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SOME CONJECTURES FOR IMMANANTS

J. R. STEMBRIDGE

ABSTRACT. We present a series of conjectures for immanants, together with the supporting evidence we possess for them. The conjectures are loosely organized into three families. The first concerns inequalities involving the immanants of totally positive matrices (*i.e.*, real matrices with nonnegative minors). This includes, for example, the conjecture that immanants of totally positive matrices are nonnegative. The second family involves the immanants of Jacobi-Trudi matrices. These conjectures were suggested by a previous conjecture of Goulden and Jackson (recently proved by C. Greene) that the immanants of Jacobi-Trudi matrices are polynomials with nonnegative coefficients. The third family involves geometric and combinatorial structures associated with total positivity and paths in acyclic digraphs.

0. **Introduction.** Immanants are matrix functions analogous to determinants in which the sign character is replaced by any irreducible character of the symmetric group. (A more explicit definition is provided in §1.) In this paper, we present a series of conjectures for immanants, together with the supporting evidence we possess for them. The conjectures are loosely organized into families. The first of these families, discussed in §2, concern inequalities involving the immanants of totally positive matrices (*i.e.*, real matrices with nonnegative minors). Perhaps the most fundamental of these is the conjecture:

Immanants of totally positive matrices are nonnegative.

A second family, discussed in §4, involves the immanants of Jacobi-Trudi matrices. One of these conjectures, concerning the Schur-positivity of immanants of Jacobi-Trudi matrices, has already been proved by Mark Haiman [6]. We should also point out that the conjectures of §4 were inspired by the work of Ian Goulden and David Jackson [4], and in particular by their conjecture that the immanants of Jacobi-Trudi matrices are polynomials with nonnegative coefficients. (This conjecture was first proved by Curtis Greene [5].) In §5, we present two refinements of the conjectures of §4 suggested by the structure of Greene's proof of the Goulden-Jackson conjecture. In §6, we formulate the last of our immanant conjectures; this one is related to the lattice path methods of Gessel and Viennot.

In order to help the reader navigate through this complicated hierarchy of conjectures, we have included a chart in the Appendix (Figure 2) which summarizes their relationships. In this chart, stronger conjectures are positioned higher than weaker conjectures.

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1. **Preliminaries.**

1.1 *Characters.* A partition of n is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l)$ of positive integers with sum n. We will write $\ell(\lambda)$ for the length of λ and $|\lambda|$ for the sum of the terms of λ.

Recall that the irreducible characters of the symmetric group S_n are indexed by the partitions of n; we will write χ^λ for the character indexed by λ. Let $d_\lambda = \deg(\chi^\lambda) = \chi^\lambda(1)$. For further information, the standard reference is [8]. The conjugacy classes of S_n (*i.e.*, cycle-types) are also indexed by partitions; for any $w \in S_n$, we will write $\rho(w)$ for the partition formed by the cycle-lengths of w.

Recall that there is a standard Hermitian inner product on the space of $\mathbf{C}$-valued class functions on S_n (or any other finite group); namely,

$$\langle f, g\rangle = \frac{1}{n!}\sum_{w \in S_n} f(w)\bar{g}(w) = \sum_{\lambda}\frac{1}{z_\lambda} f(\lambda)\bar{g}(\lambda),$$

where $\bar{g}$ denotes complex conjugation, $f(\lambda)$ denotes the value of f common to the conjugacy class indexed by λ, and z_λ denotes the order of the S_n-centralizer of any $w \in S_n$ with $\rho(w) = \lambda$. It is well-known that the irreducible characters of S_n form an orthonormal basis of the space of class functions with respect to this inner product.

A useful class of permutation characters of S_n are the ones that arise from the action of S_n by left multiplication on the cosets of a Young subgroup. (A Young subgroup is any subgroup of S_n that is generated by transpositions.) Up to conjugacy, there is only one Young subgroup of S_n for each partition of n; the ones indexed by λ are isomorphic to $S_\lambda := S_{\lambda_1} \times S_{\lambda_2} \times \cdots$. We define η^λ to be the character of this action of S_n on S_n/S_λ; *i.e.*, the induction of the trivial character from S_λ to S_n.

It is well-known and easy to show that the η^λ's form a basis for the space of class functions on S_n. We will let ϕ^λ denote the dual basis of the η^λ's (with respect to $\langle\ ,\ \rangle$), so that

$$\langle \eta^\lambda, \phi^\mu\rangle = \delta_{\lambda,\mu}.$$

The ϕ^λ's are virtual characters of S_n; *i.e.*, differences between two characters. We include the following explicit description since it does not seem to be widely known. (It should be noted that Eğecioğlu and Remmel [2] have also given a description of the ϕ^λ's. Although it is expressed in different terms, it is equivalent to this one.)

PROPOSITION 1.1. *For any $w \in S_n$, let D_w denote the set of cycles of w, regarded as a directed graph. We have*

$$\phi^\lambda(w) = (-1)^{n-\ell(\lambda)}\operatorname{sgn}(w) r_\lambda(w),$$

where $r_\lambda(w)$ denotes the number of subgraphs of D_w isomorphic to a disjoint union of directed paths having (vertex) cardinality $\lambda_1, \ldots, \lambda_l$.

In particular, when $\lambda = (1^n)$, ϕ^λ is the sign character, and when $\lambda = (n)$, $\frac{1}{n}\phi^\lambda$ is the characteristic function of the class of n-cycles. Tables are provided in the Appendix.

We will postpone the proof of this proposition to the end of this section.

1.2 *Immanants.* Let $M_n(k)$ denote the algebra of $n \times n$ matrices over some field k of characteristic zero. For each irreducible character χ^λ of S_n there is a corresponding matrix function, known as an *immanant*, defined as follows:

$$\mathrm{Imm}_\lambda(A) = \sum_{w \in S_n} \chi^\lambda(w) a_{1,w(1)} \cdots a_{n,w(n)}, \quad A \in M_n(k).$$

Thus, when $\lambda = (1^n)$ (so that χ^λ is the sign character), we obtain the determinant, and when $\lambda = (n)$ (so that χ^λ is the trivial character), we obtain the permanent. We will refer to $\mathrm{Imm}_\lambda(A)/d_\lambda$ as a *normalized immanant*; these functions all take on unit value at the identity matrix.

The immanants of a matrix were probably first considered by Schur [17], although the name "immanant" seems to have been coined by D. E. Littlewood.

Another convenient way to think about immanants is as follows. Given $A \in M_n(k)$, let

$$[A] := \sum_{w \in S_n} a_{1,w(1)} \cdots a_{n,w(n)} \cdot w$$

denote the naturally corresponding element of the group algebra kS_n. If we regard χ^λ as a linear functional on kS_n, then we have $\mathrm{Imm}_\lambda(A) = \chi^\lambda[A]$. Since there is no need to insist that the functionals applied to $[A]$ be irreducible characters, we could define the χ-immanant of A to be $\chi[A]$ for any k-linear functional χ on kS_n. We will be especially interested in ϕ^λ-immanants; we call these *monomial immanants* for reasons that will become clear momentarily.

If χ is any irreducible character of some subgroup G of S_n (extended trivially to all of S_n by setting $\chi(w) = 0$ for $w \notin G$), then the corresponding operation $A \longmapsto \chi[A]$ is known in the literature as a "generalized matrix function." In general, such characters χ will be $\mathbf{C}$-valued, and so the matrix functions will usually also be $\mathbf{C}$-valued, unless A is, say, Hermitian. Since we are primarily concerned with real inequalities satisfied by non-Hermitian matrices, these types of immanants will not be especially relevant to this investigation.

One reason that immanants remain (deservedly) obscure is that there is very little known about them. Perhaps the oldest and best known result is Schur's Dominance Theorem [17]:

THEOREM 1.2. *If $A \in M_n(\mathbf{C})$ is Hermitian positive semi-definite (*HPSD*) and χ is an irreducible character of some subgroup G of S_n, then*

$$\chi[A] \geq \deg(\chi) \det(A).$$

Perhaps the primary reason that interest in immanants has remained alive over the years is Elliott Lieb's Permanental Dominance Conjecture [14]:

CONJECTURE 1.3. *If A, χ and G are as above, then*

$$\deg(\chi)\,\mathrm{per}(A) \geq \chi[A].$$

1.3 *Symmetric functions.* Let $\Lambda = \oplus_{n\geq 0}\Lambda^n$ denote the graded k-algebra of symmetric functions in the variables $x_1, x_2, \ldots$. Following the notation of [15], we let e_r, h_r, and p_r denote the elementary, complete homogeneous, and power-sum symmetric functions of degree r, respectively; *i.e.*,

$$\begin{aligned} e_r &= \sum x_{i_1}\ldots x_{i_r}, \quad 1 \leq i_1 < \cdots < i_r \\ h_r &= \sum x_{i_1}\ldots x_{i_r}, \quad 1 \leq i_1 \leq \cdots \leq i_r \\ p_r &= x_1^r + x_2^r + \cdots. \end{aligned}$$

By convention, $e_0 = h_0 = p_0 = 1$. For each partition λ, e_λ denotes the product $e_{\lambda_1}\cdots e_{\lambda_l}$, and a similar notation is used for h_λ and p_λ. It is well-known and easy to prove that the e_r's, h_r's, and p_r's are each algebraically independent generators of Λ, and so the e_λ's, h_λ's and p_λ's each form k-bases of Λ.

There are two other important bases of Λ: the monomial symmetric functions m_λ and the Schur functions s_λ. One may define m_λ to be the sum of all distinct monomials of the form $x_{i_1}^{\lambda_1}\cdots x_{i_l}^{\lambda_l}$ with $i_1, \ldots, i_l$ distinct. For the Schur functions, there are many possible definitions; the most convenient one for us will be as follows:

$$s_\lambda = \det[h_{\lambda_i - i + j}]_{1\leq i,j\leq l},$$

using the convention that $h_{-r} = 0$ for $r > 0$. It is more usual to define the Schur function as a bialternant or as a combinatorial generating function [15]. The above definition then becomes a theorem usually referred to as the Jacobi-Trudi identity.

Given any symmetric function f with real coefficients, we will say that f is *monomial-positive* (resp., *Schur-positive*) if the coefficients of f with respect to the basis m_λ (resp., s_λ) are nonnegative. It can be shown that

$$s_\lambda = \sum_{|\mu|=n} K_{\lambda,\mu} m_\mu, \tag{1}$$

where $K_{\lambda,\mu}$ denotes the number of column-strict tableaux of shape λ and content μ [15]. Consequently, Schur-positivity is stronger than monomial-positivity.

There is a standard k-linear isomorphism, known as the *characteristic map*, between the space of class functions on S_n and the subspace Λ^n of Λ. It is defined by setting

$$\mathrm{ch}(\chi) = \frac{1}{n!}\sum_{w\in S_n}\chi(w)p_{\rho(w)} = \sum_{|\lambda|=n}\frac{1}{z_\lambda}\chi(\lambda)p_\lambda \quad \in \Lambda^n$$

for any class-function χ on S_n. In these terms, it can be shown [15, I.7] that

$$\mathrm{ch}(\chi^\lambda) = s_\lambda,\ \ \mathrm{ch}(\eta^\lambda) = h_\lambda,\ \text{ and }\ \mathrm{ch}(\phi^\lambda) = m_\lambda. \tag{2}$$

The latter identity explains why we choose to call $\phi^\lambda[A]$ a "monomial immanant."

PROOF OF PROPOSITION 1.1. We have $\phi^\lambda(\mu) = z_\mu\langle\phi^\lambda, 1_\mu\rangle$, where 1_μ denotes the characteristic function of the conjugacy class of type μ. Since the η^λ's are dual to the ϕ^λ's, it follows that $z_\mu 1_\mu = \sum_\lambda \phi^\lambda(\mu)\eta^\lambda$. An application of the characteristic map therefore implies (*cf.* (2)):

$$p_\mu = \sum_\lambda \phi^\lambda(\mu)h_\lambda.$$

To complete the proof, we show by other means that $(-1)^{n-\ell(\lambda)}\operatorname{sgn}(w)r_\lambda(w)$ is the coefficient of h_λ in p_μ, where w denotes any permutation of cycle-type μ.

First consider the case $\mu = (n)$, so that w is an n-cycle. For this, we need to show that $(-1)^{\ell(\lambda)-1}r_\lambda(w)$ is the coefficient of h_λ in p_n. By [15, (2.10)], we have

$$\sum_{n\geq 1}\frac{p_n}{n}t^n = \log\Bigl(\sum_{r\geq 0}h_rt^r\Bigr) = \sum_{r\geq 1}\frac{(-1)^{r-1}}{r}\Bigl(\sum_{j\geq 1}h_jt^j\Bigr)^r,$$

so our task reduces to verifying that

$$r_\lambda(w) = \frac{n}{\ell(\lambda)}\cdot(\#\text{ of permutations of }\lambda).$$

To see this, note that a subgraph of D_w consisting of directed paths of size $\lambda_1, \lambda_2, \ldots$ can be recovered from knowledge of the starting element of one of the paths (one out of a choice of n possibilities), together with the permutation of λ formed by the path sizes one sees in the order they appear in w, starting from the chosen element. A given subgraph can be started in $\ell(\lambda)$ different places, thus yielding the claimed formula.

Now consider the general case in which w has cycle-type μ. Let $w = w_1\cdots w_l$ denote the cycle decomposition of w, so that w_i is a μ_i-cycle. The above analysis shows that

$$\begin{aligned} p_\mu &= \prod_{i=1}^{\ell(\mu)}\Bigl(\sum_{|\lambda|=\mu_i}(-1)^{\ell(\lambda)-1}r_\lambda(w_i)h_\lambda\Bigr) \\ &= \sum_\lambda(-1)^{n-\ell(\lambda)}\operatorname{sgn}(w)h_\lambda\sum r_{\lambda^1}(w_1)\cdots r_{\lambda^l}(w_l), \end{aligned}$$

where the inner sum ranges over all ways of distributing the parts of λ into sub-partitions $\lambda^1, \ldots, \lambda^l$ so that λ^i is a partition of μ_i. This inner sum is clearly $r_\lambda(w)$, as desired. ∎

2. **Conjectures involving total positivity.** A matrix $A \in M_n(\mathbf{R})$ is *totally positive* (TP) if all minors of A are nonnegative (*i.e.*, all submatrices have nonnegative determinant). More generally, A belongs to the class TP_l if all minors of size at most l are nonnegative. Matrices of this type arise frequently in probability theory, analysis, and combinatorics; the standard reference is [10].

We are now ready to state the first conjecture.

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CONJECTURE 2.1. *Monomial immanants of totally positive matrices are nonnegative; i.e.,*

$$A \in \mathrm{TP} \Rightarrow \phi^\lambda[A] \geq 0.$$

Since (1) and (2) imply that the irreducible characters χ^λ are nonnegative sums of the ϕ^μ's, we see that this conjecture is stronger than the conjecture we mentioned in the introduction; namely,

CONJECTURE 2.2. *Immanants of totally positive matrices are nonnegative; i.e.,*

$$A \in \mathrm{TP} \Rightarrow \chi^\lambda[A] \geq 0.$$

It is interesting to note that immanants (and generalized matrix functions) of totally positive matrices have been considered before—see the recent paper by Karlin and Rinott [12].

At first it might seem that Conjecture 2.2 is the more "natural" conjecture of the two (setting aside all supporting evidence for the stronger one), but there is an instructive example which suggests that Conjecture 2.1 is the "correct" one. To explain this example, let us define J_r to be the $r \times r$ matrix of 1's, and set

$$J_\lambda = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_l}$$

for any partition λ. One may easily check that $J_\lambda \in \mathrm{TP}$, and moreover,

PROPOSITION 2.3. *For any S_n-class function χ and partition λ of n, we have*

$$\chi[J_\lambda] = \lambda_1! \cdots \lambda_l! \, \langle \chi, \eta^\lambda \rangle.$$

PROOF. By Frobenius reciprocity, we have

$$\chi[J_\lambda] = \sum_{w \in S_\lambda} \chi(w) = |S_\lambda| \cdot \langle \chi \downarrow S_\lambda, \eta_\lambda \rangle_{S_\lambda} = \lambda_1! \cdots \lambda_l! \, \langle \chi, \eta_\lambda \uparrow S_n \rangle,$$

where η_λ denotes the trivial character of the Young subgroup S_λ. The induced character $\eta_\lambda \uparrow S_n$ equals η^λ by definition. ∎

Since the ϕ^λ's are dual to the η^λ's, this result shows that Conjecture 2.1 is the best possible inequality for class functions of S_n applied to TP matrices.

If χ and ϕ are class functions on S_r and S_{n-r}, respectively, we define

$$\chi \circ \phi := (\chi \times \phi) \uparrow S_n$$

to be the induction of their outer tensor product from $S_r \times S_{n-r}$ to S_n. It is well-known [15, I.7] that the image of this product under the characteristic map corresponds to ordinary multiplication in the ring of symmetric functions; *i.e.*,

$$\mathrm{ch}(\chi \circ \phi) = \mathrm{ch}(\chi)\,\mathrm{ch}(\phi). \tag{3}$$

Let $[n] = \{1, \ldots, n\}$. For any $I, J \subseteq [n]$, let $A(I|J)$ denote the submatrix of A obtained by selecting the rows indexed by I and the columns indexed by J, in their natural order. Let I^c denote the complement of I with respect to $[n]$.

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PROPOSITION 2.4. *If χ and ϕ are as above, then*

$$(\chi \circ \phi)[A] = \sum \chi[A(I|I)]\phi[A(I^c|I^c)],$$

summed over all r-subsets I of $[n]$.

PROOF. Fix a particular r-subset I of $[n]$, and regard $S_r \times S_{n-r}$ as the subgroup of S_n that leaves I invariant. Extend the domain of $\chi \times \phi$ to S_n by defining $(\chi \times \phi)(w) = 0$ for $w \notin S_r \times S_{n-r}$. By the usual formula for induction (*e.g.*, [1, (10.3)]), we have

$$(\chi \times \phi) \uparrow S_n = \frac{1}{r!(n-r)!} \sum_{w \in S_n} (\chi \times \phi)^w,$$

where $(\chi \times \phi)^w(z) := (\chi \times \phi)(wzw^{-1})$. However,

$$(\chi \times \phi)^w[A] = \chi[A(wI|wI)]\phi[A(wI^c|wI^c)],$$

where wI denotes the image of I under w. Since the stabilizer of I is of order $r!(n-r)!$, the claimed formula follows. ∎

COROLLARY 2.5. *If χ and ϕ are nonnegative functionals on* TP_l, *then so is $\chi \circ \phi$.*

Our next task will be to show that Conjecture 2.1 is valid for $\lambda = (21^{n-2})$. Our proof is based on the following inequality, which is a special case of a more general result due to Koteljanskiĭ [13].

LEMMA 2.6. *If $A \in \mathrm{TP}$ and $I \subseteq [n]$, then* $\det A(I|I) \det A(I^c|I^c) \geq \det A$.

The special case $I = [i]$ can be found in [10, p. 88]. For the sake of completeness, we include a proof which shows that the argument in [10] also applies to the general case.

PROOF. Proceed by induction on n. For $n \leq 2$, the result is straightforward. For $n > 2$, we may assume without loss of generality that $|I| \geq 2$ (otherwise, substitute I^c for I). As a further restriction, we may also assume that A is strictly totally positive (STP); *i.e.*, that all minors of A are positive. The justification for this is provided by the fact that the set STP is dense in TP [10, p. 88].

Now choose $i_0 \in I$, and define a matrix $B = [b_{ij}]$ of order $n-1$ via

$$b_{ij} = \det A(\{i, i_0\}|\{j, i_0\}).$$

We regard the index set for the rows and columns of B to be $\{i_0\}^c$. By Sylvester's Determinant Identity [10, p. 3], one has the following formula for any $r \times r$ minor of B:

$$\det B(J|K) = a_{i_0 i_0}^{r-1} \det A(J \cup \{i_0\}|K \cup \{i_0\}).$$

In particular, B is totally positive. By the induction hypothesis, it follows that

$$\det B(I - \{i_0\}|I - \{i_0\}) \det B(I^c|I^c) \geq \det B,$$

so by application of the Sylvester identity, this can be rewritten as

$$\det A(I|I)\det A(I^c\cup\{i_0\}|I^c\cup\{i_0\})\geq a_{i_0i_0}\det A.$$

By a second application of the induction hypothesis, we know that

$$a_{i_0i_0}\det A(I^c|I^c)\geq \det A(I^c\cup\{i_0\}|I^c\cup\{i_0\}).$$

Substituting this into the previous inequality, and canceling the (positive) factor $a_{i_0i_0}$ completes the induction. ∎

We are now ready to prove the case $\lambda=(21^{n-2})$ of Conjecture 2.1.

THEOREM 2.7. *If $A\in \mathrm{TP}$, then $\phi^{(21^{n-2})}[A]\geq 0$.*

PROOF. It is easy to see, directly from the definitions, that $m_{(21^{n-2})}=e_{n-1}e_1-ne_n$. Since $e_r=s_{(1^r)}=\mathrm{ch}(\varepsilon_r)$ (where ε_r denotes the sign character of S_r), it follows that (2) and (3) imply

$$\phi^{(21^{n-2})}=\varepsilon_1\circ\varepsilon_{n-1}-n\varepsilon_n.$$

By Proposition 2.4, we therefore have

$$\phi^{(21^{n-2})}[A]=\sum_{i=1}^{n}\Big(a_{ii}\cdot\det A(\{i\}^c|\{i\}^c)-\det A\Big). \tag{4}$$

Lemma 2.6 shows that each term in this sum is nonnegative. ∎

The following result shows that Conjecture 2.1 is true whenever λ is a "rectangle"; *i.e.*, a partition with all equal parts.

THEOREM 2.8. *If $\lambda=(r^l)$, then*

$$\phi^\lambda[A]=\sum\det A(I_1|I_2)\det A(I_2|I_3)\cdots\det A(I_r|I_1),$$

where the sum ranges over all ordered partitions $(I_1,\ldots,I_r)$ of $[n]$ into disjoint l-sets.

PROOF. The term indexed by $(I_1,\ldots,I_r)$ in the above sum can be expanded as a sum over r-tuples of maps $(w_1,\ldots,w_r)$ in which w_j is a bijection from I_j to I_{j+1} (and we identify I_{r+1} with I_1). Any such r-tuple collectively defines a permutation w of $[n]$ with cycle lengths divisible by r. The term indexed by $(w_1,\ldots,w_r)$ is clearly

$$\mathrm{sgn}(w_1)\cdots\mathrm{sgn}(w_r)a_{1,w(1)}\cdots a_{n,w(n)},$$

where $\mathrm{sgn}(w_j)$ denotes the sign of w_j relative to the natural orderings of I_j and I_{j+1}.

We further claim that

$$\mathrm{sgn}(w_1)\cdots\mathrm{sgn}(w_r)=(-1)^{l(r-1)}\mathrm{sgn}(w).$$

To see this, first consider the case in which w_j is the map which assigns the ith smallest element of I_j to the ith smallest of I_{j+1}. In this case, $\mathrm{sgn}(w_j)=1$ for all j, whereas w consists of l disjoint r-cycles, so the claim is true in this case. For the general case, consider the effect of transposing the w_j-image of two elements. This will change the sign of w_j, and at the same time, it will either merge two cycles of w into one cycle, or vice-versa. In particular, this changes the sign of w, and so the claim is preserved by these transpositions. Since every r-tuple of maps can be obtained by applying transpositions to the trivial r-tuple, the claim follows.

Now suppose that w is an arbitrary permutation of $[n]$. The number of r-tuples of maps $(w_1, \ldots, w_r)$ which give rise to w is precisely the quantity $r_\lambda(w)$ introduced in Proposition 1.1 (viz., r^k, if w consists of k disjoint cycles, each of length divisible by r; 0 otherwise). Thus, the right side of the claimed result takes the form

$$\sum_{w \in S_n} (-1)^{l(r-1)} \operatorname{sgn}(w) r_\lambda(w) a_{1,w(1)} \cdots a_{n,w(n)}.$$

Proposition 1.1 shows that this quantity is $\phi^\lambda[A]$. ∎

The following is a stronger version of Conjecture 2.1 that we are less sure of.

QUESTION 2.9. *Does $A \in \mathrm{TP}_l$ and $\ell(\lambda) \leq l$ imply $\phi^\lambda[A] \geq 0$?*

The evidence we have in favor of this is as follows. First, it is trivial for $l = 1$. Second, one can show that it is true for $\lambda = (21^{n-2})$. (Indeed, if A is TP_{n-1} but not TP, then $\det(A) < 0$, so the terms of (4) are obviously nonnegative.) Third, Theorem 2.8 shows that it is true for rectangular λ. Fourth, we have verified it by direct computation for the case $l = 2, n \leq 5$.

This last case deserves further comment. Density considerations similar to [10, p. 88] show that it suffices to consider matrices with all entries positive. Since inequalities of the form $\chi[A] \geq 0$ are unaffected by multiplication of the rows or columns of A by positive scalars, we may further assume that the first entry in every row and column of A is 1. Hence, the i,j entry of A will be of the form $b_{i1} b_{i2} \cdots b_{ij}$ for suitable positive b_{ij}'s with $b_{i1} = b_{1j} = 1$. The reason for expressing the terms of A in this strange form is that the property TP_2 is equivalent to having $b_{ij} \leq b_{i+1,j}$ for all $i < n$ and $j \leq n$. By means of the computer algebra package *Maple*, we have shown for each partition λ of size at most 5 and length 2 (*i.e.*, $\lambda = (11), (21), (31), (22), (41)$, or (32)) that $\phi^\lambda[A]$, as a polynomial function of $b_{i+1,j} - b_{ij}$, has nonnegative coefficients.

The evidence we have collected here shows that Conjecture 2.1 (and Question 2.9) is true for all matrices of order $n \leq 4$. For $n = 5$, the remaining open cases are $\lambda = (311)$ and (221). Further evidence in support of these conjectures will be given in §§4–6. This evidence will involve stronger results about various special types of TP matrices.

3. **The partial order of normalized immanants.** There has been a lot of attention devoted to the Permanental Dominance Conjecture and its generalizations. An example of recent progress in this area is Peter Heyfron's proof [7] of Russell Merris's conjecture [16, §4] that

$$\operatorname{per}(A) = d_0(A) \geq d_1(A) \geq \cdots \geq d_n(A) = \det(A), \quad A \in \mathrm{HPSD},$$

where $d_r(A)$ denotes the normalized immanant $\chi^\lambda[A]/d_\lambda$ corresponding to the hook partition $\lambda = (n-r, 1^r)$.

Similarly, if one looks at the normalized immanants of totally positive matrices, there seem to exist a number of universal inequalities that hold for all of TP. To formulate this observation more precisely, let us define a partial order $\geq_{\mathrm{TP}}$ on partitions of n via

$$\lambda \geq_{\mathrm{TP}} \mu \text{ iff } \chi^\lambda[A]/d_\lambda \geq \chi^\mu[A]/d_\mu \text{ for all } A \in \mathrm{TP}.$$

For example, as pointed out to us by Charles R. Johnson [9], one can easily show that

$$(n) \geq_{\mathrm{TP}} \lambda$$

for all partitions λ of n, *i.e.*, the analogue of the Permanental Dominance Conjecture is true for totally positive matrices. Indeed, the fact that $d_\lambda \geq |\chi^\lambda(w)|$ for all $w \in S_n$ shows that if A is any nonnegative matrix, then each term in the expansion of per(A) dominates the corresponding term of $\chi^\lambda[A]/d_\lambda$.

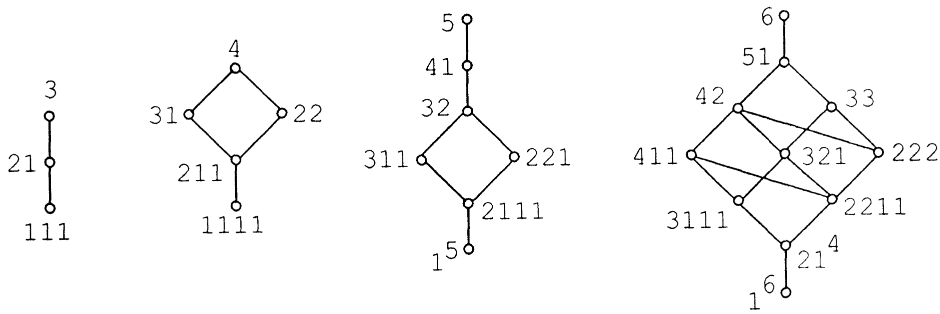

FIGURE 1. The partial order $\geq_\phi$.

Some useful information about the partial order $\geq_{\mathrm{TP}}$ can be deduced from Proposition 2.3. By choosing $A = J_\nu$, we see that $\lambda \geq_{\mathrm{TP}} \mu$ implies $\langle \chi^\lambda/d_\lambda, \eta^\nu\rangle \geq \langle \chi^\mu/d_\mu, \eta^\nu\rangle$ for all ν. Since the ϕ^ν's are dual to the η^ν's, this is equivalent to $\chi^\lambda/d_\lambda - \chi^\mu/d_\mu$ being a nonnegative linear combination of ϕ^ν's. In summary, if we define

$$\lambda \geq_\phi \mu \text{ iff } \chi^\lambda/d_\lambda - \chi^\mu/d_\mu = \text{a nonnegative sum of } \phi^\nu\text{'s},$$

then as a corollary of Proposition 2.3, we have

PROPOSITION 3.1. *$\lambda \geq_{\mathrm{TP}} \mu$ implies $\lambda \geq_\phi \mu$.*

Conjecture 2.1 is particularly interesting in this context, since it implies the converse.

We remark that the characteristic map can be used to provide an alternative description of $\geq_\phi$ in terms of Schur functions. Indeed, an immediate consequence of (2) is the fact that $\lambda \geq_\phi \mu$ if and only if $s_\lambda / d_\lambda - s_\mu / d_\mu$ is monomial-positive.

The Hasse diagram of $\geq_\phi$ for small n appears in Figure 1.

The analogue of Conjecture 2.1 for Hermitian positive semidefinite matrices is trivially false; *e.g.*, $\phi^3[3I - J_3] = -6$. Nevertheless, since $J_\lambda \in$ HPSD, we have $\lambda \geq_{\mathrm{PD}} \mu \Rightarrow \lambda \geq_\phi \mu$, where $\geq_{\mathrm{PD}}$ denotes the obvious analogue of $\geq_{\mathrm{TP}}$ for HPSD matrices.

QUESTION 3.2. *Does $\geq_\phi$ coincide with $\geq_{\mathrm{PD}}$?*

4. **Conjectures involving Jacobi-Trudi matrices.** Let H denote the (infinite) matrix of symmetric functions $[h_{j-i}]_{i,j\geq 1}$, with our usual convention that $h_{-r} = 0$ for $r > 0$. For each partition μ of length at most n (padded with trailing 0's, if necessary), define H_μ to be the $n \times n$ submatrix $[h_{\mu_i - i+j}]$, so that $s_\mu = \det H_\mu$. These submatrices of H are characterized by the fact that (1) they have no row or column of zeroes, and (2) they are selected from consecutive columns of H.

More generally, if we relax the condition on consecutive columns, then the resulting submatrices are of the form

$$H_{\mu/\nu} := [h_{\mu_i - \nu_j + j - i}]_{1 \leq i,j \leq n},$$

where μ and ν are any pair of partitions with $\ell(\nu) \leq \ell(\mu) \leq n$ and $\nu_i \leq \mu_i$. We call these *Jacobi-Trudi matrices*. The corresponding determinants

$$s_{\mu/\nu} := \det H_{\mu/\nu}$$

are called *skew* Schur functions, and are known to be Schur-positive, both by combinatorial and representation-theoretic methods [15].

CONJECTURE 4.1. *Monomial immanants of Jacobi-Trudi matrices are Schur-positive; i.e., for any partitions λ of n and θ of $|\mu| - |\nu|$, the coefficient of s_θ in $\phi^\lambda[H_{\mu/\nu}]$ is ≥ 0.*

Among the supporting evidence we have for this conjecture is the following. The fact that h_α and $s_{\mu/\nu}$ are known to be Schur-positive shows that it is true for $\lambda = (n)$ and $\lambda = (1^n)$. Second, Theorem 2.8 shows that it is true for rectangular λ. (Here we are making use of the fact that the Littlewood-Richardson rule [15, I.9] shows that products of Schur-positive functions are again Schur-positive.) Third, we have used *Maple* to verify it directly for $|\mu| \leq 12$. Fourth, in joint work with Richard Stanley [19], we have shown it to be true whenever μ/ν is a border-strip; *i.e.*, whenever $\mu_{i+1} - \nu_i = 1$ for all $i < n$.

We also remark that the case $\lambda = (21)$ is relatively easy. In fact, the Littlewood-Richardson rule can be used to show that when $A = H_{\mu/\nu}$ (and $n = 3$), each summand of (4) is Schur-positive (*cf.* Example 5.3 of [19]).

There are two natural ways to weaken Conjecture 4.1:

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CONJECTURE 4.2.
(a) Ordinary immanants of Jacobi-Trudi matrices are Schur-positive.
(b) Monomial immanants of Jacobi-Trudi matrices are monomial-positive.

A proof of Conjecture 4.2(a) by means of Kazhdan-Lusztig theory has been given by Mark Haiman [6]; the other conjecture is still open. It is worth repeating here what we mentioned in the introduction; namely, that it was Goulden and Jackson [4] who conjectured, and Greene [5] who first proved

THEOREM 4.3. *Ordinary immanants of Jacobi-Trudi matrices are monomial-positive.*

We remark that $H_{\mu/\nu} \in \mathrm{TP}$ whenever the x_i's are nonnegative (and real), so this result supports Conjecture 2.2. In §5, we will present evidence sufficient to imply the truth of Conjecture 4.2(b) for $n \leq 5$, thus providing further support for Conjectures 2.1 and 4.1. Additional support of this type will be given in §6.

There is another way to approach Conjecture 4.1 that leads to some interesting questions of combinatorics and representation theory. To describe this alternative, observe that

$$[H_{\mu/\nu}] = \sum_{w \in S_n} h_{\mu+\delta-w(\nu+\delta)} \cdot w,$$

where $\delta = (n-1, \ldots, 1, 0)$. Since the coefficient of s_θ in h_γ is the Kostka number $K_{\theta,\gamma}$ [15, (5.13)] (*cf.* also (1)), it follows that the coefficient of s_θ in $\phi^\lambda[H_{\mu/\nu}]$ is

$$\sum_{w \in S_n} K_{\theta,\mu+\delta-w(\nu+\delta)} \phi^\lambda(w). \tag{5}$$

Therefore, if we define a class function $\Gamma^\theta_{\mu/\nu}$ on S_n by setting

$$\Gamma^\theta_{\mu/\nu}(w) = \frac{n!}{|C(w)|} \sum_{w' \in C(w)} K_{\theta,\mu+\delta-w'(\nu+\delta)}, \tag{6}$$

where $C(w)$ denotes the conjugacy class of w, then the coefficient of s_θ in $\phi^\lambda[H_{\mu/\nu}]$ will be $\langle \phi^\lambda, \Gamma^\theta_{\mu/\nu} \rangle$. Bearing in mind that the dual basis η^λ of the ϕ^λ's are the permutation characters arising from the action of S_n on cosets of Young subgroups, the following is an equivalent reformulation of Conjecture 4.1:

CONJECTURE 4.1′. *$\Gamma^\theta_{\mu/\nu}$ is the character of a permutation representation of S_n of degree $n! \cdot K_{\theta,\mu-\nu}$ whose transitive components are isomorphic to the action of S_n on cosets of Young subgroups.*

The linearity of (5) with respect to ϕ^λ shows that the coefficient of s_θ in $\chi[H_{\mu/\nu}]$ is $\langle \chi, \Gamma^\theta_{\mu/\nu} \rangle$ for any class function χ. In particular, given that Haiman has proved Conjecture 4.2(a), we now know that $\langle \chi^\lambda, \Gamma^\theta_{\mu/\nu} \rangle \geq 0$; *i.e.*, $\Gamma^\theta_{\mu/\nu}$ is a character of S_n.

The characters $\Gamma^\theta_{\mu/\nu}$ corresponding to the single-rowed partition $\theta = (n)$ have an especially interesting combinatorial structure. Since $K_{(n),\gamma} = 1$ for any nonnegative sequence γ of sum n, it follows that the Kostka numbers in (6) are either 0 or 1, according to whether any component of $\mu+\delta-w'(\nu+\delta)$ is negative. This corresponds to whether the $(i,w'(i))$-entry of the matrix $H_{\mu/\nu}$ is zero for some i. It follows that for $\theta=(n)$, $\Gamma^{\theta}_{\mu/\nu}$ depends only on the pattern of zeroes in the matrix $H_{\mu/\nu}$. In general, these patterns will take the form of a Ferrers diagram (justified south and westward) that fits inside of the staircase diagram of δ. For example, if $n=5$, $\mu=(2^3 11)$, and $\nu=\emptyset$, then the pattern of zeroes is

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix},$$

which is the Ferrers diagram of $(3,2)$.

Therefore, given any partition β whose diagram fits inside of δ, let us write Γ_β as an abbreviation for the S_n-character $\Gamma^{(n)}_{\mu/\nu}$ corresponding to any Jacobi-Trudi matrix whose zero pattern is β. Upon application of the characteristic map, one finds

$$\operatorname{ch}(\Gamma_\beta)=\sum_{w\cap\beta=\emptyset} p_{\rho(w)},$$

where the notation $w\cap\beta=\emptyset$ serves to identify the set of $w\in S_n$ for which the nonzero entries of the corresponding permutation matrices avoid the zero pattern β. This shows that $\operatorname{ch}(\Gamma_\beta)$ can be viewed as the cycle-index of a set of permutations with restricted positions (*cf.* [18, 2.3–4]). Likewise, $h_\alpha=\operatorname{ch}(\eta^\alpha)$ is the cycle-index of the Young subgroup S_α divided by $\alpha_1!\,\alpha_2!\cdots$, so an interesting special case of Conjecture 4.1′ is

CONJECTURE 4.4. $\operatorname{ch}(\Gamma_\beta)$ *is a nonnegative (integer) linear combination of* h_α*'s. In other words, the cycle-index for permutations that avoid a given Ferrers pattern* β *is a nonnegative (rational) linear combination of cycle-indices of Young subgroups.*

QUESTION 4.5. *What is the combinatorial significance of the coefficient of* h_α *in* $\operatorname{ch}(\Gamma_\beta)$*?*

We remark that Conjecture 2.1 implies a more general result about permutations with restricted position. Indeed, if A is any totally positive 0–1 matrix, then Conjecture 2.1 would imply that the cycle-index for the set of permutation matrices that avoid the zeroes of A must be a nonnegative (integer) linear combination of h_α's.

We have used *Maple* to verify Conjecture 4.4 directly for $n\le 7$. Furthermore, in joint work with Richard Stanley [19], we have shown that Conjecture 4.4 is true whenever $n\ge\beta_1+\ell(\beta)$. We also provide an answer to Question 4.5 for this case.

5. **Conjectures involving cones.** For each pair of integers i,j with $1\le i<j\le n$, let us define $x_{ij}\in\mathbf{Z}S_n$ to be the sum of all permutations $w\in S_n$ that fix every $k\notin\{i,i+1,\ldots,j\}$. For example,

$$x_{24}=\mathrm{id}+(23)+(34)+(24)+(234)+(243).$$

Let Π denote the set of all finite products of x_{ij}'s, including the void product which equals the identity permutation.

There are two major steps involved in Greene's proof of Theorem 4.3. In the first step (which is actually due to Goulden and Jackson [4]), one shows that for any Jacobi-Trudi matrix $H_{\mu/\nu}$, there is a decomposition in $\Lambda \otimes \mathbf{Z}S_n$ (regarded as the group algebra of S_n with coefficients in Λ) of the form

$$[H_{\mu/\nu}] = \sum_{\pi \in \Pi} f_\pi \pi, \tag{7}$$

where each coefficient $f_\pi \in \Lambda$ is monomial-positive, and only finitely many f_π's are nonzero. Second, one shows that $\chi^\lambda(\pi) \geq 0$ for every $\pi \in \Pi$ and every irreducible character χ^λ. Greene proved this as a corollary of:

THEOREM 5.1 [5]. *In each irreducible representation of S_n, the Young seminormal form of the representing matrix for every x_{ij} is nonnegative.*[1]

Similarly, Conjecture 4.2(b) would be a corollary of (7) and the following:

CONJECTURE 5.2. $\phi^\lambda(\pi) \geq 0$ *for all* $\pi \in \Pi$.

To explain the evidence we have in favor of this conjecture, consider the cone $\mathcal{C}(\Pi)$ in $\mathbf{R}S_n$ spanned by nonnegative linear combinations of the elements $\pi \in \Pi$. It clearly suffices to verify $\phi^\lambda(\pi) \geq 0$ for the extreme rays of $\mathcal{C}(\Pi)$. For example, the identity

$$x_{12}x_{23}x_{12} = x_{13} + x_{12}$$

shows that the nonnegativity of ϕ^λ on x_{12} and x_{23} would imply the same for $x_{12}x_{23}x_{12}$.

We have used *Maple* to determine the extreme rays of $\mathcal{C}(\Pi)$ for $n \leq 5$, and to show moreover, that each ϕ^λ is nonnegative on these rays. Thus, Conjecture 5.2 is true for $n \leq 5$. It turns out that for $n \leq 4$, the cone $\mathcal{C}(\Pi)$ is simplicial (*i.e.*, the convex hull of $n!$ extreme rays), but for $n = 5$, it is the convex hull of $n! + 1$ extreme rays. Vaguely speaking, in this case there are two "exceptional" extreme rays; namely, $\pi = x_{24}x_{12}x_{35}x_{24}$ and $\pi' = x_{24}x_{13}x_{45}x_{24}$, in a position where only one might have been expected.

A list of the extreme rays of $\mathcal{C}(\Pi)$ for $n \leq 5$ is provided in Table 2 of the Appendix.

It is interesting to speculate on the extent to which the structure of Greene's proof might also be applicable to the conjectures of §2. To make this speculation more precise, let us define the *cone of total positivity* to be the cone in $\mathbf{R}S_n$ generated by all nonnegative linear combinations of the expressions $[A]$ for $A \in \mathrm{TP}$. Although I have very limited evidence to support it, I am willing to risk the following.

CONJECTURE 5.3. *The cone of total positivity is contained in* $\mathcal{C}(\Pi)$.

Since we know (Theorem 5.1) that the irreducible characters of S_n are nonnegative on Π, it follows that this conjecture implies Conjecture 2.2. Furthermore, this conjecture, together with Conjecture 5.2, would imply Conjecture 2.1.

[1] Haiman [6] has pointed out that this is also true for the Kazhdan-Lusztig representations of S_n.

QUESTION 5.4. *Is the cone of total positivity polyhedral (i.e., the convex hull of finitely many rays)? Do the extreme rays or supporting hyperplanes have a simple description?*

We remark that for $n \geq 4$, the element $x_{12}x_{24}x_{12}$ lies on an extreme ray of the cone $\mathcal{C}(\Pi)$ (*cf.* Table 2), yet there does not exist any 4×4 matrix A (totally positive or otherwise) such that $[A] = x_{12}x_{24}x_{12}$. This is a consequence of the fact in $x_{12}x_{24}x_{12}$, the number of $w \in S_n$ with nonzero coefficients is 20, whereas the corresponding number for $[A]$ can only be 24 or ≤ 18, depending on whether any entries of A are zero. Therefore, if Conjecture 5.3 is true, then the cone of total positivity must be strictly contained in $\mathcal{C}(\Pi)$ for $n \geq 4$. When $n \leq 3$, the two cones are equal.

6. **A digraph-path conjecture.** In this section, we present a conjecture that would generalize Theorem 4.3 in yet another direction. This conjecture is suggested by a standard method, due essentially to Gessel and Viennot [3], for interpreting certain types of determinants as generating functions for nonintersecting paths in a directed graph. This method applies in particular to Jacobi-Trudi matrices, thus yielding an interpretation of the Schur function $s_{\mu/\nu}$ (*i.e.*, the determinant of $H_{\mu/\nu}$) as a monomial-positive generating function. The nonintersecting paths that arise in this interpretation are well-known and easily shown to be in one-to-one correspondence with the column-strict tableaux of shape μ/ν.

First, we review the basic method. Our presentation follows [20].

Let $D = (V, E)$ be an acyclic directed graph, and choose an indeterminate z_e for each edge $e \in E$. Two (directed) paths in D are said to *intersect* if they share a common vertex. For any pair of vertices $u, v \in V$, let $\mathcal{P}(u, v)$ denote the set of paths from u to v. If $u = v$, then $\mathcal{P}(u, v)$ consists of a single path of length zero. For any such path P, we define the *weight* $\mathrm{wt}(P)$ to be $\prod_{e \in P} z_e$, and set

$$a(u, v) = \sum_{P \in \mathcal{P}(u,v)} \mathrm{wt}(P) \qquad \in \mathbf{Z}[z_e : e \in E].$$

In particular, $a(u, u) = 1$.

Given any pair $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ of n-tuples of vertices, let us define $\mathcal{P}(\mathbf{u}, \mathbf{v})$ to be the set of n-tuples of paths $\mathbf{p} = (P_1, \ldots, P_n)$ such that $P_i \in \mathcal{P}(u_i, v_i)$ for $i = 1, \ldots, n$. Extend the weight function multiplicatively to $\mathbf{p}$, so that $\mathrm{wt}(\mathbf{p}) = \mathrm{wt}(P_1) \cdots \mathrm{wt}(P_n)$. Let us also define

$$A(\mathbf{u}, \mathbf{v}) = [a(u_i, v_j)]_{1 \leq i,j \leq n}.$$

We say that $\mathbf{u}$ and $\mathbf{v}$ are *D-compatible* (following the language of [20]) if, whenever $i < j$ and $k > l$, every path $P \in \mathcal{P}(u_i, v_k)$ intersects every path $Q \in \mathcal{P}(u_j, v_l)$.

REMARK 6.1. It is well-known ([3], [4], [20]) that Jacobi-Trudi matrices are special cases of this construction. To demonstrate this, define a digraph D on the vertex set $\mathbf{Z}^2$, with an edge directed from u to v whenever $u - v = (1, 0)$ or $(0, 1)$. Specialize the indeterminates z_e by setting (1) $z_e = 1$ if e is a vertical step (*i.e.*, $u - v = (0,1)$), and (2) $z_e = x_j$ if e is a horizontal step at level j (*i.e.*, $u - v = (1,0)$ and $u = (i,j)$). Under these circumstances, it is easy to check that if $u = (j,m)$ and $v = (i,1)$, then $a(u,v) = h_{j-i}(x_1,\ldots,x_m)$. Thus, if we take $u_i = (\mu_i + n - i, m)$ and $v_i = (\nu_i + n - i, 1)$ for $i = 1,\ldots,n$, then $A(\mathbf{u},\mathbf{v})$ becomes the Jacobi-Trudi matrix $H_{\mu/\nu}$ in the variables $x_1,\ldots,x_m$. (It should also be noted that in this example, $\mathbf{u}$ and $\mathbf{v}$ are D-compatible—this is a consequence of the planarity of D.)

The following result is the fundamental theorem of the Gessel-Viennot method. For a proof, see Theorem 1 of [3], or Theorem 1.1 of [20] (*cf.* also [11] and [18, §2.7]).

THEOREM 6.2. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are D-compatible, then*

$$\det A(\mathbf{u},\mathbf{v}) = \sum_{\mathbf{p}} \mathrm{wt}(\mathbf{p}),$$

where $\mathbf{p}$ *ranges over all n-tuples of nonintersecting paths in* $\mathcal{P}(\mathbf{u},\mathbf{v})$.

Note that if $\mathbf{u}$ and $\mathbf{v}$ are D-compatible, then so are any pair of r-subsequences of $\mathbf{u}$ and $\mathbf{v}$. The preceding result therefore shows that that the minors of $A(\mathbf{u},\mathbf{v})$ are monomial-positive. In view of our previous conjectures on immanants of totally positive matrices, this suggests the following:

CONJECTURE 6.3. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are D-compatible, then monomial immanants of* $A(\mathbf{u},\mathbf{v})$ *are monomial-positive.*

Remark 6.1 shows that Conjecture 4.2(b) is a special case of this conjecture.

THEOREM 6.4. *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are D-compatible, then* $\phi^\lambda[A(\mathbf{u},\mathbf{v})]$ *is monomial-positive in the following cases:* $\lambda = (n)$, $(n-1,1)$, (r^J), (21^{n-2}), *and* (1^n).

PROOF. The case $\lambda = (n)$ is trivial, the case $\lambda = (1^n)$ is a corollary of Theorem 6.2, and the case $\lambda = (r^J)$ follows from Theorems 2.8 and 6.2.

For the case $\lambda = (21^{n-2})$, let $A_i(\mathbf{u},\mathbf{v})$ denote the submatrix obtained by deleting the ith row and column of A. Note that by Theorem 6.2, the expression $a(u_i,v_i)\cdot \det A_i(\mathbf{u},\mathbf{v})$ can be interpreted as the generating function for n-tuples $\mathbf{p} \in \mathcal{P}(\mathbf{u},\mathbf{v})$ such that only the ith path of $\mathbf{p}$ is allowed to intersect any of the other paths. It follows that the ith term in the expansion (*cf.* (4))

$$\phi^{(21^{n-2})}[A(\mathbf{u},\mathbf{v})] = \sum_{i=1}^{n}\Big(a(u_i,v_i)\cdot \det A_i(\mathbf{u},\mathbf{v}) - \det A(\mathbf{u},\mathbf{v})\Big)$$

is the generating function for such n-tuples in which the ith path *does* intersect one or more of the other paths. In particular, $\phi^{(21^{n-2})}[A(\mathbf{u},\mathbf{v})]$ is monomial-positive.

For the case $\lambda = (n-1,1)$, we henceforth assume $n \geq 3$. For any $w \in S_n$, let $\mathbf{v}^w$ denote $(v_{w(1)},\ldots,v_{w(n)})$, and define $\mathcal{P}_\mu$ to be the union of all $\mathcal{P}(\mathbf{u},\mathbf{v}^w)$'s, where w ranges over the permutations of cycle-type μ. By Proposition 1.1, we know that $\phi^{(n-1,1)}$ is nonzero only for the cycle-types $(n-1,1)$ and (n), where it takes on the values $n-1$ and $-n$, respectively. It follows that

$$\phi^{(n-1,1)}[A(\mathbf{u},\mathbf{v})] = (n-1) \sum_{\mathbf{p}\in\mathcal{P}_{(n-1,1)}} \mathrm{wt}(\mathbf{p}) - n \sum_{\mathbf{p}\in\mathcal{P}_{(n)}} \mathrm{wt}(\mathbf{p}).$$

To prove that this expression is monomial-positive, it suffices to construct a weight-preserving, n-valued map $\Psi\colon \mathcal{P}_{(n)} \longrightarrow \mathcal{P}_{(n-1,1)}$ with the property that $|\Psi^{-1}(\mathbf{p})| \le n-1$ for every $\mathbf{p} \in \mathcal{P}_{(n-1,1)}$.

To define Ψ, choose an n-cycle w, and let $\mathbf{p} = (P_1,\ldots,P_n) \in \mathcal{P}(\mathbf{u},\mathbf{v}^w)$. Assume that $w = (i_1,i_2,\ldots,i_n)$ in cycle notation, so that P_{i_1} is a path from u_{i_1} to v_{i_2}, P_{i_2} is a path from u_{i_2} to v_{i_3}, and so on. Let j denote the least index $(1\le j<n)$ such that P_{i_j} and $P_{i_{j+1}}$ intersect. To prove that j exists, let us first suppose that $i_1 < i_2$. (A similar argument applies when $i_1 > i_2$.) If P_{i_1} and P_{i_2} do not intersect, then the fact that $\mathbf{u}$ and $\mathbf{v}$ are D-compatible implies that $i_2 < i_3$. If P_{i_2} and P_{i_3} do not intersect, then we obtain $i_3 < i_4$. Continued failure to find an intersection thus leads to the condition $i_1 < i_2 < \cdots < i_n$. In that case, D-compatibility forces $P_{i_{n-1}}$ and P_{i_n} to intersect, since their respective endpoints (v_{i_n} and v_{i_1}) are in the opposite order.

Given j, we define an n-tuple of paths $\mathbf{q} = (Q_1,\ldots,Q_n)$ as follows. For $k \ne j, j+1$, we set $Q_{i_k} = P_{i_k}$. For the remaining two cases, let v denote the first vertex on $P_{i_{j+1}}$ that intersects P_{i_j}. Define Q_{i_j} and $Q_{i_{j+1}}$ to be the concatenations

$$\begin{aligned} Q_{i_j} &= P_{i_j}(u_{i_j} \longrightarrow v)P_{i_{j+1}}(v \longrightarrow v_{i_{j+2}}) \\ Q_{i_{j+1}} &= P_{i_{j+1}}(u_{i_{j+1}} \longrightarrow v)P_{i_j}(v \longrightarrow v_{i_{j+1}}), \end{aligned} \tag{8}$$

where the notation $P(u \longrightarrow v)$ indicates the segment of path P from u to v. Note that $\mathbf{q} \in \mathcal{P}(\mathbf{u},\mathbf{v}^{w'})$, where $w' = (i_{j+1})(i_1,\ldots,i_j,i_{j+2},\ldots,i_n)$ in cycle notation. Since there are n ways to put an n-cycle in cycle notation (*i.e.*, n choices for i_1), it follows that the assignment $\mathbf{p} \longmapsto \mathbf{q}$ defines an n-valued map Ψ from $\mathcal{P}_{(n)}$ to $\mathcal{P}_{(n-1,1)}$.

To complete the proof, we need to show that for any $\mathbf{q} \in \mathcal{P}_{(n-1,1)}$ there are at most $n-1$ choices of $\mathbf{p} \in \mathcal{P}_{(n)}$ such that $\mathbf{p} \longmapsto \mathbf{q}$. In fact, we will show that if we are given i_1 and $\mathbf{q}$, we can recover $\mathbf{p}$. Since i_1 must be part of the $(n-1)$-cycle associated with $\mathbf{q}$, it would follow that there are at most $n-1$ choices for i_1, and hence for $\mathbf{p}$, as desired.

For this, it suffices to show that we can recover j from knowledge of i_1 and $\mathbf{q}$. Indeed, if we know j, then we can recover w from w', we can recover v as the first vertex on $Q_{i_{j+1}}$ that intersects Q_{i_j}, and we can then recover P_{i_j} and $P_{i_{j+1}}$ from (8). To prove that we can recover j, first note that we can identify i_{j+1} as the unique fixed point of w'. We claim that $Q_{i_{j+1}}$ does not intersect $Q_{i_1},\ldots,Q_{i_{j-1}}$. Since Q_{i_j} *does* intersect $Q_{i_{j+1}}$, it would follow that j can be recovered by starting at i_1, following the $n-1$-cycle of w', and counting the number of paths until one is found that intersects $Q_{i_{j+1}}$. To prove the claim, observe that since adjacent pairs among $P_{i_1},\ldots,P_{i_j}$ do not intersect (by definition of j), the D-compatibility of $\mathbf{u}$ and $\mathbf{v}$ implies that no pair of paths in this list can intersect. This shows that the segment of $Q_{i_{j+1}}$ from v to $v_{i_{j+1}}$, which runs along P_{i_j}, cannot intersect $Q_{i_1},\ldots,Q_{i_{j-1}}$. Hence, if there is an intersection of $Q_{i_{j+1}}$ with $Q_{i_k} = P_{i_k}$ for some $k<j$, it must occur in the

segment of $Q_{i_{j+1}}$ from $u_{i_{j+1}}$ to v. However, by choice of v, this segment does not intersect P_{i_j}, so we could follow $Q_{i_{j+1}}$ from $u_{i_{j+1}}$ until it intersects P_{i_k}, and then along P_{i_k} to v_{i_k} without ever intersecting P_{i_j}. Since this would be a violation of D-compatibility, the claim follows. ∎

We remark that the cases covered by this theorem are sufficient to imply the validity of Conjecture 6.3 for $n \leq 4$.

It is interesting to note that Goulden and Jackson's proof of (7) is carried out in the language of digraph-paths, but unfortunately, it makes use of special properties of the digraphs for Jacobi-Trudi matrices that are not available in the general case. Nevertheless, we know of no counterexample that would rule out the possibility that the analogue of (7), namely that

$$[A(\mathbf{u}, \mathbf{v})] = \sum_{\pi \in \Pi} f_\pi \pi$$

for suitable monomial-positive coefficients $f_\pi \in \mathbf{Z}[z_e : e \in E]$, might be true for any D-compatible pair $\mathbf{u}, \mathbf{v}$. If so, then Theorem 5.1 would at least show that *ordinary* immanants of $A(\mathbf{u}, \mathbf{v})$ must be monomial-positive.

6.1 *Addendum.* Conjectures 2.2 and 5.3 have been proved. The details will appear in [21].

7. **Appendix.**

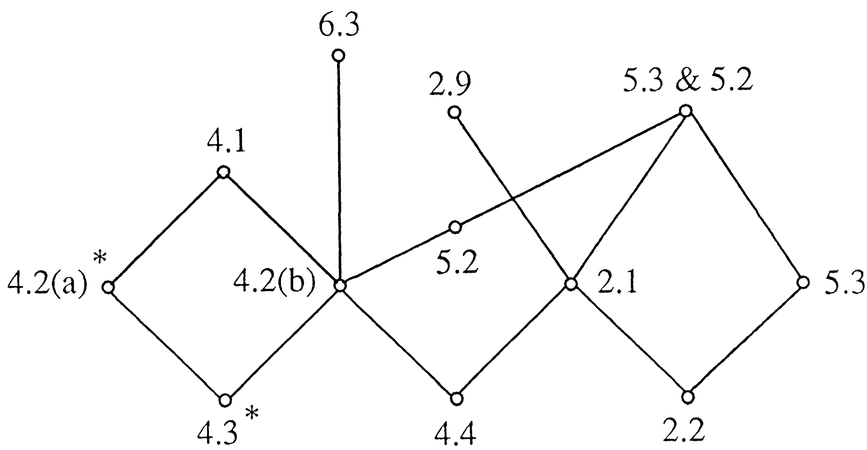

FIGURE 2. The partial order of immanant conjectures. (Theorems are marked with *'s.)

	11	2
2	0	2
11	1	−1

	1^3	21	3
3	0	0	3
21	0	2	−3
1^3	1	−1	1

	1^4	211	22	31	4
4	0	0	0	0	4
31	0	0	0	3	−4
22	0	0	4	0	−2
211	0	2	−4	−3	4
1^4	1	−1	1	1	−1

	1^5	21^3	221	311	32	41	5
5	0	0	0	0	0	0	5
41	0	0	0	0	0	4	−5
32	0	0	0	0	6	0	−5
311	0	0	0	3	−3	−4	5
221	0	0	4	0	−6	−2	5
21^3	0	2	−4	−3	5	4	−5
1^5	1	−1	1	1	−1	−1	1

	1^6	21^4	2211	2^3	31^3	321	33	411	42	51	6
6	0	0	0	0	0	0	0	0	0	0	6
51	0	0	0	0	0	0	0	0	0	5	−6
42	0	0	0	0	0	0	0	0	8	0	−6
411	0	0	0	0	0	0	0	4	−4	−5	6
33	0	0	0	0	0	0	9	0	0	0	−3
321	0	0	0	0	0	6	−18	0	−8	−5	12
31^3	0	0	0	0	3	−3	6	−4	4	5	−6
2^3	0	0	0	8	0	0	0	0	−4	0	2
2211	0	0	4	−12	0	−6	9	−2	10	5	−9
21^4	0	2	−4	6	−3	5	−6	4	−6	−5	6
1^6	1	−1	1	−1	1	−1	1	−1	1	1	−1

TABLE 1. The virtual characters ϕ^λ.

Note that $\phi^\lambda(\mu)$ is in the row indexed by λ and the column indexed by μ.

$n=2$	id	x_{12}				
$n=3$	id	x_{12}	x_{23}	x_{13}	$x_{12}x_{23}$	$x_{23}x_{12}$
$n=4$	id	x_{13}	$x_{12}x_{23}$	$x_{12}x_{24}$	$x_{13}x_{24}$	$x_{34}x_{23}x_{12}$
	x_{12}	x_{24}	$x_{23}x_{12}$	$x_{24}x_{12}$	$x_{24}x_{13}$	$x_{23}x_{12}x_{34}$
	x_{23}	x_{14}	$x_{23}x_{34}$	$x_{13}x_{34}$	$x_{12}x_{24}x_{12}$	$x_{12}x_{34}x_{23}$
	x_{34}	$x_{12}x_{34}$	$x_{34}x_{23}$	$x_{34}x_{13}$	$x_{12}x_{23}x_{34}$	$x_{23}x_{12}x_{34}x_{23}$
$n=5$	id	$x_{34}x_{23}$	$x_{14}x_{25}$	$x_{23}x_{34}x_{45}$	$x_{24}x_{12}x_{35}$	$x_{34}x_{23}x_{45}x_{34}$
	x_{12}	$x_{34}x_{45}$	$x_{25}x_{12}$	$x_{23}x_{45}x_{34}$	$x_{24}x_{45}x_{13}$	$x_{34}x_{45}x_{13}x_{34}$
	x_{23}	$x_{34}x_{13}$	$x_{25}x_{13}$	$x_{23}x_{35}x_{23}$	$x_{35}x_{23}x_{12}$	$x_{34}x_{45}x_{13}x_{24}$
	x_{34}	$x_{45}x_{34}$	$x_{25}x_{14}$	$x_{23}x_{35}x_{13}$	$x_{35}x_{13}x_{34}$	$x_{34}x_{13}x_{35}x_{23}$
	x_{45}	$x_{45}x_{13}$	$x_{12}x_{23}x_{34}$	$x_{34}x_{23}x_{12}$	$x_{35}x_{24}x_{12}$	$x_{45}x_{34}x_{23}x_{12}$
	x_{13}	$x_{45}x_{24}$	$x_{12}x_{23}x_{45}$	$x_{34}x_{23}x_{45}$	$x_{35}x_{24}x_{13}$	$x_{24}x_{12}x_{35}x_{23}$
	x_{24}	$x_{45}x_{14}$	$x_{12}x_{23}x_{35}$	$x_{34}x_{45}x_{13}$	$x_{12}x_{23}x_{34}x_{45}$	$x_{24}x_{12}x_{35}x_{24}$
	x_{35}	$x_{13}x_{34}$	$x_{12}x_{34}x_{23}$	$x_{34}x_{13}x_{35}$	$x_{12}x_{23}x_{45}x_{34}$	$x_{24}x_{45}x_{13}x_{34}$
	x_{14}	$x_{13}x_{24}$	$x_{12}x_{34}x_{45}$	$x_{45}x_{34}x_{23}$	$x_{12}x_{23}x_{35}x_{23}$	$x_{24}x_{45}x_{13}x_{24}$
	x_{25}	$x_{13}x_{35}$	$x_{12}x_{45}x_{34}$	$x_{45}x_{34}x_{13}$	$x_{12}x_{34}x_{23}x_{45}$	$x_{12}x_{23}x_{35}x_{23}x_{12}$
	x_{15}	$x_{13}x_{25}$	$x_{12}x_{45}x_{24}$	$x_{45}x_{13}x_{34}$	$x_{12}x_{45}x_{34}x_{23}$	$x_{12}x_{34}x_{23}x_{45}x_{34}$
	$x_{12}x_{23}$	$x_{24}x_{12}$	$x_{12}x_{24}x_{12}$	$x_{45}x_{13}x_{24}$	$x_{12}x_{45}x_{24}x_{12}$	$x_{23}x_{12}x_{34}x_{23}x_{45}$
	$x_{12}x_{34}$	$x_{24}x_{45}$	$x_{12}x_{24}x_{45}$	$x_{45}x_{24}x_{12}$	$x_{12}x_{24}x_{12}x_{45}$	$x_{23}x_{12}x_{45}x_{34}x_{23}$
	$x_{12}x_{45}$	$x_{24}x_{13}$	$x_{12}x_{24}x_{35}$	$x_{45}x_{24}x_{13}$	$x_{23}x_{12}x_{34}x_{23}$	$x_{34}x_{23}x_{12}x_{45}x_{34}$
	$x_{12}x_{24}$	$x_{24}x_{35}$	$x_{12}x_{35}x_{23}$	$x_{45}x_{14}x_{45}$	$x_{23}x_{12}x_{34}x_{45}$	$x_{23}x_{12}x_{34}x_{23}x_{45}x_{34}$
	$x_{12}x_{35}$	$x_{35}x_{23}$	$x_{12}x_{35}x_{24}$	$x_{45}x_{14}x_{35}$	$x_{23}x_{12}x_{45}x_{34}$	$x_{34}x_{23}x_{12}x_{45}x_{34}x_{23}$
	$x_{12}x_{25}$	$x_{35}x_{13}$	$x_{12}x_{25}x_{12}$	$x_{13}x_{34}x_{45}$	$x_{23}x_{12}x_{35}x_{23}$	
	$x_{23}x_{12}$	$x_{35}x_{24}$	$x_{12}x_{25}x_{13}$	$x_{13}x_{24}x_{45}$	$x_{23}x_{12}x_{35}x_{24}$	
	$x_{23}x_{34}$	$x_{35}x_{14}$	$x_{23}x_{12}x_{34}$	$x_{13}x_{24}x_{35}$	$x_{23}x_{35}x_{23}x_{12}$	
	$x_{23}x_{45}$	$x_{14}x_{45}$	$x_{23}x_{12}x_{45}$	$x_{13}x_{35}x_{23}$	$x_{23}x_{35}x_{13}x_{34}$	
	$x_{23}x_{35}$	$x_{14}x_{35}$	$x_{23}x_{12}x_{35}$	$x_{24}x_{12}x_{45}$	$x_{34}x_{23}x_{12}x_{45}$	

TABLE 2. The extreme rays of $C(\Pi)$.

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Department of Mathematics
University of Michigan
Ann Arbor, Michigan
U.S.A. 48109–1003