COMPOSITIO MATHEMATICA

Coarse flow spaces for relatively hyperbolic groups

A. Bartels

Compositio Math. **153** (2017), 745–779.

doi:10.1112/S0010437X16008216

Compositio Math. 153 (2017) 745–779
doi:10.1112/S0010437X16008216

Coarse flow spaces for relatively hyperbolic groups

A. Bartels

Abstract

We introduce coarse flow spaces for relatively hyperbolic groups and use them to verify a regularity condition for the action of relatively hyperbolic groups on their boundaries. As an application the Farrell–Jones conjecture for relatively hyperbolic groups can be reduced to the peripheral subgroups (up to index-2 overgroups in the L-theory case).

Introduction

Farrell and Jones [FJ86] used the geodesic flow on closed Riemannian manifolds of negative sectional curvature to prove that the Whitehead group of the fundamental group of such manifolds vanishes. This method has been extremely fruitful and has been generalized in many ways.

Among the developments following [FJ86] was the formulation of what is now known as the Farrell–Jones conjecture [FJ93]. This conjecture predicts that the K- and L-theory of group rings $R[G]$ is determined by group homology and the K- and L-theory of group rings of virtually cyclic subgroups. If the conjecture holds for a group G, then this often yields vanishing results or computational results for Whitehead groups and the manifolds structure set appearing in surgery theory. In particular, the Farrell–Jones conjecture has implications for the classification of higher-dimensional non-simply connected manifolds. We will review the precise formulation of the conjecture in § 4. More information about the Farrell–Jones conjecture and its applications can be found for example in [BLR08c, Lue10, LR05].

In many cases it is fruitful to replace the family of virtually cyclic subgroups VCyc with a bigger family of subgroups $\mathcal{F}$. There is then a formulation of the Farrell–Jones conjecture relative to $\mathcal{F}$. This version of the conjecture is particularly useful whenever the groups in the family $\mathcal{F}$ are already known to satisfy the original Farrell–Jones conjecture.

In work with Lück and Reich the geodesic flow method from [FJ86] has been successfully implemented in [BL12a, BLR08a, BLR08b] to prove the Farrell–Jones conjecture for hyperbolic groups. More generally, the results from [BL12a, BLR08b] state that the Farrell–Jones conjecture for a group G holds relative to a family $\mathcal{F}$ whenever there exists an action of G on a finite-dimenional contractible ANR satisfying a regularity condition relative to the family $\mathcal{F}$. Here, ANR stands for absolute neighborhood retract. We will review this regularity condition shortly and refer to actions satisfying it relative to $\mathcal{F}$ as finitely $\mathcal{F}$-amenable actions. In this language the main result of [BLR08a] implies that for hyperbolic groups the action on the boundary is finitely VCyc-amenable. In this paper we prove a similar result for relatively hyperbolic groups. For definitions of relatively hyperbolic groups see [Far98, Gro87, Gro93, Szc98]. We will use Bowditch's point of view [Bow12], recalled in § 2.

Received 8 April 2015, accepted in final form 4 July 2016, published online 9 March 2017.
2010 Mathematics Subject Classification 18F25, 19G24, 20F67 (primary).
Keywords: Farrell–Jones conjecture, K- and L-theory of group rings.

Theorem. *Let G be a countable group that is relatively hyperbolic to subgroups $P_1, \dots, P_n$. Let $\mathcal{P}$ be the family of subgroups of G that are either virtually cyclic or subconjugated to one of the P_i. Then the action of G on its boundary Δ is finitely $\mathcal{P}$-amenable.*

This result appears as Theorem 3.1 in § 3.

The boundary Δ is usually neither contractible nor an *ANR*. For hyperbolic groups Bestvina and Mess proved that the union of the Rips complex with the boundary is a contractible compact *ANR* [BM91]. Similarly, for relatively hyperbolic groups there is a relative Rips complex [Dah03, MY] and in the appendix we extend the Bestvina–Mess result to the relatively hyperbolic case. This is closely related to results of Dahmani [Dah03]. Using [BL12a, BLR08b] we will obtain the following application to the Farrell–Jones conjecture.

Corollary. *Let G be a countable group that is relatively hyperbolic to subgroups $P_1, \dots, P_n$. If $P_1, \dots, P_n$ satisfy the Farrell–Jones conjecture then G satisfies the Farrell–Jones conjecture.*

This corollary is more carefully formulated as Corollary 4.6. Such results are typical for relatively hyperbolic groups. For example if a group G is relatively hyperbolic to groups of finite asymptotic dimension, then G is of finite asymptotic dimension [Osi05].

For groups that are relatively hyperbolic to groups that satisfy the Farrell–Jones conjecture and are in addition residually finite Antolín *et al.* [ACG15] give an alternative proof of the above corollary using Dehn fillings.

A regularity condition

Let X be a G-space and $\mathcal{F}$ be a family of subgroups of G. An open subset $U \subseteq X$ is said to be an $\mathcal{F}$-subset if there is $F \in \mathcal{F}$ such that $gU = U$ for $g \in F$ and $gU \cap U = \emptyset$ if $g \notin F$. A cover $\mathcal{U}$ of open subsets of X is said to be G-invariant if $gU \in \mathcal{U}$ for all $g \in G$, $U \in \mathcal{U}$. A G-invariant cover $\mathcal{U}$ of X is said to be an $\mathcal{F}$-cover if the members of $\mathcal{U}$ are all $\mathcal{F}$-subsets. The order of a collection $\mathcal{U}$ of subsets of X is less than or equal to N if each $x \in X$ is contained in at most $N+1$-members of $\mathcal{U}$. If $\mathcal{U}$ is a cover, then we will often call the order of $\mathcal{U}$ the dimension of $\mathcal{U}$.

Definition 0.1. Let G be a group and $\mathcal{F}$ be a family of subgroups. An action of G on a space X is said to be N-$\mathcal{F}$-amenable if for any finite subset S of G there exists an open $\mathcal{F}$-cover $\mathcal{U}$ of $G \times X$ (equipped with the diagonal G-action) with the following properties:

(a) the dimension of $\mathcal{U}$ is at most N;
(b) for all $x \in X$ there is $U \in \mathcal{U}$ with $S \times \{x\} \subseteq U$.

An action that is N-$\mathcal{F}$-amenable for some N is said to be finitely $\mathcal{F}$-amenable.

In [BLR08b] such cover were called *wide*.

Remark 0.2. Suppose that the action $G \curvearrowright X$ is N-$\mathcal{F}$-amenable. Then all finitely generated subgroups H of G that fix a point $x \in X$ belong to $\mathcal{F}$. Indeed, if S is symmetric, contains e, generates H and satisfies $S \times \{x\} \subseteq U$ then $(e, x) \in sU$ for all $s \in S$ and therefore $U \cap sU \neq \emptyset$ for all $s \in S$.

Remark 0.3. Suppose that the action $G \curvearrowright X$ is such that there exists an $\mathcal{F}$-cover $\mathcal{V}$ for X of dimension N. Then the action is N-$\mathcal{F}$-amenable. Indeed, if we set $\mathcal{U} := \{G \times V \mid V \in \mathcal{V}\}$ then $\mathcal{U}$ is an $\mathcal{F}$-cover for $G \times X$ of dimension N and $G \times \{x\} \subseteq U = G \times V \in \mathcal{V}$ whenever $x \in V \in \mathcal{V}$. Such a cover $\mathcal{V}$ exists for example for any cellular action on an N-dimensional CW-complex whenever all isotropy groups of the action belong to $\mathcal{F}$.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Remark 0.4. Suppose that X is compact and metrizable. Then given an N-$\mathcal{F}$-amenable action of a countable group G on X there is a sequence of G-equivariant maps $f_n\colon G \times X \to K_n$ with the following properties. The space K_n is an N-dimensional simplicial complex with a simplicial G-action. The isotropy groups for this action belong to $\mathcal{F}$. The maps f_n are contracting in the G-direction: for any $g \in G$ we have $\sup_{x\in X} \|f_n(e,x) - f_n(g,x)\|_1 \to 0$ as $n \to \infty$. The maps f_n can be constructed using a partition of unity subordinated to the covers $\mathcal{U}$ appearing in the definition of N-$\mathcal{F}$-amenability, compare [BLR08b, §§ 4 and 5]. The maps $f_n(e,-)\colon X \to K_n$ are then almost G-equivariant in the following sense: for any $g \in G$ we have $\sup_{x\in X} \|gf_n(e,x) - f_n(e,gx)\|_1 \to 0$ as $n \to \infty$. The action of G on X is amenable [Oza06a] if there is such an almost equivariant sequence of maps to the space of probability measures on G. The isotropy groups for the action on the space of probability measures are the finite subgroups of G. Thus N-$\mathcal{F}$-amenability can be thought of as both stronger and weaker than amenability: stronger since a finite-dimensional target is required; weaker since the target may have larger isotropy groups.

Ozawa [Oza06b] investigated amenable actions for relatively hyperbolic groups. In particular, his results imply that if G is relatively hyperbolic to amenable groups, then the action of G on the boundary is amenable.

Flow spaces

If G is the fundamental group of a negatively curved manifold M, then the geodesic flow is a flow on the unit sphere bundle SM of the tangent bundle of M. We will say that SM together with the geodesic flow is the geodesic flow space for G. Let G now be a hyperbolic group. Mineyev [Min05] constructed an analog of the geodesic flow space and this flow space FS and its dynamic properties are key ingredients to the proof of finite VCyc-amenability for the action of G on its boundary in [BLR08a]. The proof has naturally two parts. In the first part the so-called *long and thin covers* of FS are constructed. In the second part the dynamic of the flow is used to construct maps $G \times \partial G \to FS$ under which the long and thin covers of FS pull back to the necessary covers of $G \times \partial G$.

The key property of the long and thin covers $\mathcal{U}_\alpha$ is that they are long in the direction of the flow: for each $x \in FS$ there is $U \in \mathcal{U}_\alpha$ such that x stays in U for time $t \in [-\alpha, \alpha]$. Typically the members of $\mathcal{U}$ are very thin transverse to the flow. Thus the name long and thin covers. These covers are a variation of the long and thin cell structures appearing in [FJ86]. The construction of these long and thin covers in [BLR08a] is quite involved but works for very general flow spaces. Moreover, assumptions on the order of finite subgroups of G and the structure of periodic orbits were later shown to be not necessary by Mole and Rüping [MR13] and by Kasprowski and Rüping [KR15]. Sauer [Sau09] used packing methods to prove in a measure theoretic context results that are similar to long and thin covers. Later he pointed out that such packing methods should also be applicable to the construction of long and thin covers of flow spaces. This led to a much simpler construction for the long and thin covers from [BLR08a] by Kasprowski and Rüping [KR15].

Coarse flow spaces

An observation of the present paper is that the construction of long and thin covers using the packing method works in a more general context than flow spaces. This can be used to give an alternative argument for the finitely VCyc-amenability of the actions of hyperbolic groups on their boundaries that avoids Mineyev's flow space. Moreover, this alternative argument

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

generalizes to relatively hyperbolic groups.[1] In this and the next subsection we outline this argument for hyperbolic groups. The case of relatively hyperbolic groups is treated in detail in the main text of this paper.

Let G be a hyperbolic group. We will replace Mineyev's flow space with a more easily defined *coarse flow space*. Let Γ be a Cayley graph for G. Assume that Γ is δ-hyperbolic. Let $\overline{G} := G \cup \partial G$ and $Z := \overline{G}^2$.

Definition 0.5. The coarse flow space CF for G is the subspace of $G \times Z$ consisting of triples (g, ξ_-, ξ_+) such that there exists a geodesic in Γ from ξ_- to ξ_+ that passes g within distance δ.

There is no actual flow on this coarse flow space, but there are natural analoga of the orbits of the flow on FS. These analoga are the subsets $G_{\xi_-,\xi_+} := \{g \in G \mid (g, \xi_-, \xi_+) \in CF\} \subseteq G$. Since Γ is hyperbolic each G_{ξ_-,ξ_+} is contained in a uniformly bounded neighborhood of a geodesic. Consequently, the G_{ξ_-,ξ_+} satisfy a uniform doubling property: there is D such that for any R and any subset S of a $2R$-ball in G_{ξ_-,ξ_+} the following holds: if S is R-separated, i.e. $d_G(s, s') \geqslant R$ for all $s \neq s' \in S$, then the cardinality of S is at most D. This observation is the main ingredient for the following version of long and thin covers for CF. Let d_G be a left-invariant word metric on G.

Proposition 0.6. *There is N such that for any $\alpha > 0$ there exists an* VCyc*-cover $\mathcal{W}$ of CF such that the following holds:*

(a) *for any $(g, \xi_-, \xi_+) \in CF$ there is $W \in \mathcal{W}$ such that $B_\alpha(g) \times \{(\xi_-, \xi_+)\} \cap CF \subseteq W$;*
(b) *the dimension of $\mathcal{W}$ is at most N.*

In Theorem 1.1 we prove a version of this result in a more general situation that will also be applicable to the coarse flow spaces for relatively hyperbolic groups introduced in Definition 3.4. An important assumption is again a uniform doubling property. As an application of Theorem 1.1 we obtain a version of Proposition 0.6 for relatively hyperbolic groups in Proposition 3.7. In a different direction a corollary to Theorem 1.1 is that all actions of finitely generated virtually nilpotent groups on finite dimensional normal separable spaces with isotropy in $\mathcal{F}$ are finitely $\mathcal{F}$-amenable, see Corollary 1.10. This generalizes a result for free actions of nilpotent groups by Szabó *et al.* [SWZ14].

Sketch of proof of Proposition 0.6 if $\dim \partial G = 0$. Since $\dim \overline{G}^2 = 0$ there is a basis of the topology of $\overline{G}^2$ consisting of sets that are open and closed. Choose a covering of CF by sets of the form $\{g_i\} \times V_i \cap CF$, $i \in \mathbb{N}$ where V_i is open and closed in $\overline{G}^2$. Then define inductively U_i by $U_0 := V_0$ and

$$U_i := V_i \Big\backslash \bigcup_j U_j$$

where the union is over all j with $j < i$ and $d_G(g_i, g_j) \leqslant \alpha$. Since the V_i are closed and open, the U_i are still open. The open sets $W_i := B_{2\alpha}(g_i) \times U_i \cap CF$ form then the desired cover $\mathcal{W}$ of CF. This cover is α-long in the direction of G, more or less by construction. To compute the dimension of $\mathcal{W}$ one checks that if $W_{i_1} \cap \cdots \cap W_{i_N} \neq \emptyset$, then the g_i form an α-separated set in a ball of radius 2α in one of the G_{ξ_-,ξ_+} and therefore $\dim \mathcal{W} \leqslant D - 1$.

[1] It is plausible that the argument from [BLR08a] can also be extended to relatively hyperbolic groups. A step in this direction is [Mol13].

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

In this sketch we ignored the action of G on CF. To amend this one has to choose the V_i sufficiently small in order to avoid intersections $gW_i \cap W_i$ for to many g. Moreover, in the definition of U_i and W_i the group action has to be taken into account. In order to extend the argument to the case $\dim \partial G > 0$ an induction over subspaces of CF of lower dimension can be used. □

Pulling back long and thin covers from CF to $G \times \partial G$

Let CF be the coarse flow space for the hyperbolic group G. For $W \subseteq CF$ and $\tau > 0$ we define $\iota^{-\tau} W \subseteq G \times \partial G$ to consist of all pairs (g, ξ) with the following property. If $v \in G$ belongs to a geodesic from g to ξ and satisfies $d_G(g, v) = \tau$, then $(v, g, \xi) \in W$.

One way to think about $\iota^{-\tau} W$ is as follows. First define $\iota \colon G \times \partial G \to CF$ by $\iota(g, \xi) = (g, g, \xi)$. Next apply a partially defined multi-valued geodesic flow ϕ_τ on CF: this flow takes (g, g, ξ) to the set of all (g, v, ξ) where $v \in G$ belongs to a geodesic between g and ξ and is of distance τ from g. Then $\iota^{-\tau} W$ is the pull-back of W under the composition $\phi_\tau \circ \iota$.

This construction allows us to use the long thin covers of CF from Proposition 0.6 to prove that the action of G on ∂G is finitely VCyc-amenable: if $\mathcal{W}$ is a VCyc-cover of CF then the same holds for $\iota^{-\tau} \mathcal{W} := \{\iota^{-\tau} W \mid W \in \mathcal{W}\}$. By construction $\dim \iota^{-\tau} \mathcal{W} \leqslant \dim \mathcal{W}$. Finally, if $\mathcal{W}$ is a long cover of CF (as in Proposition 0.6(a)) then for sufficiently large τ the cover $\iota^{-\tau} \mathcal{W}$ of $G \times \partial G$ is wide in the G-direction (as in Definition 0.1(b)). This last fact can be thought of as a consequence of dynamic properties of ϕ_τ and uses the hyperbolicity of Γ.

For relatively hyperbolic groups this step is carried out in detail in § 3. The main additional difficulty appearing is discussed in the next subsection.

Relatively hyperbolic groups

The precise definitions for relatively hyperbolic groups that we use will be given in § 2 and mostly follows Bowditch [Bow12]. Let G be relatively hyperbolic to the peripheral subgroups $P_1, \dots, P_n$. By definition G acts on a hyperbolic graph Γ. Unlike the Cayley graph for hyperbolic groups Γ will contain vertices of infinite valency and the isotropy groups of these vertices will be conjugated to the P_i. We write V for the set of vertices of Γ and V_∞ for the set of vertices of Γ of infinite valency. The boundary of G is defined by Bowditch as the union of $\Delta := \partial \Gamma \cup V_\infty$; this is a compact space.

The key additional property of the graph Γ used here is fineness, as introduced by Bowditch [Bow12]. Under a mild additional assumption this property can be used to define a proper metric on the set of edges of Γ, see [MY]. In particular, it is possible to measure angles in Γ. Here an angle is a pair of edges that share a vertex. In order to allow for peripheral subgroups that are not necessarily finitely generated it is better to avoid the additional assumption. To this end we take a slightly different point of view and consider G-invariant G-cofinite subsets Θ of the set of all angles. Such a subset will be called a *size for angles*. Each size for angles Θ is then locally finite in the following sense: for each edge e there are only finitely many edges e' such that $(e, e') \in \Theta$. Fineness of Γ implies that the set of all angles appearing in any non-degenerate geodesic triangle is such a size for angles.

For any size for angles Θ we define a coarse flow space $CF(\Theta)$. Its definition is similar to the hyperbolic case in Definition 0.5, where we replace G with vertices of finite valency and only use geodesics along which all angles are Θ-small. The argument outlined in the hyperbolic case above can then be used to produce wide covers of a certain subspace of $G \times \Delta$, see Proposition 3.2. In order to prove that the action of G on Δ is finitely $\mathcal{P}$-amenable, we need to extend these wide covers to all of $G \times \Delta$. This is done by an explicit construction in Proposition 3.17.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

1. Long thin covers of subspaces of $V \times Z$

Throughout this section we fix:

- a group G;
- a family $\mathcal{F}$ of subgroups of G;
- a discrete countable proper metric space V with a proper isometric G-action, where the metric of V will be denoted by d_V (we allow distances for d_V to be ∞);
- a separable metrizable space Z with an action of G by homeomorphisms;
- a closed G-invariant subspace X of $V \times Z$ (we will always use the diagonal action of G on $V \times Z$).

An example for X is the coarse flow space for hyperbolic groups from Definition 0.5. Let $\alpha > 0$. We write $B_\alpha(v) := \{w \in V \mid d_V(v,w) \leqslant \alpha\}$ for the closed α-ball around v. A subset $S \subseteq V$ is said to be *α-separated* if $d_V(s,s') > \alpha$ for any two distinct elements of S. We will say that a subset V_0 of V has the *(D,R)-doubling property* if for any $\alpha \geqslant R$ the following holds: if $S \subseteq V_0$ is α-separated and contained in a ball of radius 2α, then the cardinality of S is at most D.

For $v \in V$ we set $Z_v := \{z \in Z \mid (v,z) \in X\}$. This is a closed subset of Z. For $z \in Z$ we set $V_z := \{v \in V \mid (v,z) \in X\} \subseteq V$. Then

$$X = \bigcup_{v \in V} \{v\} \times Z_v = \bigcup_{z \in Z} V_z \times \{z\}.$$

The following is the most abstract result about long thin covers in this paper.

Theorem 1.1. *Assume that X satisfies the following assumptions:*

(A) *X is finite dimensional;*
(B) *there are $D > 0$, $R \geqslant 0$ such that for all $z \in Z$ the subspace V_z of V has the (D,R)-doubling property;*
(C) *for each $(v,z) \in X$ the isotropy group $G_z := \{g \in G \mid gz = z\}$ belongs to $\mathcal{F}$.*

Then X admits long thin covers as follows: there is a number N depending only on the dimension of X and the doubling constant D, such that for any $\alpha > 0$ there is an $\mathcal{F}$-cover $\mathcal{W}$ of X such that:

(a) *$\dim \mathcal{W} \leqslant N$;*
(b) *for any $(v,z) \in X$ there is $W \in \mathcal{W}$ such that $B_\alpha(v) \times \{z\} \cap X \subseteq W$.*

The proof of Theorem 1.1 will proceed by induction on the dimension of subspaces of X. The following proposition is the induction step. Since we can take $Y = X$ to start the induction it implies Theorem 1.1.

Proposition 1.2. *Retain the assumptions of Theorem 1.1. Let $Y \subseteq X$ be a non-empty G-invariant closed subspace. Assume that $\dim Y = n$. For any $\alpha > 0$ there is a G-invariant collection $\mathcal{W}$ of $\mathcal{F}$-subsets of X and a G-invariant closed subspace $Y' \subseteq Y$ such that:*

(a) *$\dim Y' < \dim Y$;*
(b) *the order of $\mathcal{W}$ is at most $D-1$;*
(c) *for any $(v,z) \in Y \backslash Y'$ there is $W \in \mathcal{W}$ such that $B_\alpha(v) \times \{z\} \cap X \subseteq W$.*

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Lemma 1.3. *Retain the assumptions of Proposition 1.2. Let $(v,z) \in Y$, $\alpha > 0$ and U_0 be a neighborhood of z in Z. Then U_0 contains a smaller open neighborhood U of z in Z such that:*

(a) $\{g \in G \mid d_V(gv,v) \leqslant \alpha, U \cap gU \neq \emptyset\} \subseteq G_z = \{g \in G \mid gz = z\}$;

(b) *for all $w \in V$ we have $\dim Y_w \cap \partial U < \dim Y$, where $Y_w := \{z \in Z_w \mid (w,z) \in Y\}$ and ∂U denotes the boundary of U in Z.*

Proof. We have $Y = \bigcup_{w \in V} \{w\} \times Y_w$. Since $\dim Y = n$ and V is discrete each Y_w satisfies $\dim Y_w \leqslant n$. Since Y is closed, $Y_w \subseteq Z$ is closed for each $w \in V$. The countable sum theorem in dimension theory [Pea75, Theorem 2.5, p. 125] asserts that in a normal space the dimension of the countable union of closed subspaces is the supremum of the dimension of the subspace. Since V is countable this implies that the dimension of $Y_V := \bigcup_{w \in V} Y_w$ is at most n.

Since every subspace Z' of Z is separable and metrizable, its (covering) dimension equals its (small) inductive dimension: $\dim Z' = \operatorname{ind} Z'$ [Pea75, Corollary 5.10, p. 184]. This means that for any $z \in Z'$ there are arbitrary small neighborhoods whose boundary in Z' has dimension less than $\dim Z'$.

Since d_V is proper and the action of G on V is proper, $\{g \mid d_V(gv,v) \leqslant \alpha\}$ is finite. Since Z is Hausdorff there is an open neighborhood U' of z in Z that is contained in U_0 and satisfies $gU' \cap U' = \emptyset$ for all $g \in G \backslash G_z$ with $d_V(gv,v) \leqslant \alpha$. Applying the facts from the preceding paragraph to Y_V, we obtain an open neighborhood U'' of z in Y_V that is contained in U' and satisfies $\dim \partial^{Y_V} U'' < \dim Y$, where ∂^{Y_V} denotes the boundary in Y_V. Using Lemma B.1 we can extend U'' to an open subset U of Z that is contained in U' and satisfies $U \cap Y_V = U''$, and $\partial U \cap Y_V = \partial^{Y_V} U''$, where ∂ denotes the boundary in Z.

Let $w \in V$. Since Y_w is closed in Y_V we have $\dim Y_w \cap \partial U \leqslant \dim Y_V \cap \partial U = \dim \partial^{Y_V} U'' < \dim Y$. Thus U satisfies part (b). Since $U \subseteq U'$, U also satisfies part (a). □

Lemma 1.4. *Retain the assumptions of Theorem 1.1. Let $\alpha \geqslant R$ be given. Then there exists an open G-invariant neighborhood X^{th} of X in $V \times Z$ such that for all $z \in Z$, the subspace $V_z^{th} := \{v \in V \mid (v,z) \in X^{th}\}$ of V has the following property: if $S \subseteq V_z^{th}$ is α-separated and contained in a ball of radius 2α, then the cardinality of S is at most D.*

Proof. Since the action of G on V is proper, we can pick metrics d_v, $v \in V$ on Z that are compatible with the G-action, i.e. such that we have $d_v(z,z') = d_{gv}(gz,gz')$ for all $v \in V$, $g \in G$, $z, z' \in Z$.

Let $(v,z) \in X$ (i.e. $z \in Z_v$). For $w \in V$ with $z \notin Z_w$, let $\varepsilon(v,w,z) := d_v(z, Z_w)/3$. The Z_w are closed since X is closed. Thus $\varepsilon(v,w,z) > 0$. Let $\varepsilon(v,z) := \min \varepsilon(v,w,z)$, where the minimum is taken over all $w \in V$ with $d_V(v,w) \leqslant 4\alpha$ and $z \notin Z_w$. Since d_V is proper the minimum exists and is positive. Let $B(v,w,z)$ be the open ball of radius $\varepsilon(v,z)$ around z with respect to the metric d_w. Set $B(v,z) := \bigcap_{d_V(v,w) \leqslant 4\alpha} B(v,w,z)$. Since d_V is proper $B(v,z)$ is open. Moreover, by construction,

(1.5) $d_v(z, Z_w) > 2\varepsilon(v,z)$ whenever $z \in Z_v \backslash Z_w$ and $d_V(v,w) \leqslant 4\alpha$,

(1.6) $d_v(z',z) < \varepsilon(w,z)$ whenever $z \in Z_w$, $z' \in B(w,z)$ and $d_V(v,w) \leqslant 4\alpha$.

Define $Z_v^{th} := \bigcup_{z \in Z_v} B(v,z)$ and $X^{th} := \bigcup_{v \in V} \{v\} \times Z_v^{th}$. By construction X^{th} is an open neighborhood of X in $V \times Z$. The compatibility of the metrics d_v with the G-action guarantees that X^{th} is G-invariant.

Let $S \subseteq V$ be an α-separated subset of some 2α-ball in V. Assume that the cardinality of S exceeds D. Since all V_z have the D-doubling property we have $\bigcap_{w \in S} Z_w = \emptyset$. We need to show

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

that $\bigcap_{w\in S} Z_w^{th} = \emptyset$ as well. Assume by contradiction that $z' \in \bigcap_{w\in S} Z_w^{th}$. Then we find for each $w \in S$, a point $z_w \in Z_w$ with $z' \in B(w, z_w)$. Choose $v \in S$ such that $\varepsilon(w, z_w) \leqslant \varepsilon(v, z_v)$ for all $w \in S$. Note that $d_V(w, v) \leqslant 4\alpha$ for all $w \in S$. Now (1.6) implies that for all $w \in S$

$$\begin{aligned} d_v(z_v, Z_w) \leqslant d_v(z_v, z_w) &\leqslant d_v(z_v, z') + d_v(z', z_w) \\ &\leqslant \varepsilon(v, z_v) + \varepsilon(w, z_w) \leqslant 2\varepsilon(v, z_v). \end{aligned}$$

On the other hand (1.5) implies $d_v(z_v, Z_w) > 2\varepsilon(v, z_v)$ if $z_v \notin Z_w$. Therefore $z_v \in Z_w$ for all $w \in S$. This contradicts $\bigcap_{w\in S} Z_w = \emptyset$. □

Proof of Proposition 1.2. Throughout this proof closure and boundary will always be taken with respect to Z. Let $\alpha \geqslant R$. Let X^{th} be as in Lemma 1.4. Since Y is separable we can use Lemma 1.3 and find sequences $v_i \in V$, $F_i \in \mathcal{F}$, $U_i \subseteq Z$ open, such that

(1.7) $\{g \in G \mid d_V(gv_i, v_i) \leqslant 4\alpha, U_i \cap gU_i \neq \emptyset\} \subseteq F_i$,

(1.8) for all $w \in V$ we have $\dim Y_w \cap \partial U_i < \dim Y$,

(1.9) $Y \subseteq \bigcup_i \{v_i\} \times U_i \subseteq X^{th}$.

Now we define inductively U_i' by $U_0' := U_0$ and

$$U_i' := U_i \Big\backslash \bigcup_{h,j} h\overline{U_j'}$$

where the union is over all pairs (h, j) with $h \in G$, $j < i$, and $d_V(v_i, hv_j) \leqslant \alpha$. Since d_V is proper and the action of G on V is proper, this is a finite set of pairs and U_i' is open. An easy induction shows that $\partial U_i' \subseteq \bigcup_{j\leqslant i, g\in G} g\partial U_j$. Thus with $Y'' := Y \cap V \times \bigcup_{i,g} g\partial U_i$ we have

$$Y \cap \{v\} \times g\partial U_i' \subseteq Y''$$

for all $v \in V$, $i \in \mathbb{N}$, $g \in G$. For $i \in \mathbb{N}$ let

$$W_i := B_{2\alpha}(v_i) \times U_i' \cap X$$

and set $\mathcal{W} := \{gF_iW_i \mid g \in G, i \in \mathbb{N}\}$.

Clearly, $\mathcal{W}$ consists of open subsets of X and is G-invariant. Consider $gF_iW_i \in \mathcal{W}$. If $\gamma \in gF_ig^{-1}$, then $\gamma gF_iW_i = gF_iW_i$. If $\gamma \notin gF_ig^{-1}$, then $\gamma gF_iW_i \cap gF_iW_i = \emptyset$. Indeed, if $(v, z) \in \gamma gF_iW_i \cap gF_iW_i$, then there are $a, b \in F_i$ with $d_V(v, \gamma gav_i), d_V(v, gbv_i) \leqslant 2\alpha$ and $z \in \gamma gaU_i \cap gbU_i$. Then (1.7) implies $b^{-1}g^{-1}\gamma ga \in F_i$ and therefore $\gamma \in gF_ig^{-1}$. Thus each gF_iW_i is an $\mathcal{F}$-subset.

We now prove $\dim Y'' < \dim Y$. For $v \in V$, $i \in \mathbb{N}$ and $g \in G$ we have $Y \cap \{v\} \times g\partial U_i = g(Y \cap \{g^{-1}v\} \times \partial U_i) \cong Y_{g^{-1}v} \cap \partial U_i$. By (1.8) all these spaces are of dimension less than $\dim Y$. Since they are all closed in Y the countable sum theorem [Pea75, Theorem 2.5, p. 125] implies that their union, Y'', is also of dimension less than $\dim Y$.

Next we prove that $\mathcal{W}$ is of order less than or equal to $D - 1$, i.e. that it satisfies part (b). Let $(v, z) \in X$. Suppose that $(v, z) \in gF_iW_i$. Then there is $a \in F_i$ such that $d_V(v, gav_i) \leqslant 2\alpha$ and $z \in gaU_i'$. As X^{th} is G-invariant, (1.9) implies $(gav_i, z) \in \{gav_i\} \times \{gaU_i'\} \subseteq X^{th}$. Thus $gav_i \in V_z^{th}$. If also $(v, z) \in hF_jW_j$ then there is $b \in F_j$ with $d_V(v, hbv_j) \leqslant 2\alpha$, $z \in hbU_j'$ and $hbv_j \in V_z^{th}$.

If $i = j$ then $gF_iW_i \cap hF_iW_i \neq \emptyset$ and since gF_iW_i is a $\mathcal{F}$-subset we then have $gF_iW_i = hF_jW_j$. If $i \neq j$, then $\gamma U_i' \cap U_j' = \emptyset$ for all $\gamma \in G$ with $d_V(\gamma v_i, v_j) \leqslant \alpha$. Since $b^{-1}h^{-1}gaU_i' \cap U_j' \neq \emptyset$, this implies $d_V(b^{-1}h^{-1}gav_i, v_j) = d_V(gav_i, hbv_j) > \alpha$.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

All together, we have shown that if (v,z) is contained in N distinct members in $\mathcal{W}$, then there is an α-separated set in $B_{2\alpha}(v)\cap V_z^{th}$ of cardinality N. Lemma 1.4 implies now that N is bounded by the doubling constant D appearing in the assumptions of Theorem 1.1. This implies that the order of $\mathcal{W}$ is at most $D-1$.

Next we claim that for any $(v,z)\in Y\backslash Y''$ there is $W\in\mathcal{W}$ with $B_\alpha(v)\times\{z\}\cap X\subseteq W$. Let $(v,z)\in Y\backslash Y''$. By (1.9) there is $i\in\mathbb{N}$ with $(v,z)\in\{v_i\}\times U_i$. So $v=v_i$. If $z\in U_i'$, then

$$B_\alpha(v)\times\{z\}\cap X\subseteq B_{2\alpha}(v_i)\times U_i'\cap X\subseteq F_iW_i.$$

If $z\notin U_i'$, then $z\in h\overline{U}_j'$ for some $j<i$ and $h\in G$ with $d_V(v_i,hv_j)\leqslant\alpha$. We observed earlier $Y\cap\{v\}\times h\partial U_j'\subseteq Y''$. Thus we have $z\in hU_j'$. Since $d_V(v,hv_j)=d_V(v_i,hv_j)\leqslant\alpha$ we have then

$$B_\alpha(v)\times\{z\}\cap X\subseteq B_{2\alpha}(hv_j)\times hU_j'\cap X\subseteq hF_jW_j.$$

This proves our claim.

Now observe that if $B_\alpha(v)\times\{z\}\cap X\subseteq W$, then, since W is open, $B_\alpha(v)\times\{z'\}\cap X\subseteq W$ for all z' in a neighborhood of z. Since $\mathcal{W}$ is G-invariant, this implies that there is an open G-invariant neighborhood N of $Y\backslash Y''$ such that all $(v,z)\in N$ satisfy part (c).

Now $Y':=Y\backslash N$ is G-invariant and closed. Moreover, since $Y'\subseteq Y''$, we have $\dim Y'\leqslant\dim Y''<\dim Y$ and part (a) holds. □

Corollary 1.10. *Let G be a finitely generated virtually nilpotent group. Then any action of G on a finite-dimensional separable metrizable space Z is finitely $\mathcal{F}$-amenable where $\mathcal{F}$ consists of all subgroups of G that fix a point in Z.*

Proof. This follows from Theorem 1.1 where we take $V=G$ with a word metric and $X=G\times Z$. Assumption (A) of Theorem 1.1 is satisfied since Z is finite dimensional. Finitely generated virtually nilpotent groups are of polynomial growth [Bas72]. This implies that G has a doubling property. Thus assumption (B) of Theorem 1.1 is satisfied. Assumption (C) of Theorem 1.1 holds by choice of $\mathcal{F}$. □

2. Relative hyperbolic groups

Throughout this section Γ will be a fine and hyperbolic graph in the sense of Bowditch [Bow12] and G will be a countable group equipped with a cocompact simplicial action on Γ. In particular, Γ is uniformly fine: for any α there is N_α such that for any edge e there are at most N_α circuits of length less than or equal to α containing e. (A circuit is an embedded loop in Γ.) The isotropy groups of edges for the action of G on Γ are assumed to be finite. Let $P_1,\dots,P_n$ be representatives of the conjugacy classes of the isotropy groups of the vertices of Γ of infinite valency. We will then say that G is *relatively hyperbolic to* $P_1,\dots,P_n$.[2]

We denote by Δ the union of the Gromov boundary $\partial\Gamma$ of Γ with the set V_∞ of vertices of Γ with infinite valency. By V we denote the set of all vertices of Γ and by E the set of all edges of Γ. We write $\Delta_+:=\partial\Gamma\cup V$. In other words, Δ_+ is the union of Δ with the set of all vertices of Γ of finite valency. We think of edges as subsets of V with two elements; in particular, edges are not oriented.

[2] Bowditch [Bow12, Definition 2] assumes in addition that $P_1,\dots,P_n$ are finitely generated, but this restriction will not be necessary here.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

We will use the term geodesic for finite geodesics, geodesic rays and bi-infinite geodesics. If c is a geodesic and $\xi, \xi' \in \Delta_+$ are both contained in c or endpoints of c then we write $c|_{[\xi,\xi']}$ for the restriction of c to a geodesic between ξ and ξ'.

Hyperbolicity of a graph is usually formulated in terms of geodesic triangles whose sides are finite geodesics. But it then follows that all geodesic triangles, including geodesic triangles with one or more corners at the boundary are uniformly slim [GM08, Lemma 2.11].[3] We can therefore fix a constant $\delta > 0$ such that in all geodesic triangles each side is contained in the union of the δ-neighborhood of the other two. It will be convenient to assume that δ is an integer. We will refer to δ as a hyperbolicity constant for Γ.

Angles

An unordered pair (e, e') of edges in Γ that have a vertex v in common is called an *angle* at v. If $e = e'$, then we say that the angle (e, e') is trivial. The group G acts on the set of angles; this action will usually be not cofinite. A G-invariant, G-cofinite subset Θ of the set of all angles, that contains all trivial angles will be called a *size for angles*. Members of Θ will be called Θ-*small*. Angles that are not contained in Θ will be called Θ-*large*. If c is a geodesic in Γ then c determines a non-trivial angle at every internal vertex v of c; this angle will be called the angle of c at v and sometimes be denoted by $\sphericalangle_v c$. If all these angles are Θ-small, then we will say that c is Θ-small. In particular, geodesics of length 1 are always Θ-small, as they contain no internal vertices. If Θ and Θ' are two sizes for angles then we define $\Theta + \Theta'$ to consist of all angles (e, e'') for which there is an edge e' with $(e, e') \in \Theta$ and $(e', e'') \in \Theta'$. As a consequence of the following Lemma 2.1, $\Theta + \Theta'$ is again a size for angles.

Lemma 2.1. *If Θ is a size for angles then each edge e is contained in only finitely many angles of Θ, i.e. $\Theta_e := \{e' \in E \mid (e, e') \in \Theta\}$ is finite. Conversely, a G-invariant set of angles Θ with the property that Θ_e is finite for any edge e is a size for angles.*

Proof. If the action of G on Θ is cofinite, the same holds for the action of the isotropy group G_e of e on Θ_e. Since G_e is finite, so then is Θ_e.

Conversely, since the action of G on E is cofinite, there are finitely many edges $e_1, \dots, e_n$ such that each orbit for the action of G on the set of all angles contains an angle of the form (e, e_i). Thus, if Θ_{e_i} is finite for $i = 1, \dots, n$, then the action of G on Θ is cofinite. □

Corollary 2.2. *Let Θ be a size for angles and $\alpha > 0$. Let v be a vertex and e be an edge incident to v. Consider the set V' of all vertices v' for which there exists a Θ-small geodesic from v to v' of length at most α whose initial edge e' satisfies $(e, e') \in \Theta$. Then the cardinality of V' is bounded by a number depending only on Θ and α.*

Proof. This is an immediate consequence of the first statement in Lemma 2.1 and the cofiniteness of the action of G on the set of edges. □

Definition 2.3. We define $\Theta^{(3)}$ as the set of all angles (e, e') such that there exists a geodesic triangle (possibly with some corners in $\partial\Gamma$) with sides c, c' and c'' such that c and c' determine the angle (e, e') at the corner $v \in V$, and such that c'' does not meet v.

Lemma 2.4. *The set $\Theta^{(3)}$ is a size for angles.*

[3] In this reference the proof is left as an exercise; for completeness a solution to this exercise is included in Appendix C.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Proof. Clearly, G acts on $\Theta^{(3)}$, and $\Theta^{(3)}$ contains all trivial angles. To show that $\Theta^{(3)}$ is cofinite it suffices to show, by Lemma 2.1, that for all e there are only finitely many e' with $(e,e') \in \Theta^{(3)}$. We will use hyperbolicity of Γ to show for each e' with $(e,e') \in \Theta^{(3)}$ there is a circuit of uniformly bounded length in Γ that contains both e and e'. Since Γ is fine there are only finitely many such circuits.

To construct the circuit let c, c' and c'' be sides of a geodesic triangle such that c and c' determine the angle (e,e') at the corner $v \in V$, and such that c'' does not meet v. Now pick $w \in c$ as follows. Recall that we picked the hyperbolicity constant δ to be an integer. If the length of c is greater than or equal to 3δ, then let $w \in c$ with $d_\Gamma(v,w) = 3\delta$. Otherwise, let w be the endpoint of c (not v). Similarly, pick $w' \in c'$. Hyperbolicity implies that there is a path of length less than or equal to 10δ between w and w' that misses v. Indeed, if $d_\Gamma(w,c') \leqslant \delta$ then we can connect w first to c' and then to w' along c'. If $d_\Gamma(w',c) \leqslant \delta$ then we can connect w' first to c and then to w along c. Otherwise, by hyperbolicity, we have $d_\Gamma(w,c'') \leqslant \delta$ and $d_\Gamma(w',c'') \leqslant \delta$, and we can first connect w and w' to c'', and then connect along c''. Altogether we have constructed a loop of length at most 16δ that meets v exactly once in the angle (e,e'). The loop may not be embedded, but we can shorten it to produce a circuit containing the angle (e,e'). □

Remark 2.5. By cutting geodesic n-gons in Γ into geodesic triangles one obtains a version of Lemma 2.4 for geodesic n-gons.

For $n \geqslant 3$ let $(n-2)\Theta^{(3)}$ be the $n-2$-fold sum of $\Theta^{(3)}$. Suppose that the geodesics $c_1, \ldots, c_n$ are the sides of an n-gon in Γ (possibly with some corners in $\partial\Gamma$). Let (e,e') be the angle in Γ determined by the corner $v \in V$ of the n-gon between c_1 and c_2. If $v \notin c_3 \cup \cdots \cup c_n$ then $(e,e') \in (n-2)\Theta^{(3)}$.

Similarly, if v is an internal vertex of c_i that does not belong to any of the c_j, $j \neq i$, then $\measuredangle_v c_i \in (n-1)\Theta^{(3)}$.

Lemma 2.6. *Let c and c' be two geodesics from $v \in V$ to $\xi \in \Delta_+$, $\xi \neq v$. Let e and e' be the initial edges of c and c'. Then (e,e') is $\Theta^{(3)}$-small.*

Proof. If $e \neq e'$ then we can subdivide c or c' and obtain a geodesic triangle for which (e,e') will be the angle at v and for which v does not belong to all three sides. Thus $(e,e') \in \Theta^{(3)}$. □

Lemma 2.7. *Let c and c' be two geodesics between ξ_- and $\xi_+ \in \Delta_+$. Let $v \in c$ and $v' \in c'$ be such that no geodesic between v and v' is contained in $c \cup c'$. Then there exists a $2\Theta^{(3)}$-small geodesic $\hat{c}$ between v and v'.*

Proof. Choose a geodesic $\hat{c}$ from v to v', that first travels along c, then along a geodesic c_1 that meets c and c' only in its endpoints and finally along c'. We set $c_0 := c \cap \hat{c}$ and $c_2 := c' \cap \hat{c}$. Then c_0 is disjoint from c', and c_2 is disjoint from c, since otherwise there is a geodesic from v to v' that is contained in $c \cup c'$. Now c_1 splits the bi-gone formed by c and c' in two geodesic triangles. For any internal vertex w of $\hat{c}$ we can use one of these two triangles and apply Remark 2.5 to conclude that $\measuredangle_w \hat{c}$ is $2\Theta^{(3)}$-small. □

Existence of geodesics

The following (presumably well-known) fact is implicitly used in [Bow12], but not explicitly stated.

Lemma 2.8. *For $\xi \neq \xi' \in \Delta_+$, there exists a geodesic between ξ and ξ'.*

Proof. We will use angles to extend the proofs in the locally finite case from [BH99, Lemmas 3.1, 3.2, p. 428].

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

First, we consider the case $\xi' = v \in V$. If also $\xi \in V$, then the existence of a geodesic between ξ and v is obvious. Let c be a geodesic ray from some vertex w to ξ. For $n \in \mathbb{N}$ pick finite geodesics c_n from v to $c(n)$. It will be convenient to assume that $c_n \cap c$ is a finite geodesic, i.e. c_n does not leave c after meeting c. We claim, that, as in the locally finite case, a subsequence of c_n converges pointwise to a geodesic ray c'. This ray is then a ray from v to ξ. Let e_n be the initial edge of c_n. Then we can use the geodesic triangle whose sides are c_n, c_m and $c_{[c(n),c(m)]}$ to deduce that (e_n, e_m) is $\Theta^{(3)}$-small. It follows that the e_n range only over a finite set and we can pick a subsequence $I \subseteq \mathbb{N}$ with $e = e_n$, $n \in I$ constant. Inductively we can now produce a subsequence of the c_n that converges pointwise as claimed.

Next, we consider the case $\xi, \xi' \in \partial\Gamma$. Let c and c' be geodesic rays to ξ and ξ'. By the first case we can assume $c(0) = c'(0)$. We can also assume $c \cap c' = c(0)$. Let c_n be a finite geodesic from $c(n)$ to $c'(n)$. We claim that, as in the locally finite case, there is a vertex p that belongs to infinitely many of the c_n. Since $\xi \neq \xi'$ there is $n_0 \in \mathbb{N}$ with $d_\Gamma(c(n_0), c') > \delta$. Hyperbolicity implies then $d_\Gamma(c(n_0), c_n) \leqslant \delta$ for $n \geqslant n_0$. We can therefore for $n \geqslant n_0$ pick geodesics $\tilde{c}_n$ starting in $c(n_0)$ and ending in $w_n \in c_n$ of length less than or equal to δ. We can assume $\tilde{c}_n \cap c_n = w_n$. Let w'_n be the last vertex on $\tilde{c}_n$ that is also on c. Since $d_\Gamma(w'_n, c(n_0)) \leqslant \delta$ and $w'_n \in c$ there is a subsequence $I \subseteq \mathbb{N}$ with $w' = w'_n$, $n \in I$ constant. Consider for $n \in I$ now the geodesic triangles whose sides are $c|_{[w',c(n)]}$, $c_n|_{[c(n),w_n]}$ and $\tilde{c}_n|_{[w',w_n]}$. Using Remark 2.5 it follows that $\tilde{c}_n|_{[w',w_n]}$ is $2\Theta^{(3)}$-small. It also follows that the initial edge e_n of $\tilde{c}_n|_{[w',w_n]}$ forms a $\Theta^{(3)}$-small angle with an edge e of c incident to w'. Now Corollary 2.2 implies that the w_n range only over a finite set. Thus after passing to a further subsequence we can assume that there is a vertex p contained in all c_n, $n \in I$. Now the argument used in the first case allows us to pass to a further subsequence and to assume that $c_n|_{[p,c(n)]}$, $n \in I$ converges pointwise to a geodesic ray from p to ξ and that $c_n|_{[p,c'(n)]}$, $n \in I$ converges pointwise to a geodesic ray from p to ξ'. These two rays now combine to a bi-infinite geodesic between ξ and ξ'. □

The observer topology

Bowditch [Bow12, § 8] defined a topology on Δ; this topology is sometimes called the observer topology. This topology naturally is also defined on $\Delta_+ = \partial\Gamma \cup V$. We recall a basis for the observer topology. For $\xi \in \Delta_+$ and a finite subset $V_0 \subseteq V$ let $M(\xi, V_0)$ consist of all $\xi' \in \Delta_+$ for which all geodesics between ξ and ξ' miss $V_0 \backslash \{\xi\}$. The $M(\xi, V_0)$ form an open basis for the observer topology. An important fact [Bow12, p. 51] is that the $\forall$ in the definition of $M(\xi, V_0)$ can be replaced with $\exists$ without changing the topology: the sets $M'(\xi, V_0)$, defined to consist of all ξ' for which there *exists* a geodesic between ξ and ξ' missing $V_0 \backslash \{\xi\}$, also form an open neighborhood basis for the observer topology. The $M(v, V_0)$ with $v \in V$ and $V_0 \subset V$ finite are a countable basis for the observer topology, see the discussion preceding [Bow12, Lemma 8.4]. The observer topology is compact and in particular Hausdorff [Bow12, Lemmas 8.4, 8.6]. A convenient procedure to produce convergent subsequences in Δ_+ is reviewed in Lemma 2.9. The $M(v, V_0)$ with $v \in V$ observer topology has a countable basis, As a compact space with a countable basis for the topology Δ_+ is metrizable.

Lemma 2.9. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a sequence in Δ_+. Let $v \in V$. For each n, let c_n be a geodesic from v to ξ_n. Then there exists a subsequence $I \subseteq \mathbb{N}$ such that $\xi_n \to \xi \in \Delta_+$ and moreover the c_n, $n \in I$ converge as follows:*

(a) *if $\xi \in \partial\Gamma$, then the c_n, $n \in I$ converge pointwise to a geodesic ray from v to ξ;*
(b) *if $\xi \in V$, then $\xi \in c_n$ for all $n \in I$ and each edge e incident to ξ is the initial edge for the restriction $c_n|_{[\xi,\xi_n]}$ for at most finitely many $n \in I$.*

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Proof. For each k let $c_n(k)$ be the kth vertex along c_n (starting from $v = c_n(0)$). If $d_\Gamma(v, \xi_n) < k$, then we set $c_n(k) := \xi_n$.

If $V_k := \{c_n(k) \mid n \in \mathbb{N}\}$ is finite for all k then (using a diagonal subsequence) we can pick a subsequence $I \subseteq \mathbb{N}$ such that $c(k) := c_n(k)$, $n \in I$ is eventually constant. If $d_\Gamma(v, c(k)) \to \infty$ with $k \to \infty$, $k \in \mathbb{N}$, then the c_n, $n \in I$ converge pointwise to a geodesic ray c whose endpoint ξ is also a limit for the ξ_n. In particular, part (a) holds. If $d_\Gamma(v, c(k)) \not\to \infty$, then for all sufficiently large $n \in I$, $k \in \mathbb{N}$, $\xi_n = c_n(k) = c(k) = \xi \in V$ must be constant. In this case both ξ_n and c_n, $n \in I$ are eventually constant and part (b) holds.

If not all V_k are finite, then there is k_0 such that V_k is finite for all $k < k_0$ and V_{k_0} is infinite. Then we find a subsequence $I \subseteq \mathbb{N}$ such that $c_i(k)$ is constant in i for $k < k_0$ and $c_i(k_0) \neq c_j(k_0)$ for $i \neq j \in I$. Then $\xi_i \to \xi := c_i(k_0 - 1) \in V$ for $i \in I$, $i \to \infty$ and again part (b) holds. □

Addendum 2.10. Suppose that in Lemma 2.9 in addition v is a vertex of finite valency and that the c_n are all Θ-small for some size for angles Θ. Assume also the ξ_n eventually leave every ball of finite radius in Γ. Then $\xi \in \partial\Gamma$ and, for a subsequence, the c_n converge pointwise to a geodesic ray from v to ξ.

Proof. Corollary 2.2 implies that the V_k appearing in the proof of Lemma 2.9 are all finite. Since the ξ_n leave eventually every ball of finite radius in Γ we have $d_\Gamma(v, c(k)) \to \infty$ with $k \to \infty$ in the proof of Lemma 2.9 and the result follows. □

Large angles

If the angle of a geodesic at some vertex v is large, then this often forces further geodesics to pass through v. In the next few lemmas we collect some results of this kind. Results of this form are very common in connection with relatively hyperbolic groups, see for example [MY, Lemma 3].

LEMMA 2.11. *Let c be a geodesic between ξ_- and ξ_+. Assume that v is an internal vertex of c and that $\sphericalangle_v c$ is $\Theta^{(3)}$-large. Then any other geodesic c' between ξ_- and ξ_+ will also pass through v.*

Proof. Consider the geodesic triangle whose sides are c', $c|_{[v,\xi_-]}$ and $c|_{[v,\xi_+]}$. By definition of $\Theta^{(3)}$ we have $v \in c'$. □

LEMMA 2.12. *Let ξ, ξ_1 and $\xi_2 \in \Delta_+$. Let c_1 be a geodesic between ξ and ξ_1 and c be a geodesic between ξ_1 and ξ_2. Let Θ_0 be a size for angles. Let $v \notin c$ be an internal vertex of c_1 such that the angle $\sphericalangle_v c_1$ is $\Theta_0 + 2\Theta^{(3)}$-large. Then any geodesic c_2 between ξ_2 and ξ contains v as an internal vertex and $\sphericalangle_v c_2$ is Θ_0-large.*

Proof. For the first claim, we subdivide c_1 at v and obtain a 4-gon whose corners are v, ξ, ξ_1 and ξ_2. Since $v \notin c$, we must have $v \in c_2$ for otherwise the angle of c_1 at v would need to be $2\Theta^{(3)}$-small, see Remark 2.5. Let e_1, e_1', e_2 and e_2' be edges incident to v such that e_1 points towards ξ_1 along c_1, e_1' points towards ξ along c_1, e_2 points towards ξ_2 along c_2, and e_2' points towards ξ along c_2. Thus $(e_1, e_1') = \sphericalangle_v c_1$ and $(e_2, e_2') = \sphericalangle_v c_2$. Lemma 2.6 implies that (e_1', e_2') is $\Theta^{(3)}$-small. Since $v \notin c$, we can use the geodesic triangle whose sides are c, $c_1|_{[v,\xi_1]}$ and $c_2|_{[v,\xi_2]}$ to see that (e_1, e_2) is also $\Theta^{(3)}$-small. Since (e_1, e_1') is $\Theta_0 + 2\Theta^{(3)}$-large, this implies that (e_2, e_2') is Θ_0-large. □

LEMMA 2.13. *Let Θ_0 be a size for angles. Let ξ, ξ_1 and $\xi_2 \in \Delta_+$. Let c be a geodesic between ξ_1 and ξ_2, c_1 be a geodesic between ξ_1 and ξ, c_2 be a geodesic between ξ_2 and ξ. Suppose that v is an internal vertex of c, c_1 and c_2 and that $\sphericalangle_v c$ is $2\Theta_0 + 3\Theta^{(3)}$-large. Then $\sphericalangle_v c_1$ or $\sphericalangle_v c_2$ is Θ_0-large.*

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Proof. Let $e, e', e_1, e_1', e_2, e_2'$ be edges incident to v such that e points towards ξ_1 along c, e' points towards ξ_2 along c, e_1 points towards ξ_1 along c_1, e_1' points towards ξ along c_1, e_2 points towards ξ_2 along c_2, and e_2' points towards ξ along c_2. Thus $(e, e') = \sphericalangle_v c$, $(e_1, e_1') = \sphericalangle_v c_1$, and $(e_2, e_2') = \sphericalangle_v c_2$. Lemma 2.6 implies that (e_1', e_2'), (e, e_1) and (e', e_2) are $\Theta^{(3)}$-small. By assumption (e, e') is $2\Theta_0 + 3\Theta^{(3)}$-large. Thus not both (e_1, e_1') and (e_2, e_2') can be Θ_0-small. □

Lemma 2.14. *Let Θ_0 be a size for angles. Then there exists a size X for angles such that the following holds. Assume there is a Θ_0-small geodesic c between ξ_1 and ξ_2. Let $\xi \in \Delta_+ \backslash \{\xi_1, \xi_2\}$, and c_1 be a geodesic from ξ_1 to ξ. Let $v \in V$ be an internal vertex of c_1 for which $\sphericalangle_v c_1$ is X-large and $v \neq \xi_2$. Then any geodesic c_2 between ξ_2 and ξ will also pass through v. Moreover, for any size for angles Θ we have the following:*

(a) *if $\sphericalangle_v c_1$ is Θ-small, then $\sphericalangle_v c_2$ is $\Theta + X$-small;*
(b) *if $\sphericalangle_v c_1$ is $\Theta + X$-large, then $\sphericalangle_v c_2$ is Θ-large.*

Proof. For the first statement we can take $X := \Theta_0 + 2\Theta^{(3)}$. Lemma 2.12 implies $v \in c \cup c_2$. If $v \notin c_2$ then Lemma 2.12 implies that v is an internal vertex of c and that $\sphericalangle_v c$ is Θ_0 large. But c is assumed to be Θ_0-small and thus $v \in c_2$.

For the second statement we can take $X := \Theta_0 + 3\Theta^{(3)}$. Let e_1, e_1', e_2, e_2' be the edges incident to v, such that e_1 points towards ξ_1 along c_1, e_1' points towards ξ along c_1, e_2 points towards ξ_2 along c_2, and e_2' points towards ξ along c_2. In particular $\sphericalangle_v c_1 = (e_1, e_1')$ and $\sphericalangle_v c_2 = (e_2, e_2')$. In order to prove parts (a) and (b) it will suffice to show that (e_1', e_2') is $\Theta^{(3)}$-small and that (e_1, e_2) is $\Theta_0 + 2\Theta^{(3)}$-small. The first statement follows from Lemma 2.6. For the second statement we need to distinguish the cases $v \notin c$ and $v \in c$, i.e. to distinguish whether only c_1 and c_2 intersect in v, or whether all three geodesics intersect in v. If $v \notin c$ we can use the geodesic triangle whose sides are c, $c_1|_{[v,\xi_1]}$ and $c_2|_{[v,\xi_2]}$ to deduce that (e_1, e_2) is $\Theta^{(3)}$-small. If $v \in c$, then let e and e' be the edges on c incident to v, such that e points towards ξ_1 along c, and e' points towards ξ_2 along c. Lemma 2.6 implies that the angles (e, e_1) and (e', e_2) are $\Theta^{(3)}$-small. Since c is Θ_0-small the angle (e, e') is Θ_0-small. Thus (e_1, e_2) is indeed $\Theta_0 + 2\Theta^{(3)}$-small. □

Lemma 2.15. *Let $\xi \in \Delta_+$ and $v \in V$ where $\xi \neq v$. Then there exist neighborhoods U of v and U' of ξ in Δ_+ such that any geodesic starting in U and ending in U' will meet v.*

Proof. Assume the lemma fails. Then there are $\xi_n, x_n \in \Delta_+$ with $\xi_n \to \xi$ and $x_n \to v$ for which there are geodesics b_n from ξ_n to x_n in Γ that do not meet v. Let c be a geodesic from ξ to v, a_n be geodesics from v to x_n and c_n be geodesics from ξ_n to ξ. Since $x_n \to v$, the initial edges of the a_n form an infinite set. Lemma 2.1 and Remark 2.5 imply that not all angles at v determined by c and a_n can be $2\Theta^{(3)}$-small. On the other hand we can consider the geodesic 4-gon with sides c, a_n, b_n and c_n. Since the angle at v determined by c and a_n is eventually not $2\Theta^{(3)}$-small, eventually $v \in b_n \cup c_n$. Since $\xi_n \to \xi$, we can arrange that the c_n eventually miss v. This implies that eventually $v \in b_n$, contradicting our assumption. □

Isotropy groups

Lemma 2.16. *Let $\xi \in \Delta_+$, $v \in V$ where $v \neq \xi$. There is a neighborhood U of $\xi \in \Delta_+$ such that only finitely many edges in Γ appear as the initial edge of a geodesic from v to some $\xi' \in U$.*

Proof. Assume the lemma fails. Then there are $\xi_n \in \Delta_+$, geodesics c_n from v to ξ_n such that $\xi_n \to \xi$ and the initial edges of the c_n are pairwise different. Since the initial edges of the c_n are all different it follows that $\xi_n \to v$. But $v \neq \xi$. □

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Lemma 2.17. *Let $\xi_- \neq \xi_+ \in \Delta_+$. Then the intersection of the isotropy groups $G_{\xi_-} \cap G_{\xi_+}$ is virtually cyclic.*

Proof. If $\xi_- \in V$, then, by Lemma 2.16 there are only finitely many edges that are incident to ξ_- and are part of a geodesic from ξ_- to ξ_+. The group $G_{\xi_-} \cap G_{\xi_+}$ acts on this finite set. Since the action of G on the set of all edges is proper, this implies that $G_{\xi_-} \cap G_{\xi_+}$ is finite. The same argument applies if $\xi_+ \in V$.

It remains to treat the case $\xi_-, \xi_+ \in \partial\Gamma$. Let L be the subgraph of Γ spanned by all geodesics from ξ_- to ξ_+. By hyperbolicity L is contained in a bounded neighborhood of a fixed geodesic c between ξ_- and ξ_+. By [Bow12, Lemma 8.2] the intersection of L with any bounded ball in Γ around a vertex v from L contains only finitely many vertices, in particular, L is locally finite.

Since the action of G on edges is proper, the action of $G_{\xi_-} \cap G_{\xi_+}$ on L is proper as well. Fix a vertex $v_0 \in c$. Now, if the group $G_{\xi_-} \cap G_{\xi_+}$ is infinite, then it contains elements g, with $d_\Gamma(v_0, gv_0)$ arbitrarily large. If $d_\Gamma(v_0, gv_0) \gg 0$, then, since L is contained in a δ-neighborhood of the geodesic c, it follows that either $d_\Gamma(v_0, g^2v_0) \sim 0$ or $d_\Gamma(v_0, g^2v_0) \sim 2d_\Gamma(v_0, gv_0)$. In both cases, again since L is contained in a bounded neighborhood of c, we get a good understanding of the action of g on L. In the first case g will act, up to bounded error, as reflection with respect to a point of c. In particular, g will exchange ξ_- and ξ_+. This excludes the first case. In the second cases g acts, up to bounded error, like a translation along c. It follows that $g^{\pm n}v_0 \to \xi_\pm$, and, in particular, that the action on L of the cyclic group C generated by g is cocompact. Moreover, since L is contained in a bounded neighborhood of L and since bounded balls in L are finite and since the action of G on the edges of L is proper, it follows that C has finite index in $G_{\xi_-} \cap G_{\xi_+}$. □

Remark 2.18. Suppose that the group $G_{\xi_-} \cap G_{\xi_+}$ appearing in Lemma 2.17 is infinite virtually cyclic. Then, since $G_{\xi_-} \cap G_{\xi_+}$ fixes the two ends of L, this group is virtually cyclic of type I, i.e. admits a surjection to an infinite cyclic group.

Sequences of bounded distance

Lemma 2.19. *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in V that converges in Δ_+ to $\xi \in \partial\Gamma$. If $(v'_n)_{n\in\mathbb{N}}$ is another sequence in V such that $d_\Gamma(v_n, v'_n)$ is bounded, then v'_n also converges to ξ.*

Proof. Assume the lemma fails. As Δ_+ is compact there is then a convergent subsequence $(v'_n)_{n\in I}$ with $\lim_{n\in I} v'_n =: \xi' \neq \xi$. If $\xi' \in \partial\Gamma$, then we can use Lemma 2.9 to produce geodesic rays c to ξ and c' to ξ' such that, after passing to a further subsequence, we can assume $v_i \in c$ and $v'_i \in c'$ for all $i \in I$. Then, by hyperbolicity, c and c' are asymptotic and $\xi = \xi'$.

It remains to contradict $\xi' \in V$. If $\xi' \in V$, then Lemma 2.15 allows us to assume, after passing to a further subsequence, that for all $i \in I$ any geodesic between v_i and v'_i passes through ξ'. But then $d_\Gamma(\xi', v_i)$ is bounded and this contradicts $\xi \in \partial\Gamma$. □

Addendum 2.20. Retain the assumptions of Lemma 2.19, but assume $\xi = v \in V$. Assume in addition that there is $d > 0$ and a size for angles Θ such that there are Θ-small geodesics c_n of length less than or equal to d between v_n and v'_n. Assume also that the v'_n do *not* converge to v. Then there is a subsequence $I \subseteq \mathbb{N}$ such that both v_n and v'_n are constant for $n \in I$.

Proof. We can argue as in the proof of Lemma 2.19 and assume that $\xi' = \lim_n v'_n$ exists in Δ_+ with $\xi' \neq v$. Using Lemma 2.15 as in the last paragraph of proof of Lemma 2.19, we can assume that the c_n, $n \in I$ all pass through v. Lemma 2.16 allows us, after passing to a further

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

subsequence, to assume that the initial edge of the restriction $c_n|_{[v,v'_n]}$ does not depend on $n \in I$. Since the c_n are Θ-small, it follows from Lemma 2.1 that the initial edges of the $c_n|_{[v,v_n]}$, $n \in I$, can only vary over a finite set of edges. But now, as $v_n \to v$, we must have $v_n = v$ for all sufficiently large $n \in I$. Since the c_n are Θ-small and of bounded length, Corollary 2.2 implies that the v'_n, $n \in I$ also vary only over a finite set. □

Coarse convexity

Lemma 2.21. *Let $\xi \in \Delta_+$ and U be a neighborhood of $\xi \in \Delta_+$. Then there exists a neighborhood U' of ξ with the following property. If $v_-, v_+ \in U'$ and v belongs to a geodesic between v_- and v_+ then $v \in U$.*

Proof. Assume the lemma fails. Then we find sequence of vertices $(v_-)_n, (v_+)_n, v_n$ such that $(v_-)_n \to \xi$, $(v_+)_n \to \xi$, $v_n \notin U$ and v_n belongs to a geodesic c_n between $(v_-)_n$ and $(v_+)_n$. If $\xi \in V$, then we apply Lemma 2.15 to $c_n|_{[v_n,(v_-)_n]}$ and to $c_n|_{[v_n,(v_+)_n]}$. Lemma 2.15 implies that both restrictions eventually contain ξ, which can only happen if $v_n = \xi$. Since this contradicts $v_n \notin U$ we have $\xi \in \partial\Gamma$. Hyperbolicity implies that there are v'_n such that v'_n belongs to a geodesic from $(v_-)_n$ to ξ or to a geodesic from $(v_+)_n$ to ξ and such that $d_\Gamma(v_n, v'_n)$ is uniformly bounded. The first property ensures $v'_n \to \xi$ since $(v_-)_n \to \xi$ and $(v_+)_n \to \xi$. The second property allows us to apply Lemma 2.19 and deduce that also $v_n \to \xi$. This contradicts again $v_n \notin U$. □

3. The coarse Θ-flow space for relatively hyperbolic groups

Throughout this section we use the notations and assumptions from § 2. In particular, G is a group with a simplicial cocompact action on a fine and hyperbolic graph Γ. The stabilizers of edges under this action are finite. We fix again a hyperbolicity constant $\delta \geqslant 0$ for Γ. We will use the family $\mathcal{P}$ consisting of all subgroups $H \leqslant G$ that are virtually cyclic or fix a vertex $v \in V$. The space Δ is the union $\partial\Gamma \cup V_\infty$ and equipped with the observer topology. We also fix a proper left invariant metric d_G on G.

Theorem 3.1. *The action of G on Δ is finitely $\mathcal{P}$-amenable.*

Proof. This is a consequence of Propositions 3.2 and 3.17. □

Covering $G \times_{\Theta_0} \partial\Gamma$

It will be convenient to replace Γ with its first barycentric subdivision Γ'. The set of vertices in the barycentric subdivision corresponding to the edges of the original graph will be denoted V_E. Thus the set of vertices of Γ' is the disjoint union $V \cup V_E$, where V is the set of vertices of Γ. The vertices in V_E are all of valence 2. We will give edges in Γ' the length $1/2$. Then the path length metric of Γ and Γ' coincide. We will in this subsection use $\delta' := \delta + 1$. This has the effect that we can use vertices from V_E when we apply hyperbolicity: for any geodesic triangle and any vertex v from V_E on one side of the triangle there is a vertex w from V_E on one of the two other sides with $d_\Gamma(v, w) \leqslant \delta'$. Moreover, Γ' is still fine and the considerations from § 2 and the appendix all apply to Γ' as well. In particular, we can define Δ' and Δ'_+. Since all vertices in V_E are of valence 2 we have $\Delta' = \Delta$ and $\Delta'_+ = \Delta_+ \cup V_E$. Of course, $\partial\Gamma = \partial\Gamma'$. For $v \in V$ there is a canonical bijection between angles at v with respect to Γ and angles with respect to Γ'. In this subsection sizes for angles will only be considered at vertices from V. (Since there is only one non-trivial angle at any vertex $v \in V_E$, angles at $v \in V_E$ are not very interesting.)

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

We fix a base point $v_0 \in V_E$. For a size for angles Θ we write $G \times_\Theta \partial\Gamma$ for the subset of $G \times \partial\Gamma$ consisting of all pairs $(g, \xi) \in G \times \partial\Gamma$ for which there exists a Θ-small geodesic from gv_0 to ξ. The main result of this subsection produces partial covers of $G \times \Delta$.

PROPOSITION 3.2. *There is a number N such that for any $\alpha > 0$ and any size for angles Θ_0 there exists a G-invariant collection $\mathcal{U}$ of* VCyc*-subsets of $G \times \Delta$ such that:*

(a) *the order of $\mathcal{U}$ is at most N;*
(b) *for every $(g, \xi) \in G \times_{\Theta_0} \partial\Gamma$ there is $U \in \mathcal{U}$ with $B_\alpha(g) \times \{\xi\} \subseteq U$.*

To prove Proposition 3.2 we will construct a coarse flow space and use the long and thin covers from Theorem 1.1. For a size for angles Θ, we define the metric d_Θ on V_E as follows. For $v, v' \in V_E$ we set

$$d_\Theta(v, v') := \min \sum_{i=1}^{n} d_\Gamma(w_{i-1}, w_i)$$

where the minimum is taken over all finite sequences $v = w_0, w_1, \ldots, w_n = v'$ of vertices from V_E such that there are Θ-small geodesics between w_{i-1} and w_i for all i. If there is no such sequence, then $d_\Theta(v, v') = \infty$. If there exists a Θ-small geodesic between v and v', then $d_\Theta(v, v') = d_\Gamma(v, v')$, but in general $d_\Theta(v, v') \geqslant d_\Gamma(v, v')$.

LEMMA 3.3. *The metric d_Θ on V_E is proper.*

Proof. If $d_\Theta(v, v') \leqslant n$, then there is a finite sequence $v = w_0, w_1, \ldots, w_n = v'$ in V_E with $d_\Theta(w_{i-1}, w_i) \leqslant 1$ for all i. Therefore, it suffices to check that balls of radius 1 are finite. This is a consequence of Lemma 2.1. □

DEFINITION 3.4. Set $Z := (\Delta'_+)^2$. Let Θ be a size for angles. The coarse Θ-flow space $CF(\Theta)$ for Γ is the subset of $V_E \times Z$ consisting of all triples (v, ξ_-, ξ_+) for which there exist $v' \in V_E$ and a Θ-small geodesic c between ξ_- and ξ_+ such that $v' \in c$ and $d_\Theta(v, v') \leqslant \delta'$. Moreover, we require $\xi_-, \xi_+ \in V_E \cup \partial\Gamma$.

Example 3.5. Suppose that Γ is a locally finite tree. The flow space FS from [BL12b] for Γ consists of all generalized (parametrized) geodesics $c\colon \mathbb{R} \to \Gamma$. If we use $\delta' = 0$ and all angles for Θ, then there is a natural embedding $CF(\Theta) \to FS$ that sends (v, ξ_-, ξ_+) to the generalized geodesic $c\colon \mathbb{R} \to \Gamma$ with $c(-\infty) = \xi_-$, $c(0) = v$ and $c(+\infty) = \xi_+$.

LEMMA 3.6. *Let Θ be a size for angles with $2\Theta^{(3)} \subseteq \Theta$. Then we have the following:*

(a) *$CF(\Theta) \subseteq V_E \times Z$ is closed;*
(b) *$\dim CF(\Theta) \leqslant \dim Z < \infty$;*
(c) *for $(\xi_-, \xi_+) \in Z$, $V_{\xi_-,\xi_+} := \{v \in V_E \mid (v, \xi_-, \xi_+) \in CF(\Theta)\}$ has the (D, R)-doubling property with respect to d_Θ, where D and R are independent of (ξ_-, ξ_+) and Θ;*
(d) *for (v, ξ_-, ξ_+) the isotropy group $G_{\xi_-,\xi_+} = \{g \in G \mid g\xi_- = \xi_-, g\xi_+ = \xi_+\}$ is virtually cyclic.*

Proof. (a) It suffices to show that $Z_v := \{(\xi_-, \xi_+) \mid (v, \xi_-, \xi_+) \in CF(\Theta)\}$ is closed for each $v \in V_E$. Let $((\xi_-)_n, (\xi_+)_n) \in Z_v$ with $((\xi_-)_n, (\xi_+)_n) \to (\xi_-, \xi_+)$ in Z. We need to show $(\xi_-, \xi_+) \in Z_v$. Since $((\xi_-)_n, (\xi_+)_n) \in Z_v$ there are Θ-small geodesics c_n from $(\xi_-)_n$ to $(\xi_+)_n$ and vertices $v_n \in c_n \cap V_E$

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

with $d_\Theta(v_n, v) \leqslant \delta'$. As d_Θ is proper by Lemma 3.3 we can pass to a subsequence and assume that $v_n = w$ is constant. Moreover, we can apply Lemma 2.9 to the $c_n|_{[w,(\xi_-)_n]}$ and assume that either the $c_n|_{[w,(\xi_-)_n]}$ converge pointwise to a geodesic ray from w to $\xi_- \in \partial\Gamma$, or that $\xi_- \in c_n|_{[w,(\xi_-)_n]}$ for all n. In the second case, it also follows that eventually $(\xi_-)_n = \xi_-$, as otherwise $(\xi_-)_n \to \xi_-$ would imply that the $\sphericalangle_{\xi_-} c_n$ are eventually Θ-large by Lemma 2.1. Therefore, $\xi_- \in V_E \cup \partial\Gamma$ and the $c_n|_{[w,(\xi_-)_n]}$ converge pointwise to a geodesic c_- from w to ξ_-. Similar, $\xi_+ \in V_E \cup \partial\Gamma$ and we can assume that the $c_n|_{[w,(\xi_+)_n]}$ converge pointwise to a geodesic c_+ from w to ξ_+. Now c_- and c_+ combine to a geodesic between ξ_- and ξ_+ that passes through w. Thus $(\xi_-, \xi_+) \in Z_v$.

(b) Theorem A.1 asserts that Δ'_+ is finite dimensional. Since V_E is discrete and $CF(\Theta)$ is closed in $V_E \times Z$ it follows that $\dim CF(\Theta) \leqslant \dim Z = 2 \dim \Delta'_+ < \infty$.

(c) If V_{ξ_-,ξ_+} is non-empty, then there is a Θ-small geodesic c between ξ_- and ξ_+. By hyperbolicity, any vertex $v' \in V_E$ on any other Θ-small geodesic c' from ξ_- to ξ_+ will be within distance δ' of some vertex $v \in V_E \cap c$. If $c \cup c'$ does not contain a geodesic between v and v', then Lemma 2.7 provides us with a Θ-small geodesic between v and v'. If $c \cup c'$ contains a geodesic c'' between v and v', then we can assume that c'' changes only once from c to c', at some vertex $w \in V$. Then c'' may fail to be Θ-small only at this vertex w. In this case we may assume that v is incident to w. Let now $\tilde{v} \in V_E$ be the unique vertex incident to w on $c' \backslash c''$. Then $c'|_{[v',\tilde{v}]}$ is a Θ-small geodesic of length at most δ' and thus $d_\Theta(v', \tilde{v}) \leqslant \delta'$. We can apply Lemma 2.6 to the bi-gone spanned by $c|_{[w,\xi_\pm]}$ and $c'|_{[w,\xi_\pm]}$ to deduce that the angle at w spanned by the edges $(w, \tilde{v})$ and (w, v) is $\Theta^{(3)}$-small. Thus $d_\Theta(v, \tilde{v}) = 1$ and $d_\Theta(v, v') \leqslant \delta' + 1$.

Thus any $v \in V_{\xi_-,\xi_+}$ will be within distance $2\delta' + 1$ of a vertex from $V_E \cap c$ with respect to d_Θ. Therefore, any α-separated subset S in a 2α-ball in V_{ξ_-,ξ_+} can be mapped injectively to an $\alpha - 4\delta' - 2$-separated subset S' of an $2\alpha + 2\delta' + 1$-ball in $V_E \cap c$. For sufficiently large α (for example, $\alpha > R := 24\delta' + 12$) any such set S' contains at most five elements, since $V_E \cap c$ is isometric to a subset of $\mathbb{Z}$.

(d) If $\xi_- \in V_E$ or $\xi_+ \in V_E$, then G_{ξ_-,ξ_+} is finite since the action of G on V_E is proper. If $\xi_-, \xi_+ \in \partial\Gamma$, then, by definition of $CF(\Theta)$, $\xi_- \neq \xi_+$. Lemma 2.17 implies that then G_{ξ_-,ξ_+} is virtually cyclic. □

Proposition 3.7. *There is a number N' such that for any $\alpha' > 0$ and any size for angles Θ containing $2\Theta^{(3)}$ there exists a* VCyc*-cover $\mathcal{W}$ of $CF(\Theta)$ such that:*

(a) $\dim \mathcal{W} \leqslant N'$;
(b) *for any $(v, \xi_-, \xi_+) \in CF(\Theta)$ there is $W \in \mathcal{W}$ such that $B^\Theta_{\alpha'}(v) \times \{(\xi_-, \xi_+)\} \cap CF(\Theta) \subseteq W$. Here $B^\Theta_{\alpha'}(v)$ is the α'-ball in V_E with respect to d_Θ.*

Proof. Let Θ be a size for angles. Lemmas 3.3 and 3.6 allow us to apply Theorem 1.1. We obtain a number N' such that for any $\alpha' > 0$ there exists a VCyc-cover $\mathcal{W}$ of $CF(\Theta)$ satisfying parts (a) and (b).

The number N' only depends on $\dim CF(\Theta)$ and the doubling constant. For these numbers Lemma 3.6 provides bounds that do not depend on Θ. Therefore N' does not depend on Θ. □

Lemma 3.8. *Let $\alpha > 0$ and Θ_0 be a size for angles. Then there is a size for angles Θ containing $\Theta_0 + 2\Theta^{(3)}$ with the following property. If c is a Θ_0-small geodesic from gv_0 to $\xi \in \partial\Gamma$ and $g' \in G$ with $d_G(g, g') \leqslant \alpha$, then we have the following:*

(a) *any geodesic c' from $g'v_0$ to ξ is Θ-small;*
(b) *any geodesic c'' starting on c and ending on a geodesic c' from $g'v_0$ to ξ is Θ-small.*

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Proof. Pick Θ_1 such that for any $h \in B_\alpha(e)$ there exists a Θ_1-small geodesic from hv_0 to v_0. Thus, if $g' \in G$ with $d_G(g, g') \leqslant \alpha$ then there is a Θ_1-small geodesic between gv_0 and $g'v_0$. Let X be the size for angles from Lemma 2.14. In particular, for any size for angles Y, and any geodesic triangle in Γ where the first side is Θ_0-small and the second side is Y-small, the third side will be $Y + X$-small. Then part (a) holds whenever $\Theta_1 + X \subseteq \Theta$ and part (b) holds whenever $\Theta_1 + 2X \subseteq \Theta$. □

Definition 3.9. Let $W \subseteq CF(\Theta)$ and $\tau \in \mathbb{N}$. We define

$$\iota^{-\tau} W \subseteq G \times_\Theta \partial\Gamma$$

as the subspace consisting of pairs $(g, \xi) \in G \times_\Theta \partial\Gamma$ with the following property. For every Θ-small geodesic c from gv_0 to ξ we have $(v_c, gv_0, \xi) \in W$, where v_c is the unique vertex in $V_E \cap c$ with $d_\Gamma(gv_0, v_c) = \tau$.

Lemma 3.10. *If W is open in $CF(\Theta)$, then $\iota^{-\tau} W$ is open in $G \times_\Theta \partial\Gamma$.*

Proof. Let $(g, \xi) \in \iota^{-\tau} W$. Assume that (g, ξ) does not belong to the interior of $\iota^{-\tau} W$ in $G \times_\Theta \partial\Gamma$. Then there are $\xi_n \in \partial\Gamma$ with $\xi_n \to \xi$ and Θ-small geodesics c_n from gv_0 to ξ_n such that for the vertices $v_n \in V_E \cap c_n$ with $d_\Gamma(gv_0, v_n) = \tau$ we have $(v_n, gv_0, \xi_n) \notin W$. Using Lemma 2.9(a) and passing to a subsequence we may assume that the c_n converge pointwise to a Θ-small geodesic c from gv_0 to ξ. Then eventually $v_n = v_c$ is constant and belongs to c. Since $(g, \xi) \in W$, it follows that $(v_c, gv_0, \xi) \in W$. Since W is open, eventually $(v_n, gv_0, \xi_n) \in W$, contradicting our assumption. □

Proof of Proposition 3.2. Let Θ_0 and $\alpha > 0$ be given. Let Θ be the size for angles from Lemma 3.8. Since $B_\alpha(e) \subseteq G$ is finite, we can find a number $\alpha' > 0$ such that $d_\Gamma(gv_0, ghv_0) + 2\delta' \leqslant \alpha'$ for all $g \in G$ and $h \in B_\alpha(e)$. Let $\mathcal{W}$ be the cover of $CF(\Theta)$ from Proposition 3.7. For $\tau \in \mathbb{N}$ let $\iota^{-\tau}\mathcal{W} := \{\iota^{-\tau} W \mid W \in \mathcal{W}\}$. By Lemma 3.10, the members of $\iota^{-\tau}\mathcal{W}$ are open subsets of $G \times_\Theta \partial\Gamma$. Since $\iota^{-\tau}$ is a G-equivariant operation and $\mathcal{W}$ is G-invariant and consist of VCyc-subset, the same is true for $\iota^{-\tau}\mathcal{W}$. The order of $\mathcal{W}$ is bounded by Proposition 3.7(a). Since $\iota^{-\tau}$ commutes with intersection, the order of $\iota^{-\tau}\mathcal{W}$ is bounded by the order of $\mathcal{W}$.

We claim that there exists $\tau \in \mathbb{N}$ such that $\iota^{-\tau}\mathcal{W}$ is α-wide in the G-direction, i.e. we claim that there is τ such that

(3.11) for any $(g, \xi) \in G \times_{\Theta_0} \partial\Gamma$ there is $W \in \mathcal{W}$ with $B_\alpha(g) \times \{\xi\} \subseteq \iota^{-\tau} W$.

Suppose there is no such τ. Then there is a sequence of pairs $(g_\tau, \xi_\tau)_{\tau \in \mathbb{N}}$ in $G \times_{\Theta_0} \partial\Gamma$ such that

(3.12) for all $W \in \mathcal{W}$ and all $\tau \in \mathbb{N}$ we have $B_\alpha(g_\tau) \times \{\xi_\tau\} \not\subseteq \iota^{-\tau} W$.

By definition of $G \times_{\Theta_0} \partial\Gamma$ there are Θ_0-small geodesics c_τ from $g_\tau v_0$ to ξ_τ in Γ. Let v_τ be the unique vertex on c_τ with $d_\Gamma(g_\tau v_0, v_\tau) = \tau$. Since the action of G on V is cofinite, and since the $\iota^{-\tau}\mathcal{W}$ are G-invariant we may, after passing to a subsequence $(g_\tau, \xi_\tau)_{\tau \in T}$, assume that $v_\tau = v$ is constant. Using the compactness of Δ'_+, we may, after passing to a further subsequence, assume that $\xi_- := \lim g_\tau v_0$ and $\xi_+ := \lim \xi_\tau$ exist in Δ'_+. Addendum 2.10 implies $\xi_-, \xi_+ \in \partial\Gamma$ and allows us to assume that the restrictions $c_\tau|_{[v, g_\tau v_0]}$ and $c_\tau|_{[v, \xi_\tau]}$ converge pointwise to geodesic rays c_- from v to ξ_- and c_+ from v to ξ_+. In particular, c_- can be combined with c_+ to obtain a geodesic c between ξ_- and ξ_+. This geodesic c is Θ_0-small, since the c_τ are Θ_0-small.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

By Proposition 3.7(b) there is $W \in \mathcal{W}$ such that $B^{\Theta}_{\alpha'}(v) \times \{(\xi_-, \xi_+)\} \cap CF(\Theta) \subseteq W$. Let V' be the set of $v' \in B^{\Theta}_{\alpha'}(v)$ with $d_\Theta(v', w) \leqslant \delta'$ for some $w \in c$. Then $(v', \xi_-, \xi_+) \in CF(\Theta)$ for all $v' \in V'$. Since d_Θ is proper by Lemma 3.3, the set V' is finite. Therefore we find neighborhoods U_- of ξ_- and U_+ of ξ_+, such that $V' \times U_- \times U_+ \cap CF(\Theta) \subseteq W$. Elements of $B_\alpha(g_\tau)$ can be written as $g_\tau h$ with $h \in B_\alpha(e)$. For each h, $\lim_\tau g_\tau h v_0 = \lim_\tau g_\tau v_0 = \xi_-$ by Lemma 2.19. Since $B_\alpha(e)$ is finite we find τ_0 such that $g_{\tau_0} h v_0 \in U_-$ for all $h \in B_\alpha(e)$. Moreover, we can arrange for τ_0 to satisfy $\tau_0 > \alpha'$, $\xi_{\tau_0} \in U_+$, and $c_{\tau_0} \cap B_{\alpha'}(v) = c \cap B_{\alpha'}(v)$. We claim that

$$B_\alpha(g_{\tau_0}) \times \{\xi_{\tau_0}\} \subseteq \iota^{-\tau_0} W. \tag{3.13}$$

Let $h \in B_\alpha(e)$. By Lemma 3.8(a) there is a Θ-small geodesic c' from $g_{\tau_0} h v_0$ to ξ_{τ_0}. Let $v' \in V_E \cap c'$ be the vertex with $d_\Gamma(g_{\tau_0} h v_0, v') = \tau_0$. We need to show that $(v', g_{\tau_0} h v_0, \xi_{\tau_0})$ belongs to W. To this end it suffices to show $v' \in V'$, where V' is as defined earlier. Now we use hyperbolicity to find $w \in V_E \cap c_{\tau_0}$ with $d_\Gamma(w, v') \leqslant \delta'$. Lemma 3.8(b) implies that there is a Θ-small geodesic between w and v'. Therefore $d_\Theta(w, v') \leqslant \delta'$. Since $d_\Gamma(g_{\tau_0} v_0, g_{\tau_0} h v_0) \leqslant \alpha' - 2\delta'$ and $d_\Gamma(w, v') \leqslant \delta'$ we have $d_\Gamma(g_{\tau_0} h v_0, v') - (\alpha' - \delta') \leqslant d_\Gamma(g_{\tau_0} v_0, w) \leqslant d_\Gamma(g_{\tau_0} h v_0, v') + (\alpha' - \delta')$. Since $d_\Gamma(g_{\tau_0} v_0, v) = d_\Gamma(g_{\tau_0} h v_0, v') = \tau_0$ we have $d_\Gamma(v, w) \leqslant \alpha' - \delta'$. Now $c_{\tau_0} \cap B_{\alpha'}(v) = c \cap B_{\alpha'}(v)$ implies $w \in c$. Since c is Θ_0-small and therefore Θ-small, and since $v, w \in c$, we have $d_\Theta(w, v) \leqslant \alpha' - \delta'$. Therefore $d_\Theta(v', v) \leqslant \alpha'$ and $v' \in B^{\Theta}_{\alpha'}(v)$. Altogether we have shown $v' \in V'$ and therefore (3.13). This contradicts (3.12) and finishes the proof of (3.11).

It remains to extend $\iota^{-\tau}\mathcal{W}$ to a G-invariant collection of VCyc-subset of $G \times \Delta$ of the same order. Since Δ is metrizable, there is a G-invariant metric on $G \times \Delta$. Thus the needed extension exists by Lemma B.2. □

Covering $G \times \Delta \backslash G \times_\Theta \partial\Gamma$

Definition 3.14. Let $v \in V$ and let Θ be a size for angles in Γ. Define

$$V_+(v, \Theta) \subseteq G \times \Delta$$

as the set of all (g, ξ) with the following properties:

(a) all geodesics from gv_0 to v are Θ-small;
(b) if $\xi \neq v$, then there is a geodesic c in Γ from gv_0 to ξ such that $\sphericalangle_v c$ is Θ-large. (In particular, we require $v \in c$.)

Let $V(v, \Theta)$ be the interior of $V_+(v, \Theta)$.

Lemma 3.15. *Assume that Θ is a size for angles containing $\Theta^{(3)}$.*

(a) *For $v \in V$ we have $V_+(v, \Theta) \cap G \times V_\infty \subseteq V(v, \Theta)$.*
(b) *For $v \neq v' \in V$ we have $V(v, \Theta) \cap V(v', \Theta) = \emptyset$.*
(c) *If $gV(v, \Theta) \cap V(v, \Theta) \neq \emptyset$ with $g \in G$, $v \in V$, then $gv = v$ and $gV(v, \Theta) = V(v, \Theta)$.*

Proof. (a) Let $(g, w) \in V_+(v, \Theta)$ with $w \in V_\infty$. Let c be a geodesic from gv_0 to w. We proceed by contradiction and assume that (g, w) does not belong to the interior of $V_+(v, \Theta)$. Then there is a sequence ξ_n in Δ with $\xi_n \to w$ and $(g, \xi_n) \notin V_+(v, \Theta)$. Let c_n be a geodesic from gv_0 to ξ_n. By Lemma 2.15 the c_n will eventually pass through w. In this case we can change c_n and arrange for $c_n|_{[gv_0, w]} = c$. If $w \neq v$ then these c_n prove that eventually $(g, \xi_n) \in V_+(v, \Theta)$ which contradicts our assumption. If $w = v$, then it remains to show that $\sphericalangle_v c_n$ will eventually be Θ-large. In this case the initial edges of the restriction $c_n|_{[v, \xi_n]}$ will form an infinite set since $\xi_n \to v$. This implies, by Lemma 2.1, that $\sphericalangle_v c_n$ at v will eventually be Θ-large.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

(b) Assume there is $(g,\xi) \in V(v,\Theta) \cap V(v',\Theta)$. Let c and c' be geodesics from gv_0 to ξ such that $\sphericalangle_v c$ is Θ-large (if $v \neq \xi$) and $\sphericalangle_{v'} c'$ is Θ-large (if $v' \neq \xi$). Since $\Theta^{(3)} \subseteq \Theta$, Lemma 2.11 implies $v \in c'$ and $v' \in c$. Assume $v \neq \xi$. Without loss of generality, we may assume that v is closer to gv_0 than v'. Then v is an internal vertex of $c|_{[gv_0,v']}$. Since $\sphericalangle_v c$ is Θ-large, $c|_{[gv_0,v']}$ is not Θ-small. This contradicts $(gv_0,\xi) \in V(v',\Theta)$.

(c) Clearly, $gV(v,\Theta) = V(gv,\Theta)$. Thus part (b) shows that $gV(v,\Theta) \cap V(v,\Theta) \neq \emptyset$ implies $gv = v$. □

LEMMA 3.16. *Let Θ be a size for angles. Let $v \in V$, $g \in G$, $\xi \in \Delta$. Assume that any geodesic from gv_0 to v is Θ-small. Assume that there is a geodesic c from gv_0 to ξ that passes through v and satisfies one of the following two conditions:*

(a) *$\sphericalangle_v c$ is $\Theta + 2\Theta^{(3)}$-large;*

(b) *$\sphericalangle_v c$ is Θ-large and there is an interior vertex w of c with $d_\Gamma(gv_0,v) < d_\Gamma(gv_0,w)$ such that $\sphericalangle_w c$ is $2\Theta^{(3)}$-large.*

Then $(g,\xi) \in V(v,\Theta)$.

Proof. If condition (a) holds, then set $A := \{v\}$; if condition (b) holds, then set $A := \{w\}$. Recall that $M'(\xi,A)$ consists of all ξ' for which there exists a geodesic between ξ and ξ' that misses $A\backslash\{\xi\}$. Since $M'(\xi,A)$ is part of a neighborhood basis for the observer topology, it suffices to show that $(g,\xi') \in V_+(v,\Theta)$ for all $\xi' \in M'(\xi,A)$. Let $\xi' \in M'(\xi,A)$. Let c'' be a geodesic between ξ and ξ' that misses A. Let c' be a geodesic between gv_0 and ξ'. We apply Lemma 2.12 to the geodesic triangle with sides c, c' and c''.

If condition (a) holds, then Lemma 2.12 implies that c' passes through v and that $\sphericalangle_v c'$ is Θ-large. Thus in this case $(g,\xi') \in V_+(v,\Theta)$.

If condition (b) holds, then Lemma 2.12 implies $w \in c'$. Then we can replace c' and assume $c'|_{[gv_0,w]} = c|_{[gv_0,w]}$. In particular $\sphericalangle_v c' = \sphericalangle_v c$ is Θ-large. Thus in this case $(g,\xi') \in V_+(v,\Theta)$ as well. □

PROPOSITION 3.17. *Let $\alpha > 0$. There is a size for angles Θ and a G-invariant collection $\mathcal{V}$ of $\mathcal{P}$-subsets of $G \times \Delta$ such that the following hold.*

(a) *The order of $\mathcal{V}$ is at most 2.*

(b) *For every $(g,\xi) \in G \times \Delta$ at least one of the following two statements holds:*

- *there is $V \in \mathcal{V}$ such that $B_\alpha(g) \times \{\xi\} \subseteq V$;*
- *$\xi \in \partial\Gamma$ and there is a Θ-small geodesic from gv_0 to ξ.*

Proof. Since Γ is fine there are only finitely many geodesics between any two vertices. In particular, we can pick a size for angles Θ_0 such that for all $h,h',h'' \in B_\alpha(e)$ any geodesic starting at hv_0 and ending in some vertex on another geodesic between vertices $h'v_0$ and $h''v_0$ is Θ_0-small. Let X be the size for angles appearing in Lemma 2.14. After increasing X, if necessary, we may assume $\Theta_0 + 2\Theta^{(3)} \subseteq X$. Let $\mathcal{V}_1 := \{V(v,2X) \mid v \in V\}$, $\mathcal{V}_2 := \{V(v,5X) \mid v \in V\}$ and $\mathcal{V}_3 := \{V(v,6X) \mid v \in V\}$. All three collections are G-invariant. Lemma 3.15 implies that all three collections are of order 0, and consist of $\mathcal{P}$-sets. Thus $\mathcal{V} := \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ is a G-invariant collection of $\mathcal{P}$-sets and satisfies part (a).

It remains to check that $\mathcal{V}$ also satisfies part (b) where we use $\Theta := 6X$. Since $\mathcal{V}$ is G-invariant it suffices to consider $(e,\xi) \in G \times \Delta$.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Let $\mathcal{C}$ be the set of all geodesics from some hv_0, $h \in B_\alpha(e)$ to ξ. If all $c \in \mathcal{C}$ are $6X$-small, then part (b) holds. Indeed, if $\xi \in V$, then, using Lemma 3.15(a), we obtain $B_\alpha(e) \times \{\xi\} \subseteq V(\xi, 6X)$ and part (b) holds. If $\xi \in \partial\Gamma$, then part (b) holds as well, simply since $\mathcal{C}$ contains a Θ-small geodesic from v_0 to ξ. Therefore we may assume that not all $c \in \mathcal{C}$ are $6X$-small.

Let c_0 be a geodesic from v_0 to ξ. Let W be the set of all internal vertices w of c_0 for which $\sphericalangle_w c_0$ is X-large. Lemma 2.14 implies for any $c \in \mathcal{C}$ and any size for angles Θ' that the following holds:

(3.18) c will pass through any $w \in W$;

(3.19) if v is an internal vertex of c, where $v \notin W$, then the angle of c at v is $2X$-small;

(3.20) if, for $w \in W$, the angle of c_0 at w is $\Theta' + X$-large, then the angle of c at w is Θ'-large;
if, for $w \in W$, the angle of c_0 at w is $4X$-small, then the angle of c at w is $5X$-small.

Since not all $c \in \mathcal{C}$ are $6X$-small, we conclude from (3.20) that $W \neq \emptyset$. Among all $w \in W$ we let w_0 be the one closest to v_0. Then (3.18) implies that w_0 is also closest to all hv_0, $h \in B_\alpha(e)$. In particular, any c will, on its way to ξ, meet w_0 before meeting any of the other w. Using (3.19) this implies that any geodesic starting in some hv_0, $h \in B_\alpha(e)$ and ending in w_0 will be $2X$-small. If $\sphericalangle_{w_0} c_0$ is $4X$-large, then by (3.20) for any $c \in \mathcal{C}$ the angle of c at w_0 is $3X$-large. Using Lemma 3.16(a) we see that in this case, $B_\alpha(e) \times \{\xi\} \subseteq V(w_0, 2X)$ and part (b) holds.

Therefore we may assume that $\sphericalangle_{w_0} c_0$ is $4X$-small. Then (3.20) implies that for all $c \in \mathcal{C}$ the angle at w_0 is $5X$-small. Since not all $c \in \mathcal{C}$ are Θ-small, there are $w \in W$ and $c \in \mathcal{C}$ such that the angle of c at w is $5X$-large. Among all such pairs we pick (w_1, c_1) such that w_1 is closest to v_0. As before (3.18) implies that w_1 is also closest to all hv_0, $h \in B_\alpha(e)$. Using (3.19) this implies that all geodesics starting in some hv_0, $h \in B_\alpha(e)$ and ending at w_1 are $5X$-small. Since $w_0 \neq w_1$ we can use (3.18) again, to see that for any $h \in B_\alpha(e)$ there is a geodesic from hv_0 to ξ that agrees with c_1 between w_0 and ξ. Therefore, for any $h \in B_\alpha(e)$ there is a geodesic from hv_0 to ξ for which the angle at w_1 is $5X$-large. If $\xi \in V$, then using Lemma 3.15(a), we deduce $B_\alpha(e) \times \{\xi\} \subseteq V(w_1, 5X)$ and part (b) holds.

If $\xi \in \partial\Gamma$ we need to distinguish two further cases. If the angle of c_1 at w_1 is even $\Theta = 6X$-large, then Lemma 3.16(a) implies $B_\alpha(e) \times \{\xi\} \subseteq V(w_1, 5X)$ and part (b) holds. Otherwise, we use again that not all $c \in \mathcal{C}$ are $6X$-small. Therefore, there is $w_2 \in W$ and $c_2 \in \mathcal{C}$ such that the angle of c_2 at w_2 is $6X$-large. Since w_1 was chosen closest to v_0, w_2 will be further from v_0 than w_1. Now using (3.20) twice we see that for any geodesic $c \in \mathcal{C}$ the angle at w_2 is $4X$-large. Since we already found geodesics for any $h \in B_\alpha(e)$ from hv_0 to ξ whose angle at w_1 is $5X$-large, we can now use Lemma 3.16(b) to conclude $B_\alpha(e) \times \{\xi\} \subseteq V(w_1, 5X)$. Therefore part (b) holds. □

Covering $G \times \overline{P_{d,\Theta}}$

Associated to Γ there is, for given $d > 0$ and a given size for angles Θ, a finite-dimensional simplicial complex $P_{d,\Theta}$, the relative Rips complex of Γ. The construction of $P_{d,\Theta}$ is reviewed in Definition A.3. The union $\overline{P_{d,\Theta}} := P_{d,\Theta} \cup \partial\Gamma$ carries a compact topology, see Lemma A.11(b). Theorem 3.1 has the following straight forward extension from Δ to $\overline{P_{d,\Theta}}$.

Corollary 3.21. *For any $d > 0$ and any size for angles Θ the action of G on $\overline{P_{d,\Theta}}$ is finitely $\mathcal{P}$-amenable.*

Proof. Given $\alpha > 0$ we need to construct a $\mathcal{P}$-cover of $G \times \overline{P_{d,\Theta}}$ as in Definition 0.1.

The action of G on $P_{d,\Theta}$ is simplicial and the dimension of $P_{d,\Theta}$ is finite by Lemma A.4. In particular, Remark 0.3 applies and we find a $\mathcal{P}$-cover $\mathcal{W}$ of $G \times P_{d,\Theta}$ of dimension N' such that

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

(3.22) $\dim \mathcal{W}$ is bounded by the dimension of $P_{d,\Theta}$,

(3.23) for any $(g,x) \in G \times P_{d,\Theta}$ there is $W \in \mathcal{W}$ such that $G \times \{x\} \subseteq W$.

The compact topology on $\overline{P_{d,\theta}}$ is such that $P_{d,\Theta}$ is not necessarily open in $\overline{P_{d,\theta}}$, but $P_{d,\Theta} \backslash V_\infty$ is open in $\overline{P_{d,\theta}}$. Therefore, the collection $\{W \backslash G \times V_\infty \mid W \in \mathcal{W}\}$ consists of open $\mathcal{P}$-subsets, satisfies (3.22) and satisfies (3.23) for all $(g,x) \in G \times (P_{d,\Theta} \backslash V_\infty)$.

Given $\alpha > 0$, Theorem 3.1 provides a $\mathcal{P}$-cover $\mathcal{V}$ for $G \times \Delta$ such that

(3.24) $\dim \mathcal{V}$ is bounded by a number independent of α,

(3.25) for any $(g,\xi) \in G \times \Delta$ there is $V \in \mathcal{V}$ such that $B_\alpha(g)\{\xi\} \subseteq V$.

Since $\overline{P_{d,\Theta}}$ is compact with a countable basis for the topology it is also metrizable. Therefore we find a G-invariant metric $d_{G \times \overline{P_{d,\Theta}}}$ on $G \times \overline{P_{d,\Theta}}$. Using Lemma B.2 we can extend $\mathcal{V}$ to G-invariant collection $\mathcal{V}^+$ of $\mathcal{P}$-subset of $G \times \overline{P_{d,\Theta}}$ that still satisfies (3.24) and (3.25). Altogether $\{W \backslash G \times V_\infty \mid W \in \mathcal{W}\} \cup \mathcal{V}^+$ is the desired cover of $G \times \overline{P_{d,\Theta}}$. □

4. The Farrell–Jones conjecture for relatively hyperbolic groups

Let G be a group and $\mathcal{A}$ be an additive category with a strict G-action and a strict direct sum. In [BL12a, § 4.1] such categories are called additive G-categories. Similar to (twisted) group rings there is an additive category $\int_G \mathcal{A}$ whose morphisms are formal linear combinations of group elements with morphisms from $\mathcal{A}$ as coefficients.[4] Given a family $\mathcal{F}$ of subgroups of G there is the assembly map

$$H_*^G(E_{\mathcal{F}}G; \mathbf{K}_{\mathcal{A}}) \to K_*\left(\int_G \mathcal{A}\right). \tag{4.1}$$

The K-theoretic Farrell–Jones conjecture (with coefficients) asserts that this map is an isomorphism for the family $\mathcal{F} := \mathrm{VCyc}$ of virtually cyclic subgroups of G [BR07, Conjecture 3.2]. The original formulation of the conjecture in [FJ93] can be recovered as a special case of (4.1) by using for $\mathcal{A}$ the category of finitely generated free $\mathbb{Z}$-modules; then $\int_G \mathcal{A}$ is equivalent to the category of finitely generated free $\mathbb{Z}[G]$-modules. We will say that G *satisfies the K-theoretic Farrell–Jones conjecture relative to* $\mathcal{F}$ if (4.1) is an isomorphism for all additive G-categories $\mathcal{A}$. A consequence of the transitivity principle (see [FJ93, Theorem A.10] and [BFL14, Theorem 2.10]) is the following. Suppose that G satisfies the K-theoretic Farrell–Jones conjecture relative to $\mathcal{F}$ and that every group $F \in \mathcal{F}$ satisfies the K-theoretic Farrell–Jones conjecture. Then G satisfies the K-theoretic Farrell–Jones conjecture.

If $\mathcal{A}$ is in addition equipped with a strict involution, see for example [BL12a, § 4.1], then $\int_G \mathcal{A}$ inherits an involution and there is the L-theoretic assembly map

$$H_*^G(E_{\mathcal{F}}G; \mathbf{L}_{\mathcal{A}}^{-\infty}) \to L_*^{\langle -\infty \rangle}\left(\int_G \mathcal{A}\right). \tag{4.2}$$

The L-theoretic Farrell–Jones conjecture (with coefficients) asserts that this map is an isomorphism for the family of virtually cyclic subgroups VCyc. Everything said for K-theory above has an L-theory counterpart. In particular, we will say that G *satisfies the L-theoretic Farrell–Jones conjecture relative to* $\mathcal{F}$ if (4.2) is an isomorphism for all additive G-categories $\mathcal{A}$ with involution.

[4] In [BR07, § 2] this category is denoted $\mathcal{A} *_G G/G$.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Next we give a minor reformulation of conditions from [BL12a, BLR08b] that implies the Farrell–Jones conjecture relative to $\mathcal{F}$. For a family $\mathcal{F}$ of groups we write $\mathcal{F}_2$ for the family of groups that contain a group from $\mathcal{F}$ as a subgroup of index less than or equal to 2.

Theorem 4.3. *Let G be a group that admits a finitely $\mathcal{F}$-amenable action on a compact contractible finite-dimensional ANR $\overline{X}$. Then the following hold:*

(a) *G satisfies the K-theoretic Farrell–Jones conjecture relative to $\mathcal{F}$;*
(b) *G satisfies the L-theoretic Farrell–Jones conjecture relative to $\mathcal{F}_2$.*

Proof. We start by recalling [BL12a, Definition 1.5] where a metric space Y is called *controlled N-dominated* if for every $\varepsilon > 0$ there is a finite CW-complex K of dimension at most N, maps $i\colon Y \to K$, $p\colon K \to Y$ and a homotopy $H\colon Y \times [0,1] \to Y$ between $p \circ i$ and id_Y such that the tracks $\{H(y,t) \mid t \in [0,1]\}$ are of diameter at most ε. Compact finite-dimensional ANRs have this property; this follows for example from [Bor67, Theorem 10.1, p. 122].

(a) In [BLR08b, Theorem 1.1] the K-theoretic Farrell–Jones conjecture relative to $\mathcal{F}$ is proven under very similar conditions. In this reference also an action of G on a compact metrizable space $\overline{X}$ is required. Assumption 1.4 in this reference is what we defined as finitely $\mathcal{F}$-amenability here. In this reference it is further assumed that $\overline{X}$ contains a simplicial complex X whose complement is a Z-set in $\overline{X}$. This further assumption is only used in the proof of [BLR08b, Lemma 6.9]. It is not hard to see that this Lemma also holds for controlled N-dominated metric spaces, compare [BL12a, Lemma 8.4]. Therefore, the Z-set assumption can be safely replaced with the assumption that $\overline{X}$ is a compact finite-dimensional ANR. Alternatively, part (a) can be deduced from [Weg12, Theorem 1.1]. The conditions given in this reference are more involved (using strong homotopy actions) and designed for more general situations, but can be checked to hold in our case.

(b) This follows from [BL12a, Theorem 1.1(ii)]. The assumption in this reference is that G is *transfer reducible over* $\mathcal{F}$ [BL12a, Definition 1.8]. To check that this assumption is satisfied in our situation we use the action of G on $\overline{X}$. The covers appearing in [BL12a, Definition 1.8] exist because this action is finitely $\mathcal{F}$-amenable. □

Theorem 4.4. *Let $\mathcal{P}$ be a family of subgroups of G that contains all virtually cyclic subgroups. Suppose that G is relatively hyperbolic to subgroups $P_1, \dots, P_n \in \mathcal{P}$. Then the following hold:*

(a) *G satisfies the K-theoretic Farrell–Jones conjecture relative to $\mathcal{P}$;*
(b) *G satisfies the L-theoretic Farrell–Jones conjecture relative to $\mathcal{P}_2$.*

Proof. Let $\overline{P_{d,\Theta}}$ be the space from Definition A.9. By Corollary 3.21 this space carries a finitely $\mathcal{P}$-amenable action. It is a finite-dimensional compact contractible ANR for suitable d, Θ by Lemma A.11 and Theorem A.16. Now use Theorem 4.3. □

Remark 4.5. In Theorem 4.4 it suffices to assume that $\mathcal{P}$ contains all virtually cyclic subgroups of type I, instead of all virtually cyclic subgroup. The reason for this is that no virtually cyclic subgroups of type II appeared in the previous Sections, see Remark 2.18.

For L-theory this does not strengthen Theorem 4.4, since all virtually cyclic groups contain a virtually cyclic group of type I as a subgroup of index at most 2.

For K-theory this slightly strengthens Theorem 4.4. However, it is known that such a strengthening is always possible: in the K-theoretic Farrell–Jones conjecture only virtually cyclic subgroups of type I are needed [DQR11].

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Corollary 4.6. *Let G be relatively hyperbolic to subgroups $P_1, \dots, P_n$. Then the following hold:*

(a) *G satisfies the K-theoretic Farrell–Jones conjecture, provided that the P_i satisfies the K-theoretic Farrell–Jones conjecture;*
(b) *G satisfies the L-theoretic Farrell–Jones conjecture, provided that all subgroups of G that contain one of the P_i as a subgroup of index at most 2 satisfy the L-theoretic Farrell–Jones conjecture.*

Proof. This follows from Theorem 4.4 and the transitivity principle reviewed earlier in this section. □

Remark 4.7. Many classes of groups that are known to satisfy the Farrell–Jones conjecture are closed under finite index overgroups, but there is no general result to this effect. A good formalism to circumvent this difficulty is the Farrell–Jones conjecture with wreath products considered for example in [BLRR14, FR00].

A group G is said to satisfy the Farrell–Jones conjecture with wreath products relative to a family of subgroups $\mathcal{F}$ if for any finite group F the wreath product $G \wr F$ satisfies the Farrell–Jones conjecture relative to $\mathcal{F}$. This version of the conjecture passes to finite overgroups and to finite wreath products [BLRR14, Remark 6.2]. Moreover, the conditions that we verified for relatively hyperbolic groups in the proof of Theorem 4.4 can also be used to obtain results for the Farrell–Jones conjecture with wreath products [BLRR14, Theorem 5.1]. Combining this observation with the transitivity principle we obtain the following variant of Corollary 4.6. *Suppose that G is relatively hyperbolic to subgroups $P_1, \dots, P_n$ all of which satisfy the Farrell–Jones conjecture with wreath products, then G satisfies the Farrell–Jones conjecture with wreath products.*

Here it is no longer necessary to distinguish between K- and L-theory; everything in this remark applies as stated to the K-theoretic and the L-theoretic version of the conjecture.

Remark 4.8. If G is relatively hyperbolic to infinite subgroups $P_1, \dots, P_n$, then for the action of G on Δ each P_i fixes a unique point in Δ. In particular, no overgroups of the P_i have fixed points on the space $P_2(\Delta)$ of unordered pairs in Δ used in [BL12a, § 9]. It seems plausible that this observation together with a careful analysis of the arguments in [BL12a] can be used to show that the appearance of index-2 overgroups in Theorem 4.4(b) and Corollary 4.6(b) is not necessary.

We conclude this section with some examples where Corollary 4.6 applies.

Example 4.9. Let G be the fundamental group of a complete Riemannian manifold M of pinched negative curvature and finite volume. Then G is hyperbolic relative to virtually finitely generated nilpotent groups [Bow12, Far98]. Since virtually finitely generated nilpotent groups satisfy both the K- and L-theoretic Farrell–Jones conjecture [BFL14], the K- and L-theoretic Farrell–Jones conjecture for G holds.

Example 4.10. Let G be the fundamental group of a finite graph of groups with finite edge groups. Then the associated action of G on the Bass–Serre tree reveals G as relatively hyperbolic to the vertex groups. Thus if all vertex groups satisfy the K-theoretic Farrell–Jones conjecture then G satisfies the K-theoretic Farrell–Jones conjecture. If all overgroups of vertex groups of index at most 2 satisfy the L-theoretic Farrell–Jones conjecture then G satisfies the L-theoretic Farrell–Jones conjecture.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Example 4.11. Suppose that G acts cocompactly and properly discontinuously on a systolic complex with the Isolated Flats property. Then G is relatively hyperbolic to virtually finitely generated abelian subgroups [Els08]. Since virtually finitely generated abelian groups satisfy both the K- and L-theoretic Farrell–Jones conjecture [BFL14, Qui12], the K- and L-theoretic Farrell–Jones conjecture for G holds.

Acknowledgements
I thank Adam Mole for many discussions about relatively hyperbolic groups, Roman Sauer and Daniel Kasprowski for discussions about long and thin covers and packing methods, Brian Bowditch and François Dahmani for helpful emails about relatively hyperbolic groups and their boundaries. Comments by Svenja Knopf, Timm von Puttkamer and a referee greatly improved this paper. This work has been supported by the SFB 878 in Münster.

Appendix A. The relative Rips complex and the boundary

In this appendix we prove that Δ is finite dimensional and embeds into a finite-dimensional compact contractible *ANR* with a complement homeomorphic to a simplicial complex.[5] For the boundary of hyperbolic groups both these facts are well known [BM91]. For relative hyperbolic groups closely related results have been obtained by Dahmani [Dah03] and Mineyev and Yaman [MY]. Our treatment is very similar to the one in these references, but we do not require any assumptions on the parabolic subgroups.

Throughout this appendix we use again the notation from § 2. In particular, G is a relatively hyperbolic group exhibited by a cocompact action on the fine and hyperbolic graph Γ with finite edge stabilizers. Throughout this appendix we will make the following additional assumption on Γ: no two vertices from V_∞ are adjacent. This can be easily arranged for by replacing Γ with its first barycentric subdivision.

Δ_+ is finite dimensional

Theorem A.1. *The dimension of Δ_+ is finite.*

A very similar result by Dahmani is [Dah03, Lemma 3.7], whose proof we mostly copy.

Proof. For a vertex v of finite valency let $\partial_v\Gamma \subseteq \partial\Gamma$ consist of all $\xi \in \partial\Gamma$ for which there exists a $\Theta^{(3)}$-small geodesic from v to ξ. Let U be the union of all $\partial_v\Gamma$, where we vary v over all vertices of finite valency. If $\xi \in \partial\Gamma\backslash U$, then any geodesic from a vertex of finite valency to ξ will have a $\Theta^{(3)}$-large angle at infinitely many vertices. The finite union of finite-dimensional spaces is again finite dimensional, see [HW41, p. 28]. Therefore it suffices to show that V, U and $\partial\Gamma\backslash U$ are finite dimensional subspace of Δ_+.

We recall again the countable sum theorem [Pea75, Theorem 2.5, p. 125]: the countable union of closed subsets of dimension less than or equal to n is of dimension less than or equal to n. In particular, the countable subspace $V \subseteq \Delta_+$ is of dimension 0. The spaces $\partial_v\Gamma$ are finite dimensional by Lemma A.2 below with a uniform bound on their dimensions. As a consequence of Addendum 2.10 the $\partial_v\Gamma$ are closed in $\partial\Gamma$. Thus U is finite dimensional by the countable sum theorem.

[5] The complement will be the relative Rips complex minus its vertices of infinite valence. Thus the complement is not a subcomplex of the relative Rips complex, but it is homeomorphic to a simplicial complex.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

It remains to show that $\partial\Gamma\backslash U$ is finite dimensional. In fact, we will show that it is homeomorphic to a subset of the boundary of a tree T and therefore 0-dimensional. Fix a vertex v_0 of finite valency. The tree T is a maximal subtree of Γ and can be build inductively by choosing, for each vertex at distance n from v_0, an edge of Γ that connects it to a vertex at distance $n-1$ from v_0. For any $\xi \in \partial\Gamma$ the tree T will contain a geodesic from v_0 to ξ. (To construct such a geodesic pick a sequence of vertices v_n with $v_n \to \xi$ and apply Lemma 2.9(a) to geodesics c_n from v_0 to v_n in T.) The inclusion $T \to \Gamma$ induces a continuous surjective map $f\colon \Delta_+(T) \to \Delta_+$, where $\Delta_+(T)$ is the union of the vertices of T with ∂T and is also equipped with the observer topology. For any $\xi \in \partial\Gamma\backslash U$ there is a unique geodesic in T from v_0 to ξ. Indeed, any geodesic c in Γ from v_0 to ξ will have a $\Theta^{(3)}$-large angle at infinitely many vertices and any other geodesic c' in Γ from v_0 to ξ will, by Lemma 2.11, also pass through these vertices. Therefore $c = c'$ if both are in T. It follows that the restriction of f to the preimage of $\partial\Gamma\backslash U$ is bijective. We claim that, since $\Delta_+(T)$ is compact, this restriction of f is a homeomorphism. To prove this claim we need to show that the inverse of f on $\partial\Gamma\backslash U$ is continuous. Let $\xi_n \to \xi$ be a convergent sequence in $\partial\Gamma\backslash U$. Let ξ'_n, ξ' be the unique preimages in $\Delta_+(T)$ of the ξ_n, ξ. We need to show $\xi'_n \to \xi'$ in $\Delta_+(T)$. Assume this fails. Then, as $\Delta_+(T)$ is compact, there is a subsequence $I \subseteq \mathbb{N}$ with $\lim_{n\in I} \xi'_n = \xi'' \neq \xi'$ in $\Delta_+(T)$. By continuity of f, we have $f(\xi'') = \lim_{n\in I} f(\xi'_n) = \lim_{n\in I} \xi_n = \xi$. In particular, ξ'' belongs to the preimage of $\partial\Gamma\backslash U$ under f. Now we use that the restriction of f to this preimage is injective to contradict $\xi'' \neq \xi'$. Thus $\xi'_n \to \xi'$ in $\Delta_+(T)$. □

Lemma A.2. *There is N such that $\dim \partial_{v_0}\Gamma \leqslant N$ for all vertices v_0 of finite valency.*

Proof. Let V_{v_0} be the set of all vertices v of Γ for which there exists a $\Theta^{(3)}$-small geodesic from v_0 to v. Hyperbolicity of Γ implies that this is a quasi-convex subset of Γ: there is $r > 0$ such that for $v, v' \in V_{v_0}$ any geodesic between v and v' will be contained in the r-neighborhood of the union of geodesics from v_0 to v and v'. Consequently, V_{v_0} is hyperbolic in the metric d_{v_0} inherited as a subspace of Γ. Let Γ_{v_0} be the graph whose vertices are V_{v_0} and for which there is an edge between v and v' whenever $d_{v_0}(v, v') \leqslant 2r+1$. Of course, the metric $d_{\Gamma_{v_0}}$ induced by Γ_{v_0} on V_{v_0} satisfies $d_{v_0} \leqslant (2r+1)d_{\Gamma_{v_0}}$. Conversely, since any $\Theta^{(3)}$-small geodesic in Γ starting in v_0 defines also a path in Γ_{v_0}, we can as before apply hyperbolicity to any geodesic c in Γ between points $v, v' \in V_{v_0}$ and replace c with a path in Γ_{v_0} of length equal to the length of c. Thus $d_{\Gamma_{v_0}}$ and d_{v_0} are Lipschitz equivalent. In particular, Γ_{v_0} is hyperbolic.

Let $v, v' \in V_{v_0}$. Let Θ be a size for angles such that $3\Theta^{(3)} \subseteq \Theta$. Since v_0 is a vertex of finite valency we can, after enlarging Θ, assume that all angles at the vertex v_0 are Θ-small. We claim that any geodesic c in Γ between v and v' is Θ-small. If this fails, then for some internal vertex w of c the angle $\sphericalangle_w c$ is Θ-large. Let c_v and $c_{v'}$ be $\Theta^{(3)}$-small geodesics from v_0 to v and v'. We apply Lemma 2.12 to the geodesic triangle whose sides are c, c_v and $c_{v'}$. Since c_v and c'_v are $\Theta^{(3)}$-small we conclude that $w = v_0$. But our choice of Θ guarantees that $\sphericalangle_w c$ is Θ-small. Therefore c is Θ-small.

For $v \in V_{v_0}$ the ball $B_r(v)$ of radius r around v in V_{v_0} with respect to d_{v_0} is contained in the set of vertices of Γ that can be connected to v by a Θ-small geodesic of length r. If v is of finite valency, then this set is finite by Corollary 2.2. Moreover, as the action of G on the vertices of Γ is cofinite, the number of vertices in $B_r(v)$ is bounded by a number depending only on r. Any vertex $w \in V_{v_0}$ of infinite valency is adjacent to a vertex $v \in V_{v_0}$ of finite valency. (Consider the $\Theta^{(3)}$-small geodesic between w and v_0 and recall that we assumed that no two vertices of infinite valency are adjacent.) Since $B_r(w) \subseteq B_{r+1}(v)$ whenever $v, w \in V_{v_0}$ are adjacent, we can now conclude that V_{v_0} is uniformly proper with respect to d_{v_0} (or $d_{\Gamma_{v_0}}$): the number of elements

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

in a ball is bounded by a number only depending on the radius of the ball. The dimension of the boundary of a hyperbolic graph can be estimated in terms of the number of vertices in balls of a fixed radius (depending on the hyperbolicity constant); this is a standard fact (see for example [BLR08a, Proof of Proposition 9.3(ii)]). Consequently, the boundary of Γ_{v_0} is finite dimensional. Since this boundary agrees with $\partial_{v_0}\Gamma$ it follows that $\partial_{v_0}\Gamma$ is finite dimensional. The action of G on the set of vertices of finite valency is cocompact. It follows that the maximum of the dimension of the $\partial_{v_0}\Gamma$ is finite. □

The relative Rips complex

We will say that a geodesic is (d,Θ)-small if it is Θ-small and of length at most d.

Definition A.3. Let Θ be a size for angles and $d > 0$. The relative Rips complex $P_{d,\Theta}$ of Γ has V as the set of vertices. A finite set σ of vertices spans a simplex for $P_{d,\Theta}$ if and only if between any two vertices $v, v' \in \sigma$ there exists a (d,Θ)-small geodesic.

Lemma A.4. *The relative Rips complex $P_{d,\Theta}$ is finite dimensional. For $n \geqslant k \geqslant 1$ each k-simplex is contained in only finitely many n-simplices.*

Proof. Let $v \in V$ and e be an edge incident to v. Because of Lemma 2.1 there are only finitely many vertices v' for which there exists a (d,Θ)-small geodesic from v to v' whose first edge is e. Since G acts cocompactly on the set of edges the number of such geodesics is uniformly bounded.

Fix $v' \neq v$ such that there exists a Θ-small geodesic c between v and v'. Let $W(v,v')$ be the set of vertices w for which there exist (d,Θ)-small geodesics, c_w between w and v and c'_w between w and v'. We need to show that the number of elements in $W(v,v')$ is uniformly bounded. Let e be the initial edge of c, starting at v. Let f be the initial edge of c_w, starting at v. We claim that (e,f) is $\Theta + 2\Theta^{(3)}$ small. Lemma 2.1 implies then that the number of such f is uniformly bounded and the first paragraph of this proof implies then that the cardinality of $W(v,v')$ is uniformly bounded.

It remains to prove that (e,f) is $\Theta + 2\Theta^{(3)}$ small. If c'_w does not meet v, then (e,f) is $\Theta^{(3)}$ small. Otherwise $\sphericalangle_v c'_w$ is Θ-small and Lemma 2.6 implies then that (e,f) is $\Theta + 2\Theta^{(3)}$-small. □

In the next statement the positive integer δ will be again a hyperbolicity constant for Γ.

Proposition A.5. *Assume that $d \geqslant 4\delta$ and $7\Theta^{(3)} \subseteq \Theta$. Then $P_{d,\Theta}$ is contractible. More precisely the following holds.*

Let K be a finite subcomplex of $P_{d,\Theta}$. Let L be the subcomplex of $P_{d,\Theta}$ spanned by all vertices $\tilde{v}$ for which there are vertices $v, v' \in K$ such that $\tilde{v}$ belongs to a geodesic between v and v'. Then K is contractible within L.

Proof. Let V_K be the set of vertices of K. Fix $v_0 \in V_K$. Let $\alpha := \max_{v \in V_K} d_\Gamma(v_0, v)$. Let a be the number of vertices in V_K for which $d_\Gamma(v_0, v) = \alpha$. For $v \in V_K$ let $W(v)$ be the set of vertices $w \in V$ such that there exists a geodesic c from v_0 to v such that c passes through w such that $\sphericalangle_w c$ is $2\Theta^{(3)}$-large. Let $\beta := \max_{v \in V_K, w \in W(v)} d_\Gamma(v_0, w)$. Let b be the number of vertices $v \in V_K$ for which there is $w \in W(v)$ with $d_\Gamma(v_0, w) = \beta$. Clearly, $\alpha \geqslant \beta$. If $\alpha = 0$, then K consists of a single vertex and there is nothing to prove.

For the general case we will use induction on $(\alpha + \beta, a + b)$. We claim that there is a vertex $v \in V_K$ and a vertex $\tilde{v}$ on a geodesic from v_0 to v such that:

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

(A.6) there is a (d,Θ)-small geodesic from v to $\tilde{v}$;

(A.7) if there is a (d,Θ)-small geodesic from v to v', with $v' \in V_K$, then there is also a (d,Θ)-small geodesic from $\tilde{v}$ to v';

(A.8) either $d_\Gamma(v_0,v) = \alpha$ and $d_\Gamma(v_0,\tilde{v}) < \alpha$ or there is $w \in W(v)$ with $d_\Gamma(v_0,w) = \beta$, and $d_\Gamma(v_0,\tilde{w}) < \beta$ for all $\tilde{w} \in W(\tilde{v})$. (In fact, in the second case we will use $\tilde{v} = w$.)

Given this claim there is a homotopy in L that replaces v by $\tilde{v}$. The effect on $(\alpha+\beta, a+b)$ is then that we either reduce $\alpha+\beta$ or that we reduce $a+b$ but do not increase $\alpha+\beta$. This will complete the induction step modulo our claim.

To prove the claim, we distinguish two cases. In the first case we assume $\alpha \geqslant \beta + d$. Then choose $v \in V_K$ such that $d_\Gamma(v_0,v) = \alpha$. As δ is an integer, we can pick a vertex $\tilde{v}$ with $d_\Gamma(v,\tilde{v}) = 2\delta$ that belongs to a geodesic c between v_0 and v. Clearly (A.8) holds. The geodesic $c|_{[v,\tilde{v}]}$ is of length $2\delta \leqslant d$. Let w be an internal vertex of $c|_{[v,\tilde{v}]}$. If $\sphericalangle_w c$ is Θ-large, then, since $2\Theta^{(3)} \subseteq \Theta$, $w \in W(v)$ and $\beta \geqslant d_\Gamma(v_0,w) > d_\Gamma(v_0,\tilde{v}) \geqslant \alpha - 2\delta \geqslant \alpha - d$. Since this contradicts $\alpha \geqslant \beta + d$, the angle $\sphericalangle_w c$ at w is Θ-small. Thus $c|_{[v,\tilde{v}]}$ is Θ-small and (A.6) holds. In fact, this argument proves that $c|_{[v,\tilde{v}]}$ is $2\Theta^{(3)}$-small. To prove (A.7) consider $v' \in V_K$ such that there exists a (d,Θ)-small geodesic c' between v and v'. Hyperbolicity of Γ implies that there is $w \in V$ with $d_\Gamma(\tilde{v},w) \leqslant \delta$ such that w belongs to a geodesic between v_0 and v' or to a geodesic between v and v'. In the first case $d_\Gamma(w,v') = d_\Gamma(v_0,v') - d_\Gamma(w,v_0) \leqslant d_\Gamma(v_0,v') - (d_\Gamma(v_0,\tilde{v}) - d_\Gamma(w,\tilde{v})) \leqslant \alpha - (\alpha - 2\delta - \delta) = 3\delta \leqslant d - \delta$. In the second case $d_\Gamma(w,v') = d_\Gamma(v,v') - d_\Gamma(w,v) \leqslant d_\Gamma(v,v') - (d_\Gamma(v,\tilde{v}) - d_\Gamma(w,\tilde{v})) \leqslant d - (2\delta - \delta) = d - \delta$. Thus $d_\Gamma(\tilde{v},v') \leqslant d$ in both cases. Let $\tilde{c}$ be a geodesic from v' to $\tilde{v}$. It remains to show that $\tilde{c}$ is Θ-small. Assume this fails. Then there is an internal vertex w of $\tilde{c}$ such that $\sphericalangle_w \tilde{c}$ is Θ-large. We can then apply Lemma 2.12 to the geodesics $c|_{[v,\tilde{v}]}$, c' and $\tilde{c}$. Since $c|_{[v,\tilde{v}]}$ is $2\Theta^{(3)}$-small and $2\Theta^{(3)} + 2\Theta^{(3)} \subseteq \Theta$, Lemma 2.12 implies $w \in c'$. In particular, $d_\Gamma(w,v) < d$ and therefore $d_\Gamma(v_0,w) \geqslant d_\Gamma(v_0,v) - d_\Gamma(w,v) > \alpha - d \geqslant \beta$. Let c'' be a geodesic from v' to v_0. Consider the geodesic triangle whose sides are $\tilde{c}$, c'' and $c|_{[v_0,\tilde{v}]}$. Since $\sphericalangle_w \tilde{c}$ is Θ-large, we can again apply Lemma 2.12 and deduce $w \in c'' \cup c|_{[v_0,\tilde{v}]}$. Moreover, since $2 \cdot 2\Theta^{(3)} + 3\Theta^{(3)} \subseteq \Theta$ we can combining this Lemma with Lemma 2.13 and deduce that the angle of c'' or of $c|_{[v_0,\tilde{v}]}$ at w will be $2\Theta^{(3)}$-large. In the first case $w \in W(v')$ in the second $w \in W(v)$. Both cases imply $d_\Gamma(w,v_0) \leqslant \beta$. But this is a contradiction since we proved earlier $d_\Gamma(v_0,w) > \beta$.

It remains to prove our claim under the assumption $\alpha < \beta + d$. If $\beta = 0$ then there are (d,Θ)-small geodesics between v_0 and any $v \in V_K$. Thus we can set $\tilde{v} := v_0$ if $\beta = 0$. So assume $\beta > 0$ and pick $v \in V_K$, a geodesic c between v_0 and v and an internal vertex $\tilde{v}$ of c such that $\sphericalangle_{\tilde{v}} c$ is $2\Theta^{(3)}$-large and $d_\Gamma(v_0,\tilde{v}) = \beta$. Clearly (A.8) holds. The geodesic $c|_{[v,\tilde{v}]}$ is of length less than or equal to $\alpha - \beta \leqslant d$. Let w be an internal vertex of $c|_{[v,\tilde{v}]}$. If $\sphericalangle_w c$ is Θ-large, then, since $2\Theta^{(3)} \subseteq \Theta$, $w \in W(v)$ and we obtain the contradiction $\beta \geqslant d_\Gamma(v_0,w) > d_\Gamma(v_0,\tilde{v}) = \beta$. Thus $c|_{[v,\tilde{v}]}$ is Θ-small and (A.6) holds. In fact, this argument proves that $c|_{[v,\tilde{v}]}$ is $2\Theta^{(3)}$-small. To prove (A.7) let $v' \in V_K$ such that there exists a (d,Θ)-small geodesic c' between v and v'. Let c'' be a geodesic between v_0 and v'. Consider the geodesic triangle whose sides are c, c' and c''. Since $\sphericalangle_{\tilde{v}} c$ is $2\Theta^{(3)}$-large, Lemma 2.12 implies $\tilde{v} \in c' \cup c''$. If $\tilde{v} \in c'$, then $c'|_{[v',\tilde{v}]}$ is a (d,Θ)-small geodesic between v' and $\tilde{v}$ and (A.7) holds. If $\tilde{v} \in c''$, then we use the restriction $c''|_{[v',\tilde{v}]}$. This restriction is Θ-small, since otherwise we could argue as in the proof of (A.6) and find $w \in W(v')$ with $d_\Gamma(v_0,w) > \beta$. Similarly, in this case $d_\Gamma(\tilde{v},v') = d_\Gamma(v',v_0) - d_\Gamma(\tilde{v},v_0) \leqslant \alpha - \beta \leqslant d$. □

From now on we will assume that d and Θ satisfy the assumptions of Proposition A.5.

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Definition A.9. Let $\overline{P_{d,\Theta}} := P_{d,\Theta} \cup \partial\Gamma$. For an open subset U of Δ_+ we define $P_{d,\Theta}(U) \subseteq P_{d,\Theta}$ as the subcomplex spanned by the vertices of Γ that belong to U. For $\xi \in \overline{P_{d,\Theta}}$ we define collections of subsets $\mathcal{N}_\xi$ of $\overline{P_{d,\Theta}}$ as follows.

(a) If $\xi \in P_{d,\Theta} \backslash V_\infty$, then $\mathcal{N}_\xi$ consists of all open neighborhoods W of ξ in $P_{d,\Theta} \backslash V_\infty$.

(b) If $\xi \in \partial\Gamma$, then $\mathcal{N}_\xi$ consists of all sets of the form $P_{d,\Theta}(U) \cup U$, where U is a neighborhood of ξ in Δ_+.

(c) If $\xi \in V_\infty$, then $\mathcal{N}_\xi$ consists of all sets of the form $P_{d,\Theta}(U) \cup U \cup W$, where U is a neighborhood of ξ in Δ_+, and W is a neighborhood of ξ in $P_{d,\Theta}$ (i.e. the intersection of W with any simplex of $P_{d,\Theta}$ is open in the simplex).

We will use the topology on $\overline{P_{d,\Theta}}$ which for $U \subseteq \overline{P_{d,\Theta}}$ is open if and only if for every $\xi \in U$ there is $N \in \mathcal{N}_\xi$ with $N \subseteq U$.

Lemma A.10. *Each $N \in \mathcal{N}_\xi$ is a (not necessarily open) neighborhood of ξ in $\overline{P_{d,\Theta}}$.*

Proof. If $\xi \in P_{d,\Theta} \backslash V_\infty$, then $\mathcal{N}_\xi$ consists of open subsets of $\overline{P_{d,\Theta}}$. Before we discuss the cases $\xi \in \partial\Gamma$ and $\xi \in V_\infty$ we point out a collection of open subsets of $\overline{P_{d,\Theta}}$. Consider $U' \subseteq \Delta_+$ open, $W' \subseteq P_{d,\Theta}$ open and assume that $P_{d,\Theta}(U') \subseteq W'$ and that $V_\infty \cap W' \subseteq U'$. It is not difficult to check that then $W' \cup U'$ is open in $\overline{P_{d,\Theta}}$.

Let $\xi \in \partial\Gamma$ and $N = P_{d,\Theta}(U) \cup U \in \mathcal{N}_\xi$. Lemma 2.19 implies that there exists a neighborhood U' of ξ in Δ_+ with the following property: if v is a vertex in Γ for which there exists a geodesic of length less than or equal to d starting in v and ending in U', then $v \in U$. It follows that $P_{d,\Theta}(U')$ is contained in the interior $P^\circ_{d,\Theta}(U)$ of $P_{d,\Theta}(U)$. Set now $W' := P^\circ_{d,\Theta}(U) \backslash (V_\infty \backslash U')$, i.e. we remove all vertices of infinite valency from $P_{d,\Theta}(U)$ that do not belong to U'. Then $W' \cup U'$ satisfies the conditions from the first paragraph of the proof and thus is open in $\overline{P_{d,\Theta}}$. Since $\xi \in W' \cup U' \subseteq N$, N is a neighborhood of ξ.

Let $\xi \in V_\infty$ and $N = P_{d,\Theta}(U) \cup U \cup W \in \mathcal{N}_\xi$. Addendum 2.20 implies that there exists a neighborhood U' of ξ in Δ_+ with the following property: if v is a vertex in Γ for which there exists a (d, Θ)-small geodesic starting in v and ending in $U' \backslash \xi$, then $v \in U$. It follows that $P_{d,\Theta}(U') \backslash \{\xi\}$ is contained in $P^\circ_{d,\Theta}(U)$. Thus $P_{d,\Theta}(U') \subseteq P^\circ_{d,\Theta}(U) \cup W$. Set $W' := (P^\circ_{d,\Theta}(U) \cup W) \backslash (V_\infty \backslash U')$. Again $W' \cup U'$ is open in $\overline{P_{d,\Theta}}$. Since $\xi \in W' \cup U' \subseteq N$, N is a neighborhood of ξ. □

Lemma A.10 implies that the inclusions $\Delta \to \overline{P_{d,\Theta}}$ and $P_{d,\Theta} \backslash V_\infty \to \overline{P_{d,\Theta}}$ are homeomorphisms onto their images. The canonical map $i\colon P_{d,\Theta} \to \overline{P_{d,\Theta}}$ is continuous, but i is not a homeomorphism onto its image (unless $V_\infty = \emptyset$). For the continuity of i it is important to include the sets W in the definition of the neighborhood basis $\mathcal{N}_\xi$ for $\xi \in V_\infty$, see Definition A.9(c).

Lemma A.11. *We have the following:*

(a) *the topology on $\overline{P_{d,\Theta}}$ is second countable;*
(b) *$\overline{P_{d,\Theta}}$ is compact and metrizable;*
(c) *$\overline{P_{d,\Theta}}$ is finite dimensional;*
(d) *for any $\xi \in \overline{P_{d,\Theta}}$, and every neighborhood U of ξ in $\overline{P_{d,\Theta}}$ there is a neighborhood U' of ξ in $\overline{P_{d,\Theta}}$ such that for any map $f\colon S^k \to i^{-1}(U')$ there is a map $\hat{f}\colon D^{k+1} \to i^{-1}(U)$ such that $\hat{f}|_{S^k} = f$.*

Proof. (a) Since G is countable and acts cocompactly on Γ there are only countably many simplices in $P_{d,\Theta}$. By Lemma A.4 every point in $P_{d,\Theta} \backslash V_\infty$ is contained in only finitely

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

many simplices. Therefore, $P_{d,\Theta}\backslash V_\infty$ is second countable and we need for a basis of the topology on $\overline{P_{d,\Theta}}$ only countable many open subsets as in part (a) from Definition A.9. Similarly, since Δ is second countable we need only countable many open subsets as in part (b). Finally, for part (c), since V_∞ is countable, we need to argue that each $\xi \in V_\infty$ has a countable neighborhood basis. It suffices to vary U over a countable neighborhood basis of $\xi \in \Delta$. Addendum 2.20 implies that for each such U almost all vertices of Γ that are incident to ξ (in $P_{d,\Theta}$) are contained in U. Using Lemma A.4 it follows that all but finitely many of the simplices of $P_{d,\Theta}$ that contain ξ are contained in $P_{d,\Theta}(U)$. Therefore, for each U it suffices to vary W over countable collection of neighborhoods of $\xi \in P_{d,\Theta}$.

(b) Let $\xi \in \Delta_+$, $v \in V_\infty$. Using Addendum 2.20 we find open neighborhoods U of ξ in Δ_+ and W of v in $P_{d,\Theta}$ such that $P_{d,\Theta}(U) \cap W = \emptyset$. Combining this observation with the fact that Δ_+ and $P_{d,\Theta}$ are Hausdorff we see that $\overline{P_{d,\Theta}}$ is also Hausdorff.

Since the topology on $\overline{P_{d,\Theta}}$ has a countable basis it suffices to prove sequential compactness. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\overline{P_{d,\Theta}}$. We will produce a convergent subsequence. If $x_n \in \partial\Gamma$ for infinitely many n, then we can use the compactness of Δ. Therefore we assume that $x_n \in P_{d,\Theta}$ for all n. Since $P_{d,\Theta}$ is finite dimensional we find k and vertices $v_{0,n}, \dots, v_{k,n}$ such that for each n the point x_n belongs to the simplex spanned by $v_{0,n}, \dots, v_{k,n}$. Since Δ_+ is compact we can assume that $\xi_j := \lim_n v_{j,n}$ exists in Δ_+ for each j. We now apply Lemma 2.19 and Addendum 2.20 to the $v_{0,n}, \dots, v_{k,n}$ as sequences in n. It follows that either the $\xi_j =: \xi$ all coincide or that we find a subsequence $I \subseteq \mathbb{N}$ such that for each j, $v_{j,n} =: w_j$ is constant in $n \in I$. In the first case $\lim_{n\in\mathbb{N}} x_n = \xi$. In the second case we find a subsequence of x_n that converges to a point on the simplex spanned by the w_j.

Since $\overline{P_{d,\Theta}}$ is compact and has a countable basis for the topology it is metrizable.

(c) This follows from Theorem A.1 and Lemma A.4 and the fact that the finite union of finite-dimensional spaces is finite dimensional [HW41, p. 28].

(d) If $\xi \in \overline{P_{d,\Theta}}\backslash\Delta = P_{d,\Theta}\backslash V_\infty$, then ξ has arbitrarily small contractible neighborhoods. If $\xi \in \Delta$, then we may assume that $P_{d,\Theta}(U_+) \subseteq U$ for some neighborhood U_+ of $\xi \in \Delta_+$. Using Lemma 2.21 we find a smaller neighborhood U'_+ of ξ such that all vertices on geodesics between points of U'_+ belong to U_+. Proposition A.5 implies that any finite subcomplex of $P_{d,\Theta}(U'_+)$ is contractible within $P_{d,\Theta}(U_+)$. If $\xi \in \partial\Gamma$, we can use the interior of $P_{d,\Theta}(U'_+) \cup U'_+$ for U'. If $\xi \in V_\infty$, then we pick in addition an neighborhood W' of ξ in $P_{d,\Theta}$ such that $W' \subseteq U$. Moreover, since $P_{d,\Theta}(U'_+)$ is a subcomplex of $P_{d,\Theta}$, we can assume that $W' \cap P_{d,\Theta}(U'_+)$ is a deformation retract of W'. Now we can choose the interior of $P_{d,\Theta}(U'_+) \cup U'_+ \cup W'$ for U'. □

Let K be a simplicial complex and $\mathcal{U}$ be a cover of a space X. A map $f\colon K' \to X$ defined on a subcomplex K' of K, containing the 0-skeleton $K^{(0)}$ of K, is said to be a *partial $\mathcal{U}$-realization* if for every simplex σ of K, there is a member of $\mathcal{U}$ that contains $f(K' \cap \sigma)$. If $K' = K$, then f is called a *full $\mathcal{U}$-realization*.

Lemma A.12. *Let $\mathcal{U}$ be an open cover of $\overline{P_{d,\Theta}}$ and $n \in \mathbb{N}$. Then there exists an open cover $\mathcal{U}'$ of $\overline{P_{d,\Theta}}$ such that for any finite simplicial complex K of dimension less than or equal to n, any partial $i^{-1}\mathcal{U}'$-realization of K in $P_{d,\Theta}$ extends to a full $i^{-1}\mathcal{U}$-realization $K \to P_{d,\Theta}$.*

Proof. Using Lemma A.11(d) we find a sequence of successively smaller covers $\mathcal{U} = \mathcal{U}_n, \dots, \mathcal{U}_0 = \mathcal{U}'$ such that for any $U' \in \mathcal{U}_k$ there is $U \in \mathcal{U}_{k+1}$ such that any map $S^k \to i^{-1}(U')$ extends to $D^{k+1} \to i^{-1}(U)$. Inductively, any partial $i^{-1}\mathcal{U}'$-realization can then be extended to a $i^{-1}\mathcal{U}$-realization $K \to P_{d,\Theta}$. □

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Let $\mathcal{U}$ be a cover of a space X. Maps $f, f' \colon Z \to X$ are said to be *$\mathcal{U}$-close* if for any $x \in Z$ there is $U \in \mathcal{U}$ containing both $f(x)$ and $f'(x)$. The *mesh* of a cover of a metric space is the supremum of the diameter of its members.

Lemma A.13. *For any open cover $\mathcal{W}$ of $\overline{P_{d,\Theta}}$ and any map $f \colon K \to \overline{P_{d,\Theta}}$ defined on a finite complex there is a map $f' \colon K \to P_{d,\Theta}$ such that f and $i \circ f'$ are $\mathcal{W}$-close.*

Proof. Pick a metric $d_{\overline{P}}$ for $\overline{P_{d,\Theta}}$ and $\varepsilon > 0$ such that 3ε is a Lebesgue number for $\mathcal{W}$. Let $\mathcal{U}$ be a cover of mesh less than or equal to ε. Pick $\mathcal{U}'$ as in Lemma A.12 with respect to $\mathcal{U}$ and $n := \dim K$. Pick $\delta > 0$ such that 3δ is a Lebesgue number for $\mathcal{U}'$ and $\delta < \varepsilon$. Now subdivide K until the diameter of the image of each simplex in $\overline{P_{d,\Theta}}$ is at most δ. Since $i(P_{d,\Theta})$ is dense in $\overline{P_{d,\Theta}}$ we find $f_0' \colon K^{(0)} \to P_{d,\Theta}$ defined on the 0-skeleton $K^{(0)}$ of K such that $d_{\overline{P}}(i \circ f_0'(x), f(x)) \leqslant \delta$ for any $x \in K^{(0)}$. It follows that f_0' is a partial $i^{-1}\mathcal{U}'$ realization and therefore extends to a full $i^{-1}\mathcal{U}$-realization $f' \colon K \to P_{d,\Theta}$. By construction for any $x \in K$, $d_{\overline{P}}(f(x), f'(x)) \leqslant 2\delta + \varepsilon \leqslant 3\varepsilon$. Thus f and $i \circ f'$ are $\mathcal{W}$-close. □

Lemma A.14. *Let $f \colon K \to \overline{P_{d,\Theta}}$ be a map defined on a finite simplicial complex. Then there exists a homotopy $H \colon K \times (0,1] \to P_{d,\Theta}$ such that*

$$(x,t) \mapsto \begin{cases} f(x) & t = 0, \\ i(H(x,t)) & t > 0, \end{cases}$$

is a continuous homotopy $K \times [0,1] \to \overline{P_{d,\Theta}}$.

Proof. Let $d_{\overline{P}}$ be again a metric on $\overline{P_{d,\Theta}}$. Let $\mathcal{U}_l$ be a sequence of open covers of $\overline{P_{d,\Theta}}$ with $\operatorname{mesh} \mathcal{U}_l \to 0$. Let $\mathcal{U}_l'$ be a cover as in Lemma A.12 with respect to $\mathcal{U}_l$ and $n := \dim K + 1$. Let λ_l be a Lebesgue number for $\mathcal{U}_l$. Let ε_l be a sequence such that $\varepsilon_l + \varepsilon_{l+1} \leqslant \lambda_l$ for all l. Using Lemma A.13 we find maps $f_l' \colon K \to P_{d,\Theta}$ such that f and $i \circ f_l'$ are ε_l-close, i.e. $d_{\overline{P}}(f(x), i(f_l'(x))) < \varepsilon_l$ for all $x \in K$. For $t = 1/l$ set $H(x, 1/l) := f_l'(x)$. In order to define $H(x,t)$ for $1/l + 1 < t < 1/l$, we can view $f_{l+1}' \coprod f_l' \colon K \times \{1/l + 1, 1/l\} \to P_{d,\Theta}$ as a partial $\mathcal{U}_l'$-realization for an appropriate triangulation of $K \times [1/l + 1, 1/l]$ and use Lemma A.12. Since $\operatorname{mesh} \mathcal{U}_l \to 0$ the map H defined this way has the desired continuity property. □

Lemma A.15. *The space $\overline{P_{d,\Theta}}$ is locally n-connected for each n, i.e. for each $\xi \in \overline{P_{d,\Theta}}$ and each open neighborhood U of ξ, there is a smaller neighborhood U' of ξ such that any map $f \colon S^k \to U'$ with $k \leqslant n$ extends to a map $D^{k+1} \to U$.*

Proof. Pick U' as in Lemma A.11(d). Lemma A.14 implies that any map $f \colon S^k \to U'$ is homotopic to a map of the form $i \circ f'$ with $f' \colon S^k \to i^{-1}(U')$. Now f' extends to a map $D^{k+1} \to i^{-1}(U)$ by Lemma A.11(d) and this yields the desired extension of f as well. □

Recall that we fixed d and Θ such that Proposition A.5 applies.

Theorem A.16. *The space $\overline{P_{d,\Theta}}$ is a contractible ANR.*

Proof. From Lemmas A.11 and A.15 we know that $\overline{P_{d,\Theta}}$ is compact, finite dimensional and locally n-connected for any n. Since this is a characterization of finite-dimensional compact *ANR*s [Bor67, Theorem 10.3, p. 122], $\overline{P_{d,\Theta}}$ is an *ANR*. Lemma A.14 together with the contractibility of $P_{d,\Theta}$ (Proposition A.5) imply that $\overline{P_{d,\Theta}}$ is weakly contractible. Since finite-dimensional *ANR*s are retracts of simplicial complexes [Bor67, Theorem 10.1, p. 122] the contractibility of $\overline{P_{d,\Theta}}$ follows. □

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

Remark A.17. Bestvina and Mess [BM91] proved Theorem A.16 for hyperbolic groups and the argument in this section is closely modeled on their argument. For hyperbolic groups Bestvina and Mess showed moreover that the boundary is a Z-set in the analog of $\overline{P_{d,\Theta}}$, but the methods of this section do not give this stronger statement. The reason for this is, that the contractions from Proposition A.5 may not work in $\overline{P_{d,\Theta}}\backslash\Delta = P_{d,\Theta}\backslash V_\infty$. For example, if Γ is a tree and if we use for Θ only the trivial angles (e,e), then $P_{d,\Theta} = \Gamma$. Nevertheless, it seems likely that in many cases Δ_+ is a Z-set in $\overline{P_{d,\Theta}}$ for suitable d, Θ.

Appendix B. Extending open sets

Here we review a convenient procedure to extend open subset to open subsets of an ambient space. Let (X, d_X) be a metric space. Let $X_0 \subseteq X$ be a subspace. For an open subset U of X_0 we define $U^+ := \{x \in X \mid d_X(x, U) < d_X(x, X_0\backslash U)\}$. This construction has the following properties (compare [BL12b, Lemma 4.14]): U^+ is open; $X_0 \cap U^+ = U$; $(U \cap V)^+ = U^+ \cap V^+$. We record the following two consequences.

Lemma B.1. *For any open subset $U \subseteq X_0$, the open subset $U^+ \subseteq X$ satisfies $U = X_0 \cap U^+$ and $\partial_0 U = X_0 \cap \partial U^+$ where ∂_0 is the boundary in X_0 and ∂ is the boundary in X.*

Proof. We already know that U^+ is open and satisfies $U = X_0 \cap U^+$. This implies $\partial_0 U \subseteq X_0 \cap \partial U \subseteq X_0\backslash U$. Let V be the complement of the closure of U in X_0. Then V^+ is open and $V^+ \cap U^+ = \emptyset$. Thus $V^+ \cap \partial U^+ = \emptyset$. Thus $X_0 \cap \partial U^+ \subseteq X_0\backslash(U \cap V) = \partial_0 U$. □

Suppose now that G acts isometrically on X and that $X_0 \subseteq X$ is G-invariant.

Lemma B.2. *Let $\mathcal{F}$ be a family of subgroups of G. Let $\mathcal{U}$ be a G-invariant collection of $\mathcal{F}$-subsets of X_0. Then $\mathcal{U}^+ := \{U^+ \mid U \in \mathcal{U}\}$ is a G-invariant collection of $\mathcal{F}$-subsets of X extending the $U \in \mathcal{U}$. The order of $\mathcal{U}$ and $\mathcal{U}^+$ agree.*

Proof. We have $(gU)^+ = g(U^+)$ as the action is isometric. This implies that $\mathcal{U}^+$ consists of $\mathcal{F}$-subsets and that $\mathcal{U}^+$ is G-invariant. The equation $(U \cap V)^+ = U^+ \cap V^+$ implies that the orders of $\mathcal{U}$ and $\mathcal{U}^+$ agree. □

Appendix C. Ideal triangles are slim

Lemma C.1. *Let Γ be a δ-hyperbolic graph, i.e. all finite geodesic triangles are δ-slim. Then all geodesic triangles, including those with one or more corners on the boundary $\partial\Gamma$ are 5δ-slim.*

Proof. Let c, c' and c'' be the three sides of a geodesic triangles. Write ξ, ξ' and ξ'' for the three corners of the triangle, i.e. c is geodesic between ξ' and ξ'', c' is a geodesic between ξ and ξ'', and c'' is a geodesic between ξ and ξ'. We pick vertices $y \in c$, $y' \in c'$ and $y'' \in c''$, We write $c_{\xi'}$ for the restriction of c to a geodesic from y to ξ'. Similarly, we define $c_{\xi''}$, c'_ξ, $c'_{\xi''}$, c''_ξ and $c''_{\xi'}$. These six geodesics are asymptotic in three pairs. Thus we find $A > 0$ such that the Hausdorff distances between c'_ξ and c''_ξ, and between $c_{\xi'}$ and $c''_{\xi'}$, and between $c_{\xi''}$ and $c'_{\xi''}$ are all three less than or equal to A.

Let now v be a vertex on c. We pick a geodesic a of length at most A that starts on c' and ends on c''. If $\xi \in \partial\Gamma$, then we can assume in addition that $d_\Gamma(v, a) \geqslant 100(A + \delta)$. If ξ is a vertex of Γ then we assume that a is the constant geodesic at ξ. We pick a vertex x on a. Similarly,

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press

we pick geodesics a' and a'' with vertices $x' \in a'$ and $x'' \in a''$. We can assume that along c the vertex v is between $a' \cap c$ and $a'' \cap c$; if not then move a' towards ξ' and a'' towards ξ''. We also pick geodesics b from x' to x'', b' from x to x'', and b'' from x to x'.

Consider now the geodesic 4-gon whose four sides are b, and restrictions of a', c and a''. Since a' and a'' are either constant or far from v we can apply hyperbolicity and find a vertex $w_1 \in b$ with $d_\Gamma(v, w_1) \leqslant 2\delta$. Next we use the geodesic triangle with sides b, b' and b''. By hyperbolicity there is $w_2 \in b' \cup b''$ with $d_\Gamma(w_1, w_2) \leqslant \delta$. Without of loss of generality we assume $w_2 \in b'$. Now we consider the geodesic 4-gon whose four sides are b' and restrictions of a, c' and a''. Since a and a'' are either constant or far from v (and then also far from w_2) we can apply hyperbolicity and find a vertex $v' \in c'$ with $d_\Gamma(w_2, v') \leqslant 2\delta$. Altogether, $d_\Gamma(v, v') \leqslant 5\delta$ and $v' \in c' \cup c''$. □

References

ACG15 Y. Antolín, R. Coulon and G. Gandini, *Farrell–Jones via Dehn fillings*, Preprint (2015), arXiv:1510.08113v1.

BFL14 A. Bartels, F. T. Farrell and W. Lück, *The Farrell–Jones conjecture for cocompact lattices in virtually connected Lie groups*, J. Amer. Math. Soc. **27** (2014), 339–388.

BL12a A. Bartels and W. Lück, *The Borel conjecture for hyperbolic and CAT(0)-groups*, Ann. of Math. (2) **175** (2012), 631–689.

BL12b A. Bartels and W. Lück, *Geodesic flow for CAT(0)-groups*, Geom. Topol. **16** (2012), 1345–1391.

BLR08a A. Bartels, W. Lück and H. Reich, *Equivariant covers for hyperbolic groups*, Geom. Topol. **12** (2008), 1799–1882.

BLR08b A. Bartels, W. Lück and H. Reich, *The K-theoretic Farrell–Jones conjecture for hyperbolic groups*, Invent. Math. **172** (2008), 29–70.

BLR08c A. Bartels, W. Lück and H. Reich, *On the Farrell–Jones conjecture and its applications*, J. Topol. **1** (2008), 57–86.

BLRR14 A. Bartels, W. Lück, H. Reich and H. Rüping, *K- and L-theory of group rings over $GL_n(\mathbf{Z})$*, Publ. Math. Inst. Hautes Études Sci. **119** (2014), 97–125.

BR07 A. Bartels and H. Reich, *Coefficients for the Farrell–Jones conjecture*, Adv. Math. **209** (2007), 337–362.

Bas72 H. Bass, *The degree of polynomial growth of finitely generated nilpotent groups*, Proc. Lond. Math. Soc. (3) **25** (1972), 603–614.

BM91 M. Bestvina and G. Mess, *The boundary of negatively curved groups*, J. Amer. Math. Soc. **4** (1991), 469–481.

Bor67 K. Borsuk, *Theory of retracts*, Monografie Matematyczne, vol. 44 (Państwowe Wydawnictwo Naukowe, Warsaw, 1967).

Bow12 B. H. Bowditch, *Relatively hyperbolic groups*, Internat. J. Algebra Comput. **22** (2012), 1250016, 66.

BH99 M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Die Grundlehren der mathematischen Wissenschaften, vol. 319 (Springer, Berlin, 1999).

Dah03 F. Dahmani, *Classifying spaces and boundaries for relatively hyperbolic groups*, Proc. Lond. Math. Soc. (3) **86** (2003), 666–684.

DQR11 J. F. Davis, F. Quinn and H. Reich, *Algebraic K-theory over the infinite dihedral group: a controlled topology approach*, J. Topol. **4** (2011), 505–528.

Els08 T. Elsner, *Systolic groups with isolated flats*, Preprint (2008).

Far98 B. Farb, *Relatively hyperbolic groups*, Geom. Funct. Anal. **8** (1998), 810–840.

FJ86 F. T. Farrell and L. E. Jones, *K-theory and dynamics. I*, Ann. of Math. (2) **124** (1986), 531–569.

FJ93 F. T. Farrell and L. E. Jones, *Isomorphism conjectures in algebraic K-theory*, J. Amer. Math. Soc. **6** (1993), 249–297.

FR00 F. T. Farrell and S. K. Roushon, *The Whitehead groups of braid groups vanish*, Int. Math. Res. Not. IMRN **10** (2000), 515–526.

Gro87 M. Gromov, *Hyperbolic groups*, in *Essays in group theory* (Springer, New York, 1987), 75–263.

Gro93 M. Gromov, *Asymptotic invariants of infinite groups*, in *Geometric group theory, Vol. 2 (Sussex, 1991)* (Cambridge University Press, Cambridge, 1993), 1–295.

GM08 D. Groves and J. F. Manning, *Dehn filling in relatively hyperbolic groups*, Israel J. Math. **168** (2008), 317–429.

HW41 W. Hurewicz and H. Wallman, *Dimension theory*, Princeton Mathematical Series, vol. 4. (Princeton University Press, Princeton, NJ, 1941).

KR15 D. Kasprowski and H. Rüping, *Long and thin covers for cocompact flow spaces*, Preprint (2015), arXiv:1502.05001.

Lue10 W. Lück, *K- and L-theory of group rings*, in *Proceedings of the international congress of mathematicizans, Vol. II* (Hindustan Book Agency, New Delhi, 2010), 1071–1098.

LR05 W. Lück and H. Reich, *The Baum–Connes and the Farrell–Jones conjectures in K- and L-theory*, in *Handbook of K-theory, Vols. 1, 2* (Springer, Berlin, 2005), 703–842.

Min05 I. Mineyev, *Flows and joins of metric spaces*, Geom. Topol. **9** (2005), 403–482 (electronic).

MY I. Mineyev and A. Yaman, *Relative hyperbolicity and bounded cohomology*, Preprint, http://www.math.uiuc.edu/∼mineyev/math/art/rel-hyp.pdf.

Mol13 A. Mole, *Extending a metric on a simplicial complex*, Preprint (2013), arXiv:1309.0981.

MR13 A. Mole and H. Rüping, *Equivariant refinements*, Preprint (2013), arXiv:1308.2799.

Osi05 D. Osin, *Asymptotic dimension of relatively hyperbolic groups*, Int. Math. Res. Not. IMRN **35** (2005), 2143–2161.

Oza06a N. Ozawa, *Amenable actions and applications*, in *International congress of mathematicians, Vol. II* (European Mathematical Society, Zürich, 2006), 1563–1580.

Oza06b N. Ozawa, *Boundary amenability of relatively hyperbolic groups*, Topology Appl. **153** (2006), 2624–2630.

Pea75 A. R. Pears, *Dimension theory of general spaces* (Cambridge University Press, Cambridge, 1975).

Qui12 F. Quinn, *Algebraic K-theory over virtually abelian groups*, J. Pure Appl. Algebra **216** (2012), 170–183.

Sau09 R. Sauer, *Amenable covers, volume and L^2-Betti numbers of aspherical manifolds*, J. Reine Angew. Math. **636** (2009), 47–92.

SWZ14 G. Szabó, J. Wu and J. Zacharias, *Rohklin dimension for actions of residually finite groups*, Preprint (2014), arXiv:1408.6096.

Szc98 A. Szczepański, *Relatively hyperbolic groups*, Michigan Math. J. **45** (1998), 611–618.

Weg12 C. Wegner, *The K-theoretic Farrell–Jones conjecture for CAT(0)-groups*, Proc. Amer. Math. Soc. **140** (2012), 779–793.

A. Bartels a.bartels@wwu.de
Westfälische Wilhelms-Universität Münster, Mathematisches Institut,
Einsteinstr. 62, D-48149 Münster, Germany

https://doi.org/10.1112/S0010437X16008216 Published online by Cambridge University Press