2$, we omit it.

We can now give a characterization of $W_0^{1,p}(\Lambda)$.

THEOREM 3. *Suppose that* $\partial\Lambda$ *satisfies a* θ*-cone condition for some* $\theta \in (0, \pi/2)$, *and let* $p \in [1, \infty)$. *Then*

$$W_0^{1,p}(\Lambda) = \{f \in W^{1,p}(\Lambda): f/d \in L^p(\Lambda)\}.$$

Proof. That

$$W_0^{1,p}(\Lambda) \subset \{f \in W^{1,p}(\Lambda): f/d \in L^p(\Lambda)\}$$

follows immediately from Corollary 1 and Proposition 1. The reverse inclusion holds with no conditions on $\partial\Lambda$, as has already been explained.

Now that these results have been established, we may turn to Orlicz-Sobolev spaces.

THEOREM 4. *For all* $f \in C_0^\infty(\Lambda)$,

(5) $\qquad \|f/m_\nu\|_{(\phi),\Lambda} \leqq \nu' \|\operatorname{grad} f\|_{(\phi),\Lambda},$

where $\nu' = \nu/(\nu - 1)$.

Proof. For all $\lambda > 0$,

$$\int_\Lambda \phi\left(\frac{f(x)}{\lambda m_\nu}\right) dx = \int_\Lambda \sum_{j=1}^\infty \frac{1}{j!} \left|\frac{f(x)}{\lambda m_\nu}\right|^{j\nu} dx$$

$$\leqq \sum_{j=1}^\infty \frac{1}{j!} \int_\Lambda \left|\frac{f(x)}{\lambda m_\nu}\right|^{j\nu} dx$$

$$\leqq \sum_{j=1}^\infty \frac{1}{j!\lambda^{j\nu}} \left(1 - \frac{1}{j\nu}\right)^{-j\nu} \|\operatorname{grad} f\|_{j\nu,\Lambda}^{j\nu},$$

by Theorem 2. Hence

$$\int_\Lambda \phi\left(\frac{f(x)}{\lambda m_\nu}\right) dx \leqq \sum_{j=1}^\infty \frac{(\nu')^\nu}{j!\lambda^{j\nu}} \|\operatorname{grad} f\|_{j\nu,\Lambda}^{j\nu}$$

$$= \sum_{j=1}^\infty \frac{1}{j!(\lambda/\nu')^{j\nu}} \|\operatorname{grad} f\|_{j\nu,\Lambda}^{j\nu} \leqq 1$$

if $\lambda/\nu' \geqq \|\operatorname{grad} f\|_{(\phi),\Lambda}$, for

$$\int_\Lambda \phi\left(\frac{|\operatorname{grad} f(x)|}{\mu}\right) dx = \sum_{j=1}^\infty \frac{1}{j!\mu^{\nu j}} \|\operatorname{grad} f\|_{j\nu,\Lambda}^{j\nu} \leqq 1$$

if

$$\mu \geqq \|\operatorname{grad} f\|_{(\phi),\Lambda}.$$

The result follows immediately.

COROLLARY 2. *If* $f \in W_0^1 E_\phi(\Lambda)$, *then* $f/m_\nu \in L_\phi(\Lambda)$.

Proof. Since (5) clearly holds for all $f \in W_0^1 E_\phi(\Lambda)$, the result is obvious.

COROLLARY 3. *Suppose that* $\partial\Lambda$ *is of class* C^∞. *Then*

$$W_0^1 E_\phi(\Lambda) = \{u \in W^1 E_\phi(\Lambda) : u/d \in L_\phi(\Lambda)\}.$$

Proof. The corollary is a direct consequence of Theorem 1, Proposition 1, and Corollary 2.

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We now extend these results to higher-order spaces, and begin with an analogue of (2).

PROPOSITION 2. *Let $a, b \in \mathbf{R}$, $a < b$, let $p \in \mathbf{N}$ be even, $p \geqq 2$, and let $k \in \mathbf{N}$. Then for all $f \in C_0^\infty((a, b))$,*

$$(6) \quad \int_a^b |f(t)|^p/(\rho(t))^{kp}dt \leqq \left(\frac{p}{p-1}\right)^{kp} \int_a^b |f^{(k)}(t)|^p dt.$$

Proof. Let $j \in \{0, 1, \dots, k-1\}$ and put $c = \frac{1}{2}(a + b)$. Exactly as in the proof of (2), we find that

$$\int_a^c |f^{(j)}(t)|^p/(t-a)^{(k-j)p}$$
$$\leqq \left(\frac{p}{p-1}\right)^p \int_a^c |f^{(j+1)}(s)|^p (s-a)^{-(k-j-1)p} ds,$$

with a corresponding inequality for the integrals from c to b. Hence

$$(7) \quad \int_a^c |f^{(j)}(t)|^p/(\rho(t))^{(k-j)p}$$
$$\leqq \left(\frac{p}{p-1}\right)^p \int_a^b |f^{(j+1)}(t)|^p (\rho(t))^{-(k-j-1)p} dt.$$

Successive application of (7) now shows that (6) holds.

THEOREM 5. *Let p be even, $p \geqq 2$, let $k \in \mathbf{N}$ and suppose that $f \in C_0^\infty(\Lambda)$. Then*

$$(8) \quad \int_\Lambda |f(x)|^p/(m_{pk}(x))^{pk} dx$$
$$\leqq \left(\frac{p}{p-1}\right)^{kp} \int_\Lambda \left(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\right)^{p/2} dx.$$

Proof. Just as in the proof of Theorem 2, and with the same notation, we see from (6) that

$$\int |f(x)|^p/\rho_i^{pk}(x)dx_i \leqq \left(\frac{p}{p-1}\right)^{kp} \int |D_i^k f(x)|^p dx_i.$$

Thus,

$$\int_\Lambda |f(x)|^p/\rho_i^{pk}(x)dx \leqq \left(\frac{p}{p-1}\right)^{pk} \int_\Lambda \left(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\right)^{p/2} dx,$$

and so for any $\xi \cup \mathbf{R}^n$ with $|\xi| = 1$,

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$$\int_\Lambda |f(x)|^p/\rho_\xi^{pk}(x)dx \leqq \left(\frac{p}{p-1}\right)^{pk} \int_\Lambda \left(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\right)^{p/2} dx.$$

Hence

$$\int_{|\xi|=1} \int_\Lambda |f(x)|^p/\rho_\xi^{kp}(x)dxd\xi$$

$$\leqq \omega_n \left(\frac{p}{p-1}\right)^{kp} \int_\Lambda \left(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\right)^{p/2} dx,$$

from which (8) follows directly.

COROLLARY 4. *Let p be even, $p \geqq 2$, and let $k \in \mathbf{N}$. Then if $f \in W_0^{k,p}(\Lambda)$, $f/m_p^k \in L^p(\Lambda)$.*

To remove the hypothesis that p is even needs a little more effort.

THEOREM 6. *Let $p \in [2, \infty)$, let $k \in \mathbf{N}$, suppose $\partial\Lambda$ is of class C^∞ and let $f \in C_0^\infty(\Lambda)$. Then there are constants C, C_1, depending only on k, n, and Λ, such that*

$$(9) \quad \int_\Lambda |f(x)|^p/(d(x))^{pk}dx \leqq C_1^k \|f\|_{k,p,\Lambda}^p$$

$$\leqq C_1^k C(k, n, \Lambda) \int_\Lambda \left(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\right)^{p/2} dx.$$

Proof. There is a unique $m \in \mathbf{N}$ such that $2m \leqq p < 2(m + 1)$. In view of Theorem 5, Proposition 1, and the Poincaré inequality [1], (9) holds when $p = 2m$ or $2(m + 1)$. Thus the map

$$T_q: f \mapsto f/d^k$$

of $W_0^{k,q}(\Lambda)$ in $L^q(\Lambda)$ is continuous when $q = 2m$ and $q = 2(m + 1)$. By interpolation theory (cf. [8], Theorems 4.3.1 and 4.3.2), the map $u \mapsto u/d^k$ of $W_0^{k,p}(\Lambda)$ in $L^p(\Lambda)$ is continuous and has norm bounded above by

$$\|T_{2m}\|^{1-\theta}\|T_{2(m+1)}\|^\theta$$

for an appropriate $\theta \in (0, 1)$. The result now follows easily.

COROLLARY 5. *Let $\partial\Lambda$ be of class C^∞, let $p \in [2, \infty)$ and suppose that $k \in \mathbf{N}$. Then*

$$W_0^{k,p}(\Lambda) = \{f \in W^{k,p}(\Lambda): f/d^k \in L^p(\Lambda)\}.$$

This result is also given in [5].

Finally, we deal with higher-order Orlicz-Sobolev spaces.

THEOREM 7. *Let $\partial\Lambda$ be of class C^∞, let $k \in \mathbf{N}$ and let $f \in C_0^\infty(\Lambda)$. Then there is a constant C, which depends only on k, n and Λ, such that*

$$\|f/d^k\|_{(\phi),\Lambda} \leqq C\|D^k f\|_{(\phi),\Lambda},$$

where

$$|D^k f(x)| = \Big(\sum_{|\alpha|=k} |D^\alpha f(x)|^2\Big)^{1/2}.$$

Proof. For any $\lambda > 0$,

$$(10) \qquad \int_\Lambda \phi\left(\frac{f(x)}{\lambda d^k(x)}\right)dx \leqq \sum_{j=1}^{\infty} \frac{1}{j!\lambda^{j\nu}} \int_\Lambda \left|\frac{f(x)}{d^k(x)}\right|^{j\nu} dx.$$

Now suppose that $\nu \geqq 2$. Since we may, and shall, assume without loss of generality that the constants C_1 and C of (9) are both greater than or equal to 1, the right-hand side of (10) may be majorized by

$$\sum_{j=1}^{\infty} \frac{1}{j!\lambda^{\nu j}} C_1^k C\|D^k f\|_{\nu j,\Lambda}^{\nu j} \leqq \sum_{j=1}^{\infty} \frac{1}{j!(\lambda C_1^{-k} C^{-1})^{\nu j}} \|D^k f\|_{\nu j,\Lambda}^{\nu j},$$

from which it is easy to see that

$$\int_\Lambda \phi\left(\frac{f(x)}{\lambda d^k(x)}\right)dx \leqq 1 \quad \text{if } \lambda(C_1^k C)^{-1} \geqq \|D^k f\|_{(\phi),\Lambda};$$

that is,

$$\|f/d^k\|_{(\phi),\Lambda} \leqq CC_1^k\|D^k f\|_{(\phi),\Lambda}.$$

If $1 < \nu < 2$, the first term in the series expansion requires separate treatment. In this case,

$$\begin{aligned}
&\int_\Lambda |f(x)/d^k(x)|^\nu dx \\
&\leqq |\Lambda|^{1/2}\Big(\int_\Lambda |f(x)/d^k(x)|^{2\nu}\Big)^{1/2} \\
&\leqq C_1^{k/2}C^{1/2}|\Lambda|^{1/2}\|D^k f\|_{2\nu,\Lambda}^\nu.
\end{aligned}$$

Hence

$$\int_\Lambda \phi\left(\frac{f(x)}{\lambda d^k(x)}\right)dx \leqq (\lambda C_1^{-k}\widetilde{C}^{-1})^{-\nu}\|D^k f\|_{2\nu,\Lambda}^\nu$$

$$+ \sum_{j=2}^{\infty} \frac{1}{j!}(\lambda C_1^{-k}\widetilde{C}^{-1})^{-\nu j}\|D^k f\|_{\nu j,\Lambda}^{\nu j},$$

where $\widetilde{C} = C \max(1, |\Lambda|)$. Since

$$\int_{\Lambda} \phi\left(\frac{|D^k f(x)|}{\mu}\right) dx = \sum_{j=1}^{\infty} \frac{1}{j!\mu^{\nu j}} \|D^k f\|_{\nu j,\Lambda}^{\nu j} \leqq 1$$

if

$$\mu \geqq \|D^k f\|_{(\phi),\Lambda},$$

it follows that

$$(\lambda C_1^{-k} \tilde{C}^{-1})^{-2\nu} \|D^k f\|_{2\nu,\Lambda}^{2\nu} \leqq \frac{2}{9}$$

and

$$\sum_{j=2}^{\infty} \frac{1}{j!} \left(\lambda C_1^{-k} \tilde{C}^{-1}\right)^{-\nu j} \|D^k f\|_{\nu j,\Lambda}^{\nu j} \leqq \frac{1}{9}$$

if

$$\lambda C_1^{-k} \tilde{C}^{-1} \geqq 3\|D^k f\|_{(\phi),\Lambda}.$$

Thus,

$$\int_{\Lambda} \phi\left(\frac{f(x)}{\lambda d^k(x)}\right) dx \leqq \frac{1}{9} + \frac{1}{3}\sqrt{2} < 1$$

if

$$\lambda \geqq 2C_1^k \tilde{C} \|D^k f\|_{(\phi),\Lambda},$$

and so

$$\|f/d^k\|_{(\phi),\Lambda} \leqq 3C_1^k \tilde{C} \|D^k f\|_{(\phi),\Lambda}.$$

The proof is complete.

COROLLARY 6. *Let $\partial\Lambda$ be of class C^{∞}. Then*

$$W_0^k E_{\phi}(\Lambda) = \{f \in W^k E_{\phi}(\Lambda): f/d^k \in L_{\phi}(\Lambda)\}.$$

Proof. The corollary is an immediate consequence of Theorems 1 and 7.

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REFERENCES

1. R. A. Adams, Sobolev spaces (Academic Press, New York, 1975).
2. E. B. Davies, *Some norm bounds and quadratic form inequalities for Schrödinger operators II*, J. Operator Theory *12* (1984), 177-196.

3. T. Donaldson and N. Trudinger, *Orlicz-Sobolev spaces and embedding theorems*, J. Functional Anal. *8* (1971), 52-75.

4. D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators* (forthcoming book).

5. D. E. Edmunds, A. Kufner and J. Rakosnik, *Embeddings of Sobolev spaces with weights of power type*, Z. Analysis Anwendungen *4* (1985), 25-34.

6. A. Friedman, *Partial differential equations* (Holt, Rinehart, Winston, New York, 1968).

7. E. M. Stein, *Singular integrals and differentiability properties of functions* (Princeton University Press, Princeton, 1970).

8. H. Triebel, *Interpolation theory, function spaces, differential operators* (VEB Deutscher Verl. Wissenschaften, Berlin, 1978, and North-Holland Publishing Co., Amsterdam, New York, Oxford, 1978).

9. ——— *Remark on a paper by D. E. Edmunds, A. Kufner and J. Rakosnik, "Embeddings on Sobolev spaces with weights of power type"*, Z. Analysis Anwendungen *4* (1985), 35-38.

University of Sussex,
Brighton, U.K.,
University College,
Cardiff, U.K.