ON THE ZEROS OF A CLASS OF CANONICAL PRODUCTS OF INTEGRAL ORDER

by N. A. BOWEN

(Received 8th April 1963)

1. Introduction

In (1) I obtained † an asymptotic formula for the number of zeros of an arbitrary canonical product $\Pi(z)$ of integral order but not of mean type, all of whose zeros lie on a single radius, from a knowledge of the asymptotic behaviour of (i) $\log|\Pi(z)|$ as $|z| = r \to \infty$ along another radius l, with certain side conditions. After proving the analogous theorem in which $\log|\Pi(z)|$ in (i) is replaced by $\mathscr{I}\{\log \Pi(z)\}$, I show in this note that, at a cost of replacing l by two radii l_1 and l_2, both of these theorems may be generalised to include a class of canonical products of integral order whose zeros lie along a *whole* line. In one of the resulting theorems ‡ (Theorem II) I find the asymptotic number of zeros on each half of the line of zeros; another theorem (Theorem III) includes a previous result of mine.§

2. Notation, Reference Formulae and Lemmas

In this paper (a_n), (b_n) denote non-decreasing sequences of positive numbers; $j(r) \geqq 0$, $k(r) \geqq 0$ ($j(0) = 0 = k(0)$) denote the numbers of a_n, b_n respectively in $|z| \leqq r$, and $n(r) \equiv j(r)+k(r)$; J, K are non-negative constants; $S(z, a, \gamma, q)$ is the canonical product of genus q defined by

$$S(z, a, \gamma, q) = \prod_{n=1}^{\infty}\left(1+\frac{ze^{i\gamma}}{a_n}\right)\exp\left\{-\frac{ze^{i\gamma}}{a_n}+\ldots+\frac{(-1)^q}{q}\left(\frac{ze^{i\gamma}}{a_n}\right)^q\right\},$$

and

$$P(z, a, b, \gamma, q) \equiv S(z, a, \gamma, q)S(z, b, -\gamma, q),$$

where γ is real.

The following formulae may be found useful for reference:

$$\mathscr{R}\{\log S(re^{i\alpha}, a, \gamma, q)\} = (-1)^q \int_0^{\infty} \frac{r^{q+1}\{t\cos\overline{q+1}\,\overline{\alpha+\gamma}+r\cos\overline{q\alpha+\gamma}\}j(t)dt}{t^{q+1}(t^2+r^2+2tr\cos\overline{\alpha+\gamma})},$$

$$\mathscr{I}\{\log S(re^{i\alpha}, a, \gamma, q)\} = (-1)^q \int_0^{\infty} \frac{r^{q+1}\{t\sin\overline{q+1}\,\overline{\alpha+\gamma}+r\sin\overline{q\alpha+\gamma}\}j(t)dt}{t^{q+1}(t^2+r^2+2tr\cos\overline{\alpha+\gamma})},$$

$$(\alpha+\gamma \neq \pi).$$

† (1), p. 299, Theorem 2, part (ii).
‡ For statements and proofs of the theorems, see sections **4**, **5**, **6**.
§ (1), p. 313, Theorem 3, part (ii).

https://doi.org/10.1017/S0013091500010890 Published online by Cambridge University Press

I suppose also that $V(t)$ is any function of the form

$$V(t) = (\log t)^{S_1}(\log_2 t)^{S_2} \dots (\log_m t)^{S_m}, \quad t \geqq t_0,$$

where the $S_u (u = 1, 2, \dots, m)$ are real and not all zero, and t_0 is chosen large enough to ensure that $V(t)$ is positive and monotonic.

Let p be a non-negative integer and let

$$h(z) \equiv (-1)^{p-1} z^p V(z) \text{ if } V(r) \downarrow 0 \text{ as } r \uparrow \infty, \qquad \dots\dots(1)$$

$$h(z) \equiv (-1)^p z^p V(z) \text{ if } V(r) \uparrow \infty \text{ as } r \uparrow \infty. \qquad \dots\dots(2)$$

3. We shall need the following results.

Lemma 1. *Let $\psi(z)$ be an analytic function of $z = re^{i\theta}$, regular for $|\arg z| < \pi$ and on the negative real axis with the possible exception of logarithmic singularities.*‡ *Suppose also that $\psi(z)$ is real on the positive real axis and that*

$$\text{(i)} \quad |\psi(z)| = o(r^{s-1}) \text{ as } |z| = r \to 0,$$

$$\text{(ii)} \quad \int_{-\pi}^{\pi} |\psi(re^{i\theta})| \, d\theta = o(r^s) \text{ as } r \to \infty,$$

where s is an integer.

Then for $|\arg z| < \pi$ we have

$$\text{(iii)} \quad \psi(z) = \frac{(-1)^{s+1}}{\pi} \int_A^\infty \frac{z^s \mathscr{I}\{\psi(te^{i\pi})\} dt}{t^s(t+z)} + O(|z|^{s-1}), \quad (z \neq 0)$$

where A is any positive constant.

Lemma 2. *If the indices* $(s-1)$, s *in* (i), (ii) *of Lemma* 1 *are replaced by* $(s-\frac{1}{2})$, $(s+\frac{1}{2})$ *respectively, then the analogue of* (iii) *is*

$$\psi(z) = \frac{(-1)^s}{\pi} \int_A^\infty \frac{z^{s+\frac{1}{2}} \mathscr{R}\{\psi(te^{i\pi})\} dt}{t^{s+\frac{1}{2}}(t+z)} + O(|z|^{s-\frac{1}{2}}), \quad (z \neq 0).$$

Proofs of Lemmas 1 and 2. Lemma 1 is obtained by considering the contour integral

$$\psi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{z^s \psi(\zeta) d\zeta}{\zeta^s(\zeta - z)},$$

where Γ is the contour formed by the radii $\arg z = \pm\pi$ joined by the circumferences of the circles $|\zeta| = A$, $|\zeta| = R$, and z lies within this contour. Lemma 2 is obtained in the same way,† the indices s in the contour integral being replaced by $(s+\frac{1}{2})$.

Lemma § **3. Case I.** *Let $V(r) \downarrow 0$ as $r \uparrow \infty$. Then from*

$$\log S(re^{i\alpha}, a, 0, q) \sim Jh(re^{i\alpha}) \quad (\alpha \text{ constant}, 0 \leqq \alpha < \pi) \qquad \dots\dots(3)$$

† By "logarithmic singularities," which do not arise in this paper but arose in (2) from which the analogous Lemma 2 below is taken, I mean the singularities of $\psi(z)$ at the zeroes of $p(z)$, say, where $\psi(z) \equiv \log p(z)$ and $p(z)$ is an arbitrary canonical product.

‡ For details see (2), p. 116.

§ (1), p. 299, Theorem 2, part (i).

with the $h(z)$ *of* (1) *follows the relation*

$$q = p - 1.$$

Case II. *Let* $V(r)\uparrow\infty$ *as* $r\uparrow\infty$. *Then from* (3) *with the* $h(z)$ *of* (2) *follows the relation*

$$q = p.$$

In both cases the asymptotic formula

$$j(r) \sim Jr^p \left| \frac{dV(r)}{d(\log r)} \right|$$

follows.

Lemma 4.

$$(a)\ \lim_{r\to\infty} |V(re^{i\theta})|/V(r) = 1.$$

$$(b)\ \mathscr{R}\{V(re^{i\theta})\} \sim V(r).$$

$$(c)\ \mathscr{I}\{V(re^{i\theta})\} \sim \theta \frac{dV(r)}{d(\log r)}.$$

4. Theorem I. *Let*

$$|\alpha| < \pi$$

and let the canonical product $S(z, a, 0, q)$ *of at most order* $(q+1)$, *convergent type, satisfy*

$$\mathscr{I}\{\log S(re^{i\alpha})\} \sim J\mathscr{I}\{h(re^{i\alpha})\} \quad (\alpha \text{ constant}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

with the $h(z)$ *of* (1) *if* $V(r)\downarrow 0$ *as* $r\uparrow\infty$ (*Case I*), *with the* $h(z)$ *of* (2) *if* $V(r)\uparrow\infty$ *as* $r\uparrow\infty$ (*Case II*).

Suppose also that

$$s\pi/p < |\alpha| < (s+1)\pi/p \quad (s \geqq 0 \text{ integral})$$

and

$$j(r) = o(r^{(s+1)\pi/|\alpha|}).$$

Then

$$q = p - 1 \ (\textit{Case I}), \quad q = p \ (\textit{Case II});$$

and, in both cases,

$$j(r) \sim Jr^p \left| \frac{dV(r)}{d(\log r)} \right|. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Proof of Theorem I. It is not necessary to give details as the proof follows the same lines as that of Theorem 2, part (ii), in (1), p. 299; we use here Lemma 1 to obtain (3) for $\alpha = 0$, and the result follows from Lemma 3.

Note on Theorem I. If, in Theorem I, the possibility $\sin p\alpha = 0$ were permitted, the corresponding right-hand side of (4) would be $o(r^p V(r)$, and consequently no precise information like (5) could be expected, since canonical products of the form $S(z, a, 0, q)$ with $Jr^p\left|\dfrac{dV(r)}{d(\log r)}\right|$ zeros would for *all* $J(\geqq 0)$ satisfy † $\mathscr{I}\{\log S(re^{i\alpha})\} = o(r^p V(r))$.

5. Theorem II. *Let*

$$\left.\begin{array}{ll} 0<\gamma<\pi & (\gamma \textit{ constant}) \\ |\alpha|<\pi-\gamma & (\alpha \textit{ constant}) \end{array}\right\} \quad \dots\dots(6)$$

and let the canonical product ‡ $P(z, a, b, \gamma, q)$ *of at most order* $(q+1)$, *convergent type, satisfy*

$$\mathscr{I}\{\log P(re^{i\alpha})\}\sim J\mathscr{I}\{h(re^{i(\alpha+\gamma)})\}+K\mathscr{I}\{h(re^{i(\alpha-\gamma)})\} \quad \dots\dots(7)$$

and

$$\mathscr{I}\{\log P(re^{-i\alpha})\}\sim J\mathscr{I}\{h(re^{i(-\alpha+\gamma)})\}+K\mathscr{I}\{h(re^{i(-\alpha-\gamma)})\} \quad \dots\dots(8)$$

where §

$$\sin p(\alpha+\gamma) \neq 0, \quad \sin p(\alpha-\gamma) \neq 0, \quad \dots\dots(9)$$

and the $h(z)$ *of* (1) *is used if* $V(r)\downarrow 0$ *as* $r\uparrow\infty$ (*Case I*), *the* $h(z)$ *of* (2) *is used if* $V(r)\uparrow\infty$ *as* $r\uparrow\infty$ (*Case II*).

Suppose also that

$$\left.\begin{array}{ll} s\pi/p<|\alpha|<(s+1)\pi/p & (s\geqq 0 \textit{ integral}) \\ \textit{with} \quad n(r) = o(r^{(s+1)\pi/|\alpha|}), & \end{array}\right\} \quad \dots\dots(10)$$

and **either**

$$\left.\begin{array}{l} 0<\gamma<\pi/2p \\ \textit{with} \quad n(r) = o(r^{\pi/2\gamma}) \end{array}\right\} \quad \dots\dots(11a)$$

or

$$\left.\begin{array}{ll} (2k-1)\pi/2p<\gamma<(2k+1)\pi/2p & (k>0 \textit{ integral}) \\ \textit{with} \quad n(r) = o(r^{(2k+1)\pi/2\gamma}) & \end{array}\right\} \quad \dots\dots(11b)$$

Then

$$q = p-1 \ (\textit{Case I}), \quad q = p \ (\textit{Case II}); \quad \dots\dots(12)$$

and, in both cases

$$n(r)\sim(J+K)r^p\left|\frac{dV(r)}{d(\log r)}\right|. \quad \dots\dots(13)$$

If, in addition,

$$\sin p(\alpha+\gamma) \neq \sin p(\alpha-\gamma) \quad \dots\dots(14)$$

and, with $|\alpha|$ *and* γ *interchanged,* (10) *with either* (11a) *or* (11b) *holds, then*

$$j(r)\sim Jr^p\left|\frac{dV(r)}{d(\log r)}\right| \quad k(r)\sim Kr^p\left|\frac{dV(r)}{d(\log r)}\right| \quad \dots\dots(15)$$

† For the estimate for $\mathscr{I}\{\log S(re^{i\alpha})\}$ we should use (**1**), p. 298, Theorem 1, part (iv).
‡ Defined in section **2**.
§ The note at the end of section **4** explains the necessity of the assumption (9).

Proof of Theorem II, Case I. Setting

$$Q(z) \equiv P(z, a, b, \gamma, q)P(z, b, a, \gamma, q),$$

we observe that $Q(z)$ is regular in $|\arg z| < \pi-\gamma$, real on the real axis, and, by (7), (8), the definition (1) of $h(z)$, and the properties of $V(z)$ as listed in Lemma 4 of section 3, $Q(z)$ satisfies

$$\mathscr{I}\{Q(re^{i\alpha})\} \sim 2(-1)^{p-1}(J+K)r^p\Big\{V(r)\cos p\gamma \sin p\alpha - (\alpha\cos p\gamma\cos p\alpha - \gamma \sin p\gamma \sin p\alpha)\frac{dV}{d(\log r)}\Big\}. \quad \text{.........(16)}$$

Now if

$$p\alpha = s\pi, \text{ or } p\gamma = (2k-1)\pi/2, \text{ or both}, \quad \text{.....................(17)}$$

the right-hand side of (16) is $o(r^pV(r))$ and we can expect the method to give no precise information about $n(r)$.

If (17) does *not* hold, then (16) is equivalent to

$$\mathscr{I}\{Q(re^{i\alpha})\} \sim 2(-1)^{p-1}(J+K)r^pV(r)\cos p\gamma \sin p\alpha,$$

from which, on putting $\psi(z) \equiv \log Q(z^{\alpha/\pi})$, assuming (as we obviously may at this stage, since $Q(r)$ is real) that $\alpha > 0$, and using Lemma 1 of section 3, we find that

$$\log Q(r) \sim 2(-1)^{p-1}(J+K)r^pV(r)\cos p\gamma \quad \text{.....................(18)}$$

by the argument used in (1), pp. 309-310, the " order " requirements of the lemma following simply from (10).

Now

$$\log Q(r) \equiv 2(-1)^q \int_0^\infty \frac{r^{q+1}\{t\cos \overline{q+1}\,\gamma + r\cos q\gamma\}\{j(t)+k(t)\}dt}{t^{q+1}\,|\,t+re^{i\gamma}\,|^2}$$

$$= 2\log|P_1(re^{i\gamma})|,$$

where $P_1(z)$ is any canonical product of genus q, having only negative zeros, $n(r) = j(r)+k(r)$ in number. To $P_1(z)$ we now apply † Lemma 2 of section 3 with $\psi(z) \equiv \log P_1(z^{\gamma/\pi})$, using (11*a*) or (11*b*), and (18). We get

$$\log P_1(r) \sim (-1)^{p-1}(J+K)r^pV(r)$$

and hence, by Lemma 3 (Case I) of section 3,

$$q = p-1$$

and

$$n(r) \sim (J+K)r^p \left|\frac{dV(r)}{d(\log r)}\right| \quad \text{...........................(13)}$$

as required.

For the other part of Theorem II (Case I), we also have ‡

$$\sin p(\alpha+\gamma) - \sin p(\alpha-\gamma) \neq 0,$$

† For details, see proof of Theorem 2, part (ii), in (1), p. 299.

‡ We notice that, if $\sin p(\alpha+\gamma)$ were equal to $\sin p(\alpha-\gamma)$, the result (15) with specific J, K could not be expected, since the right-hand sides of (7) and (8) would involve $(J+K)$ and so be unaltered for an infinite choice of non-negative J, K having a constant sum.

that is,

$$\sin p\gamma \neq 0, \quad \cos p\alpha \neq 0. \qquad \text{(19)}$$

Let

$$I(0, \infty, \alpha, \gamma, h) \text{ denote } (-1)^q \int_0^\infty \frac{r^{q+1}\{t \sin \overline{q+1}\,\overline{\alpha+\gamma} + r \sin q\,\overline{\alpha+\gamma}\}h(t)dt}{t^{q+1}(t^2+r^2+2tr\cos\overline{\alpha+\gamma})}.$$

Then, on substituting from (13) into (7) we get †

$$I(0, \infty, \alpha, \gamma, j) - I(0, \infty, \alpha, -\gamma, j) \sim J\mathscr{I}\{h(re^{i(\alpha+\gamma)}) - h(re^{i(\alpha-\gamma)})\},$$

that is,

$$I(0, \infty, \gamma, \alpha, j) + I(0, \infty, \gamma, -\alpha, j) \sim J\mathscr{I}\{h(re^{i(\gamma+\alpha)}) + h(re^{i(\gamma-\alpha)})\}, \quad \text{...(20)}$$

and

$$I(0, \infty, -\gamma, \alpha, j) + I(0, \infty, -\gamma, -\alpha, j) \sim J\mathscr{I}\{h(re^{i(-\gamma+\alpha)}) + h(re^{i(-\gamma-\alpha)}\} \quad \text{...(21)}$$

follows similarly from (13) and (8).

Since (20) with (21) is a particular case of (7) with (8), with $|\alpha|$ and γ interchanged,‡ the argument used to prove the first part of the theorem here yields

$$j(r) \sim Jr^p \left| \frac{dV(r)}{d(\log r)} \right|$$

and hence, by (13),

$$\left.\begin{aligned} j(r) &\sim Jr^p \left| \frac{dV(r)}{d(\log r)} \right| \\ k(r) &\sim Kr^p \left| \frac{dV(r)}{d(\log r)} \right| \end{aligned}\right\}. \qquad \text{(15)}$$

Proof of Theorem II, Case II. The same method applies.

6. Theorem III. Let

$$0 < \gamma < \pi \qquad (\gamma \textit{ constant})$$

$$|\alpha| < \pi - \gamma \quad (\alpha \textit{ constant})$$

and let canonical product§ $P(z, a, b, \gamma, q)$ *of at most order* $(q+1)$, *convergent type, satisfy*

$$\mathscr{R}\{\log P(re^{i\alpha})\} \sim J\mathscr{R}\{h(re^{i(\alpha+\gamma)})\} + K\mathscr{R}\{h(re^{i(\alpha-\gamma)})\} \qquad \text{(22)}$$

and

$$\mathscr{R}\{\log P(re^{-i\alpha})\} \sim J\mathscr{R}\{h(re^{i(-\alpha+\gamma)})\} + K\mathscr{R}\{h(re^{i(-\alpha-\gamma)})\}$$

where ||

$$\cos p(\alpha+\gamma) \neq 0, \quad \cos p(\alpha-\gamma) \neq 0, \qquad \text{(23)}$$

and the $h(z)$ *of* (1) *is used if* $V(r)\downarrow 0$ *as* $r\uparrow\infty$ (*Case I*), *the* $h(z)$ *of* (2) *is used if* $V(r)\uparrow\infty$ *as* $r\uparrow\infty$ (*Case II*).

Suppose also that

either

$$0 < \gamma < \pi/2p$$

† Using (**1**), p. 298, Theorem 1, part (iv).
‡ The radii which now have to be excepted are those given by (19).
§ Defined in section **2**.
|| The note at the end of section **4** explains the necessity of the assumption (23).

with

$$n(r) = o(r^{\pi/2\gamma});$$

or

$$(2k-1)\pi/2p < \gamma < (2k+1)\pi/2p \quad (k > 0 \textit{ integral})$$

with

$$n(r) = o(r^{(2k+1)\pi/2\gamma});$$

and **either**

$$\alpha = 0;$$

or

$$0 < |\alpha| < \pi/2p$$

with

$$n(r) = o(r^{\pi/2|\alpha|});$$

or

$$(2s-1)\pi/2p < |\alpha| < (2s+1)\pi/2p \quad (s > 0 \textit{ integral})$$

with

$$n(r) = o(r^{(2s+1)\pi/2|\alpha|}).$$

Then

$$q = p-1 \ (\textit{Case I}), \quad q = p \ (\textit{Case II});$$

and, in both cases,

$$n(r) \sim (J+K)r^p \left| \frac{dV(r)}{d(\log r)} \right|.$$

Proof of Theorem III. This follows the same lines as the proof of the first part of Theorem II, except that here Lemma 1 of section **2** is not required, Lemma 2 being used for both of the main steps in the argument. In the special case † in which $\alpha = 0$, however, only one application of Lemma 2 is needed.

REFERENCES

(1) N. A. Bowen, *Proc. London Math. Soc.* (3), **12** (1962), 297-314.

(2) N. A. Bowen and A. J. Macintyre, *Trans. Amer. Math. Soc.*, **70** (1951), 114-126.

The University
Leicester

† If $\alpha = 0$ and $\gamma = \frac{1}{2}\pi$, then $p = 2k$ and hence (1), p. 313, Theorem 3, part (ii) is included in Theorem III above. Part (i) however is *not* given, since, by the properties of $V(z)$, the right-hand side of (22) here, for $\alpha = 0$, $\gamma = \frac{1}{2}\pi$ and *odd p*, is of magnitude $O\left(r^p \left| \frac{dV(r)}{d(\log r)} \right|\right)$ and not $O(r^p V(r))$ as in the theorem quoted.

https://doi.org/10.1017/S0013091500010890 Published online by Cambridge University Press