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A NOTE ON ASYMPTOTIC UNIQUENESS FOR SOME NONLINEARITIES WHICH CHANGE SIGN

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In this paper, we prove the uniqueness of the decaying positive solution on all of R^n for certain second order non linear elliptic equations. This improves earlier work of a number of authors. These problems occur in the theory of peak solutions. In particular, our results apply to a number of non-smooth nonlinearities which occur as limiting equations in population problems.

The purpose of this paper is to improve a uniqueness result in Section 2 of Dancer [**6**]. The improvement is of particular interest because it applies to the asocial nonlinearity or its variants which occurs in population models (as in Conway, Gardner and Smoller [**1**], Dancer [**2, 4**], Du [**8**]. Once again, the uniqueness results are new even for balls.

We consider the problem

$$\begin{aligned} -\varepsilon\Delta u &= g(u) \text{ in } D \\ u &= 0 \text{ on } \partial D. \end{aligned} \tag{1}$$

More precisely, as in Dancer [**6**], we assume $g : R \to R$ is C^1 and that there exist $0 < a < b$ such that $g(0) = g(b) = 0$, $g(y) < 0$ on $(0,a)$, $g(y) > 0$ on (a,b), and $\int_0^b g(y)dy > 0$. In addition, we assume that there is $\delta > 0$ such that $g'(y) < 0$ on $(0,\delta)$, $g'(y) \leqslant 0$ on $(b-\delta,b)$ and, if $n \geqslant 4$, we assume that there exist $\tau \leqslant (n-1)/(n-3)$ and $k_1 > 0$ such that $g(y) \geqslant k_1|y-a|^\tau$ for y near a and $y > a$. These are the basic assumptions on g in Dancer [**6**]. Define $\phi \in (a,b)$ by $\int_0^\phi g(y)dy = 0$. In addition, assume that there exists $\widetilde{\gamma} \in [a,b)$ such that $(y-\widetilde{\gamma})g'(y) \leqslant g(y)$ on $(\widetilde{\gamma},b)$. If $\widetilde{\gamma} > \phi$, we assume $\theta(y) = yg'(y)/g(y)$ has the following properties: $\theta(y) \leqslant \theta(\widetilde{\gamma})$ on $[0,a]$, $\theta(y) \geqslant \theta(\phi)$ on (a,ϕ), θ is non increasing on $(\phi,\widetilde{\gamma})$ and $\theta(y) \leqslant \theta(\widetilde{\gamma})$ on $(\widetilde{\gamma},b)$. Note that, compared with [**6**], we have weakened (at least partially) the condition on g on (a,b) at the expense of placing conditions on g on $(0,a)$. Note also that our assumptions imply that $\theta(\widetilde{\gamma}) > 1$ if $\widetilde{\gamma} > \phi$. We shall need this below. To see this, note that since $(y-\alpha)g'(y) < g(y)$ is true close to α (for $y > \alpha > a$ and to b for $y < b$

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(since $g(y) > 0$ and $g'(y) \leqslant 0$), when we choose $\widetilde{\gamma}$ minimal, $(y - \widetilde{\gamma})g'(y) - g(y)$ must be zero at some point $\gamma_1 \in (\widetilde{\gamma}, b)$. Then $\theta(\gamma_1) = \gamma_1/(\gamma_1 - \widetilde{\gamma}) > 1$. Hence $\theta(\widetilde{\gamma}) \geqslant \theta(\gamma_1) > 1$. Note also that the existence of ϕ is necessary for the existence of solutions (by Dancer and Schmitt [**7**]).

We consider a bounded domain D in R^n such that $0 \in D$, D has C^3 boundary, D is invariant under the n reflections in the coordinate planes and such that, in addition, if $1 \leqslant i \leqslant n$ and if $0 < t < s < \widetilde{t}_i$, then $(I - P_i)D_{i,t} \supseteq (I - P_i)D_{i,s}$. Here P_i is the orthogonal projection onto span e_i, $D_{i,s} = \{x \in D : x_i = s\}$, $\widetilde{t}_i = \sup\{x_i : x \in D\}$ and $\{e_i\}$ denotes the usual basis for R^n. We say that such a domain is of type R_n.

THEOREM 1. *Assume that the above assumptions on g hold and D is of type R_n and $n \geqslant 2$. Then (1) has exactly two positive solutions u with $0 < \|u\|_\infty < b$, for all small positive ε.*

PROOF: The proof is a straight forward adaption of the proof of Dancer [**6**, Theorem 2]. The only difficulty is to prove the uniqueness and weak non-degeneracy (in the sense of Section 1 there) of the positive solution u_0 of

$$-r^{1-n}\left(r^{n-1}u'(r)\right)' = g\left(u(r)\right) \text{ on } (0,\infty) \tag{2}$$
$$u'(0) = 0, \qquad 0 < \|u\|_\infty < b, \qquad u(r) \to 0 \text{ on } r \to \infty.$$

The proof of this combines the ideas in the proof of Dancer [**6**, Lemma 1] with those in Kwong and Zhang [**10**]. We let $w(r)$ denote the solution of

$$-r^{1-n}\left(r^{n-1}w'\right)' = g'\left(u_0(r)\right)w(r)$$
$$w(0) = 1, \qquad w'(0) = 0\,.$$

As in Dancer [**6**], $u_0(0) > a$ and $u_0'(r) < 0$ on $(0,\infty)$. Choose $\widehat{r} > 0$ such that $u_0(\widehat{r}) = \widetilde{\gamma}$. (If no such $\widehat{r}$ exists, the argument is simpler.) Note that $u_0(r) > \widetilde{\gamma}$ on $(0,\widehat{r})$. Hence $-r^{1-n}\left(r^{n-1}(u_0(r) - \widetilde{\gamma})'\right)' = g\left(u_0(r)\right) \geqslant g'\left(u_0(r)\right)\left(u_0(r) - \widetilde{\gamma}\right)$ on $(0,\widehat{r})$ by our assumptions on g. Since $u_0(r) - \widetilde{\gamma} > 0$ on $(0,\widehat{r})$, a standard Sturm comparison theorem argument ensures that $w(r) > 0$ on $[0,\widehat{r})$. Note that $u_0(r) < \widetilde{\gamma}$ on $(\widehat{r},\infty)$. Our argument here follows the one in Dancer [**6**] fairly closely. We first assume that, if r_1 denotes the first positive zero of w, then $u_0(r_1) \geqslant \phi$. We shall prove in a little while that r_1 exists. Thus $\phi \leqslant u_0(r_1) \leqslant \widetilde{\gamma}$. By our assumptions on g, it follows easily (see the proof of Kwong and Zhang [**10**, Lemma 7]) that there is $\gamma > 1$ such that

$$\gamma g(y) - yg'(y) \begin{cases} \leqslant 0 & \text{if } 0 \leqslant y \leqslant u_0(r_1) \\ \geqslant 0 & \text{if } u_0(r_1) \leqslant y \leqslant b\,. \end{cases} \tag{3}$$

(That $\gamma > 1$ follows because $\theta(u_0(r_1)) \geqslant \theta(\widetilde{\gamma}) > 1$.) As in Dancer [6] or Kwong and Zhang [10], we define β by $\gamma = 1 + 2\beta^{-1}$. If r_2 denotes the second zero of w (where we take $r_2 = \infty$ if there is no second zero), we see from (3) that

$$(4) \qquad \Big(\gamma g(u_0(r)) - u_0(r) g'(u_0(r))\Big) w(r) \geqslant 0 \text{ on } [0, r_2) \; .$$

(Note that $w(r) > 0$ on $(0, r_1)$ and $w(r) < 0$ on (r_1, r_2). As in Step 1 of the proof of Dancer [6, Lemma 1], it follows easily that $W(r)$ is increasing and positive on $(0, r_2)$, where $W(r) = r^{n-1}(v(r)w'(r) - w(r)v'(r))$ and $v(r) = ru_o'(r) + \beta u_0(r)$. As in Dancer [6], this leads to a contradiction when $r_2 = \infty$ if w is bounded and $n = 2$ or if $w(r) \to 0$ as $r \to \infty$ and $n \geqslant 3$. Thus w cannot have exactly one zero and satisfy the boundary condition at infinity (the one in the definition of weak nondegeneracy).

For future reference, note that we can apply the same argument to the finite problem

$$-r^{1-n}\left(r^{n-1}u'(r)\right)' = g(u(r)) \; , \;\; u'(0) = 0,$$
$$u(R) = 0 \;\; u(r) > 0 \;\; \text{on} \;\; (0, R).$$

If we define w as before, we find that the first zero r_1 of w if it exists satisfies $u(r_1) \leqslant \widetilde{\gamma}$ and if the second zero r_2 of w exists, then $W(r_2) > 0$. If $u(r_1) \geqslant \phi$, this implies that $r_2 \neq R$ because, if $w(r_2) = 0$, $w'(r_2) > 0$ while $v(R) = Ru'(R) < 0$ (since $u(R) = 0$). If $r_2 = R$, this then implies that $W(R) = R^{n-1}v(R)w'(R) < 0$ which contradicts what we have proved above. This nearly always precludes the second zero of w being at R.

We now return to the problem on $[0, \infty)$. First note that the Wronskian argument at the end of Step 1 of the proof of Dancer [6, Lemma 1] implies that w must have a zero in $(0, \infty)$. Thus r_1 always exists.

Next we need to consider the possibility that $u_0(r_1) < \phi$. Here we need to use an argument in Kwong and Zhang [6] and Kaper and Kwong [9]. We can simply apply Kwong and Zhang [6, Lemmas 15 and 16] on (r_1, ∞) to deduce that for the finite problem $w(r)$ has no zero in $(r_1, R]$ if $u(r_1) < \phi$ and $u(R) = 0$, while for the infinite problem (that is, on $[0, \infty)$) we find that, if $u_0(r_1) < \phi$, w is not the principal solution of the linearised equation (in the sense there). This means that w does not decay at infinity if $n \geqslant 3$ and w is not bounded at infinity if $n = 2$ (by a slight variant of Kwong and Zhang [6, Lemma 4]).

Hence we see that, in all cases, for the finite problem on $[0, R]$, no solution of the linearised equation can have its second zero at R while, for the infinite problem there can be no solution w of the linearised equation with $w'(0) = 0$, w satisfies our boundary condition at infinity and w has at most one positive zero.

Let α denote

$$\inf\left\{t > 0 : u(r,t) > 0 \text{ on } (0,\infty),\ u\,(r,t) \to 0 \text{ as } r \to \infty\right\},$$

where $u(r,t)$ is the solution of the initial value problem for (2) which satisfies $u(0,t) = t$, $u_1'(0,t) = 0$. It is well known and easy to prove that $\alpha \geqslant \phi$ and the infimum is achieved. By a similar argument to that in the proof of Kwong and Zhang [**10**, Lemma 6] (which is basically a comparison argument starting from $u(r,a)$), we see that, if $u_0 = u(\ ,\alpha)$, then w has exactly one zero in $(0,\infty)$. By our arguments above this implies that w does not satisfy the boundary condition at infinity and thus $u(r,\alpha)$ is weakly non-degenerate in the sense of Dancer [**6**]. As at the end of Section 1 in Dancer [**6**], we see that there is a branch of positive solutions of

$$(5) \qquad -\varepsilon\Delta u = g(u) \text{ in } B_1, \qquad u = 0 \text{ on } \partial B_1, \qquad u > 0 \text{ in } B_1$$

for small ε with $u(0)$ close to α (a unique such solution for each small positive ε). By continuous dependence arguments (as in Dancer [**6**]) the solution $\widetilde{w}$ of the linearised equation at these solutions has exactly one positive zero and hence as there these solutions have degree -1 as fixed points (of the map $u \to (-\varepsilon\Delta + cI)^{-1}\big(g(u) + cu\big)$ on $C(B_1)$ for suitable large c). The connected branch $\mathcal{B}$ of solutions of (5) containing this curve does not continue to large ε (as in Dancer [**6**]) and hence for each small ε the sum of the indices of solutions in $\mathcal{B}$ must be zero (by homotopy invariance). On the other hand by a simple connectedness argument and our earlier arguments, if $(u,\varepsilon) \in \mathcal{B}$ and $\widehat{w}$ is the corresponding solution of the linearised equation (of (5)) with $\widehat{w}(0) = 1$ then $\widehat{w}$ has at most one zero in $[0,1]$. Now $\mathcal{B}$ is a smooth curve parametrised by $u(0)$ (Dancer [**3**]) and hence the other "end" of $\mathcal{B}$ (rescaled) must approach another solution $\widetilde{u}$ of (2) as $\varepsilon \to 0$ or must satisfy $u(0) \to b$ as $\varepsilon \to 0$. We show that the first case can not occur. If it did, our remarks above on w and continuous dependence would imply that the radial solution of $-\Delta\widetilde{w} = g'(\widetilde{u})\widetilde{w}$, satisfying $\widetilde{w}(0) = 1$ would have at most 1 zero on $(0,\infty)$. By remarks earlier in the proof, $\widetilde{w}$ must have a zero and $\widetilde{w}$ is not bounded if $n = 2$ and does not tend to zero as r tends to infinity if $n \geqslant 3$. Thus $\widetilde{u}$ is weakly non-degenerate and since $\widetilde{w}$ has exactly one zero, as before we see that solutions of (1) nearby (for $D = B_1$) for small positive ε have index -1. This is impossible since the index sum is zero. Hence $\{u(0) : (u,\varepsilon) \in \mathcal{B}\}$ is a connected set in (α, b) with α and b in its closure. Thus, if $\alpha < t < b$, $u(r,t)$ is zero for a finite r and hence $u(r,\alpha)$ is the unique solution of (2). Since the weak non-degeneracy of $u(\ ,\alpha)$ has been obtained earlier in the proof, we see from our comment at the beginning of the proof that this completes the proof of Theorem 1. (The last part of the proof could be completed in a number of ways). $\square$

REMARK 1. Our methods can be extended to cover cases where g has 3 positive zeros $0 < a_1 < a_2 < a_3$ by a simple combination of the techniques here with those in Dancer [**6**]. More precisely, we assume that the basic assumptions of Theorem 3 in there hold (not the assumption of part (iii)) but assume that $g'(y) < 0$ in a deleted neighbourhood of a_1. (This assumption was omitted there). In addition, assume that there exists $\widetilde{\gamma} \in (a_2, a_3)$ such that $(y-\widetilde{\gamma})g'(y) < g(y)$ on $(\widetilde{\gamma}, a_3)$ while, if $\widetilde{\gamma} > \phi$, $\theta(y) = (y-a_1)g'(y)/g(y)$ has the following properties : $\theta(y) \leqslant \theta(\widetilde{\gamma})$ on (a_1, a_2), $\theta(y) \geqslant \theta(\phi)$ on (a_2, ϕ), θ is non-increasing on $(\phi, \widetilde{\gamma})$ and $\theta(y) \leqslant \theta(\widetilde{\gamma})$ on $(\widetilde{\gamma}, a_3)$. Here $\phi \in (a_2, a_3)$ is defined by $\int_{a_1}^{\phi} g(y)dy = 0$. The result is then that, if the above conditions hold and D is of type R_n, then (1) has exactly 2 positive solutions u with $a_1 < \|u\|_\infty < a_3$ for all small positive ε.

REMARK 2. We can weaken very slightly the condition that g is C^1 by allowing one positive point (and probably finitely many positive points) where g is continuous but g' has a jump discontinuity (usually a jump down if our assumptions are to be satisfied) and our assumptions are satisfied in a natural way. To see this, note that, if the jump discontinuity happens at ℓ, then a solution u of (2) can only take the value ℓ once (since u is decreasing) while it follows easily from the Gidas-Ni-Nirenberg theorem that a solution of (1) only takes the value ℓ on a set of zero measure. This ensures that the linearised equations still make sense. One can then carefully work through and check all our arguments are valid. There is one point which needs extra explanation. Assume that $\widetilde{u}_i$ and $\widetilde{v}_i$ are positive solutions of $-\Delta u = g(u)$ on $\varepsilon_i^{-1/2}D$ for Dirichlet boundary conditions where $\varepsilon_i \to 0$ as $i \to \infty$ such that $\widetilde{u}_i$ and $\widetilde{v}_i$ both converge uniformly to the same solution u_0 of (2). Let $w_i = \left(\|\widetilde{u}_i - \widetilde{v}_i\|_\infty\right)^{-1}(\widetilde{u}_i - \widetilde{v}_i)$. Since g is Lipschitz one easily check that a subsequence of w_i converges locally in $W^{2,p}$ to w. It remains to check that $-\Delta w = g'(u_0)w$ almost everywhere. (The rest of the argument is as before). If $u_0(z) = \ell$, and $\|x\| > z$, $\widetilde{u}_i(x) < \ell$ and $\widetilde{v}_i(x) < \ell$ for large i and hence

$$\begin{aligned} \frac{g(\widetilde{u}_i(x)) - g(\widetilde{v}_i(x))}{\widetilde{u}_i(x) - \widetilde{v}_i(x)} &= g'(\widetilde{\theta}(x)) \\ \big(\text{where } \widetilde{\theta}(x) \text{ is between } &\widetilde{u}_i(x) \text{ and } \widetilde{v}_i(x)\big) \\ &\to g'(u_0(z)) \end{aligned}$$

and this holds locally uniformly for $\|x\| \geqslant z + \delta$. Since we can use a similar argument for $\|x\| < z$, we can easily use this to show that w satisfies the required equation. The rest of the argument is a straightforward adaption of the proof of Theorem 1. This extension is of interest because frequently when one has a system and one equation is singularly perturbed, the limit equation has exactly this type of non-smoothness. (See Conway, Gardner and Smoller [**1**], Dancer [**2**] and [**5**].) Indeed, some of our results could be combined with results in Dancer [**5**] to obtain exact multiplicity results for some systems occurring in population problems. (We discuss this in a little more detail below).

REMARK 3. In the case of a ball B, our proof shows that, if u is a positive solution of $-\Delta u = \lambda g(u)$ in B, $u = 0$ on ∂B which is degenerate than the corresponding solution of the linearised equation must be positive on B. This may be of use for studying the bifurcations. This idea is useful for some other problems, for example, for $g(y) = y^p - y^q$ where $1 < p < q < (n+2)/(n-2)$.

REMARK 4. Note that most of Dancer [**6**, Lemma 4] was proved in Peletier and Serrin [**11**] by a different method.

REMARK 5. Our positive solutions can be shown to be non-degenerate for small positive ε and the dimension of the unstable manifold of the smaller solution is equal to the dimension of the unstable manifold for u_0 (in the space of radial functions). By an example in Dancer [**6**], we would not expect the result to be true for all domains. Note that the theorem is new even for the case of a ball.

Lastly, we want to discuss briefly how the assumptions of Theorem 1 can be verified in examples. Firstly, it is nearly always true that $g'(y) < 0$ on $\left(b - \widetilde{\beta}, b\right)$ where $\widetilde{\beta} > 0$. It is then easy to see that $\widetilde{\gamma} \leqslant b - \widetilde{\beta}$. If g is C^2, and if $\widetilde{\beta}$ exists, we consider intervals (α, b) (with $\alpha > a$) where the condition $g'(y)(y - \alpha) \leqslant g(y)$ holds. This inequality is clearly true if $y > b - \widetilde{\beta}$ and is clearly true for y close to α (if $\alpha > a$). Thus as we decrease α from $b - \widetilde{\beta}$, it must clearly first fail at an interior point of (α, b) and hence there exists $\alpha < y_0 < b$ such that $g(y_0) - g'(y_0)(y_0 - \alpha) = 0$ while $h(y) = g(y) - g'(y)(y - \alpha) \geqslant 0$ on (α, b). Thus $h'(y_0) = 0$ and $h''(y_0) \geqslant 0$ if $g'''(y_0)$ exists. By an easy calculation we see that $g'(y_0) > 0$, $g''(y_0) = 0$ and $g'''(y_0) \leqslant 0$ if $g'''(y_0)$ exists and $\alpha = y_0 - \left(g(y_0)/g'(y_0)\right)$. Hence we see that, if $\widetilde{\gamma} > a$, $\widetilde{\gamma} \leqslant \sup\left\{y - \left(g(y)/g'(y)\right) : a < y \leqslant b - \widetilde{\beta},\ g'(y) > 0,\ g''(y) = 0\right\}$ and if g has a third derivative, we only consider y's with $g'''(y_0) \leqslant 0$. This is a useful estimate for $\widetilde{\gamma}$ which is usually best possible. We now consider θ. Note that the last inequality on θ is trivially satisfied when $g'(y) < 0$ since, as we saw earlier, $\theta(\widetilde{\gamma}) > 1$.

Consider the case $g(y) = y(y - a)(1 - y)$ where $0 < a < 1/2$. If $1/2 \leqslant a < 1$, ϕ does not exist and by Dancer and Schmitt [**7**], there are no positive solutions. Note that we can rescale the case $a > 1$ back to the case $a \leqslant 1$. Then

$$\theta(y) = y(\ell n\, g)' = 1 - \frac{y}{1-y} + \frac{y}{y-a}.$$

Since $y(y - a)^{-1}$ is decreasing on $(a, 1)$ while $y(1 - y)^{-1}$ is increasing on $(0, 1)$, we see that θ decreases on $(a, 1)$. Hence the last 3 inequalities on θ always hold. To prove that $\theta(y) < \theta(\widetilde{\gamma})$ on $(0,a)$ it suffices to prove that $\theta(y) < 1$ on $(0,a)$ (since $\theta(\widetilde{\gamma}) > 1$). By a simple calculation we see that it suffices to prove that $1 + a - 2y > 0$ on $(0,a)$ and this is trivial to check. My result in this case was quoted in Yihong Du [**8**], and indeed, a question of Yihong Du motivated the present work.

By Remark 2, our methods can be easily applied to cover nonlinearities $g(y) = ys(y)$ where $s(0) < 0$, s is piecewise linear and continuous with a single jump discontinuity of s' at α and $s'(y) < 0$ for large y. Note that in this case $\theta(y) = 1 + \big(ys'(y)/s(y)\big)$, $\theta(y) < 1$ on $(0,a)$, $\theta(y) = 1 + \big(y/(y-a)\big)$ on (a,α) and hence is decreasing there and $\theta(y) = 1 + \big(y/(y-b)\big)$ on (α,b) which is also decreasing. Here s vanishes at a and b. These nonlinearities occur for the limit equation of competitive systems as in Conway, Gardner and Smoller [**1**], Dancer [**4**] or [**5**].

Our results also apply to nonlinearities $\widetilde{g}(y) = y\big((1-y)(y-a) - \mu(y-\alpha)^+\big)$ with $\mu > 0$. This occurs in a singular limit of a population system with an asocial nonlinearity (Conway, Gardner and Smoller [**1**] and Dancer [**5**]). We sketch the proof of this. Note that the nonlinearity is of the form $y\widetilde{p}(y)$ where $\widetilde{p}$ is a quadratic for $y \leqslant \alpha$ and is a different quadratic if $y > \alpha$. There is nothing to prove if $\alpha \leqslant 0$ or if $\alpha \geqslant 1$ or if ϕ does not exist. (Note that ϕ may not exist if $\alpha < \widehat{\phi}$ where $\widehat{\phi}$ is ϕ for $\mu = 0$ and that, if $\alpha \leqslant 0$, $y^{-1}\widetilde{g}(y)$ is a quadratic for $y \geqslant 0$.) Since $p_1(y) > (1-y)(y-a)$ for $y < \alpha$ with inequality reversed if $y > \alpha$ where $p_1(y) = (1-y)(y-a) - \mu(y-\alpha)$, p_1 must vanish at two points a_1, a_2 where $a_1 < a$ if $\alpha > a$ and $a_2 < 1$. (If p_1 does not vanish at all in $[\alpha, 1]$, ϕ cannot exist.) Since p_1 is quadratic, one can easily argue as earlier to see that $yr_1'(y)/r_1(y)$ decreases on (a_1,a_2) where $r_1(y) = yp_1(y)$. From this and earlier, we see that $\theta(y) = y\widetilde{g}'(y)/\widetilde{g}(y)$ decreases on $[\beta, a_2]$ where β is the first positive zero of $\widetilde{g}$. (Note that the jump discontinuity of $\theta(y)$ is downward if it occurs in $[\beta, a_2]$.) Hence we see that our assumptions on $\widetilde{g}$ are satisfied if $\theta(y) \leqslant 1$ on $[0,\beta]$. If $\alpha > a$, this follows from earlier. If $0 < \alpha < \beta$, p_1 is a quadratic on $[0,\beta]$, and so we can use the same argument as earlier to prove that $yp_1'(y)/p_1(y) < 1$ on $[0,\beta]$. This and our earlier argument for the pure quadratic case (that is with $\mu = 0$) imply that $\theta(y) < 1$ on $[0,\beta]$. This proves that our assumptions hold. Hence we have exactly 2 positive solutions of

$$\begin{aligned} -\varepsilon\Delta u &= \widetilde{g}(u) \ \text{ in } D \\ u &= 0 \quad \text{on } \partial D \end{aligned}$$

if ε is small and positive, D is of type R_n and ϕ exists. We can combine this with the theory in Dancer [**5**] to prove that the predator prey system with asocial nonlinearity has exactly 2 positive solutions in certain parameter ranges if D is of type R_n (for a large and ε very small compared with a^{-1} in the notation there.) Note that we need to ensure that the solution $\widetilde{u}_0$ of (2) (for g replaced by $\widetilde{g}$) satisfies $\|\widetilde{u}_0\|_\infty > \alpha$ to know there are exactly 2 solutions. Note that α is $-d^{-1}e$ in the notation there and b is used there where we use a here and that the above condition on $\widetilde{u}_0$ is equivalent to assuming $\|u_0\|_\infty > \alpha$ provided ϕ still exists for $\widetilde{g}$. $\square$

References

[1] E. Conway, R. Gardner and J. Smoller, 'Stability and bifurcation of steady state solutions for predator-prey equations', *Adv. in Appl. Math.* **3** (1982), 288–334.

[2] E.N. Dancer, 'On positive solutions of some pairs of equations', *Trans. Amer. Math. Soc.* **284** (1984), 729–743.

[3] E.N. Dancer, 'On the structure of solutions of an equation in catalysis theory when a parameter is large', *J. Differential Equations* **37** (1980), 404–437.

[4] E.N. Dancer, 'On positive solutions of some pairs of equations II', *J. Differential Equations* **60** (1985), 236–258.

[5] E.N. Dancer, 'On uniqueness and stability for solutions of singularly perturbed predator-prey type equations with diffusion', *J. Differential Equations* **102** (1993), 1–32.

[6] E.N. Dancer, 'On positive solutions of some singularly perturbed problems where the nonlinearity changes sign', *Topol. Methods Nonlinear Anal.* **5** (1995), 141–175.

[7] E.N. Dancer and K. Schmitt, 'On positive solutions of semilinear elliptic problems', *Proc. Amer. Math. Soc.* **101** (1987), 445–452.

[8] Y. Du, 'Bifurcation from semitrivial solution bundles and application to certain equation systems', *Nonlinear Anal.* **27** (1996), 1407–1435.

[9] H. Kaper and M.H. Kwong, 'Uniqueness of non-negative solutions of a class of semilinear elliptic equations', in *Nonlinear diffusion equations and their equilibrium states II*, (W-M Ni, L.A. Peletier and J. Serrin, Editors) (Springer-Verlag, Berlin, Heidelberg, New York, 1988), pp. 1–17.

[10] M.H. Kwong and L. Zhang, 'Uniqueness of the positive solution of $\Delta u + f(u) = 0$ in an annulus', *Differential Integral Equations* **4** (1991), 583–599.

[11] L.A. Peletier and J. Serrin, 'Uniqueness of non-negative solutions of semilinear equations in R^n', *J. Differential Equations* **61** (1986), 380–397.

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