A CHARACTERIZATION OF INTRINSIC FUNCTIONS ON $\mathfrak{Q}_n$

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1. Introduction. Let $\mathfrak{A}$ be an associative algebra over the field $\mathscr{F}$ and let $\mathfrak{G}$ be the group of all automorphisms and anti-automorphisms of $\mathfrak{A}$ which leave $\mathscr{F}$ elementwise invariant. A function F with domain $\mathfrak{D}$ and range contained in $\mathfrak{A}$ is called an *intrinsic function* on $\mathfrak{D}$ if (i) $\Omega\mathfrak{D} = \mathfrak{D}$ for each Ω in $\mathfrak{G}$ and (ii) $F(\Omega Z) = \Omega F(Z)$ for every Z in $\mathfrak{D}$.

Rinehart (**5**) has introduced and motivated the study of the class of intrinsic functions on $\mathfrak{A}$, and has characterized these functions for the cases in which $\mathfrak{A}$ is the algebra $\mathfrak{Q}$ of real quaternions, the algebra $\mathscr{C}_n$ of $n \times n$ complex matrices, or the algebra $\mathscr{R}_n$ of $n \times n$ real matrices (**5**; **6**). The algebras listed above, along with the algebra $\mathfrak{Q}_n$ of $n \times n$ quaternion matrices, constitute the full list of possibilities for the simple direct summands of any semi-simple algebra over $\mathscr{R}$ or $\mathscr{C}$; see (**2**).

In (**2**), Cullen attempted to characterize intrinsic functions on $\mathfrak{Q}_n$, but, as pointed out in (**1**), there are some flaws in that characterization. Our aim in the present paper is to provide the above-mentioned characterization.

We denote the generators of $\mathfrak{Q}$ by $1, i_1, i_2, i_3$ ($i_1^2 = i_2^2 = -1, i_1 i_2 = -i_2 i_1 = i_3$) and do not distinguish between the real field $\mathscr{R}$ and the subfield of $\mathfrak{Q}$ generated by 1, nor do we distinguish between the complex field $\mathscr{C}$ and the subfield of $\mathfrak{Q}$ generated by 1 and i_1.

The usual notions of eigenvectors and eigenvalues (characteristic roots) for matrices over a field have been extended to $\mathfrak{Q}_n$ by Lee (**4**). Specifically, $\lambda \in \mathscr{C}$ is an eigenvalue of $A \in \mathfrak{Q}_n$ if there exists a non-zero $n \times 1$ quaternion matrix X (the eigenvector associated with λ) satisfying

$$AX = X\lambda.$$

Lee (**4**) has shown that the eigenvalues of A occur in conjugate pairs and are precisely the eigenvalues, in the classical sense, of the $2n \times 2n$ complex matrix

$$\phi(A) = \begin{bmatrix} A_1 & -\bar{A}_2 \\ A_2 & \bar{A}_1 \end{bmatrix}, \tag{1.1}$$

where A_1 and A_2 are the unique $n \times n$ complex matrices satisfying

$$A = A_1 + i_2 A_2.$$

The mapping ϕ, defined above, is known to be an isomorphism of $\mathfrak{Q}_n$ into $\mathscr{C}_{2n}$ (see **4**) and has been used by Wiegmann (**8**) to establish an analogue of the

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Jordan canonical form for $\mathfrak{Q}_n$. A corollary to Wiegmann's result asserts that for every $A \in \mathfrak{Q}_n$ there exists a non-singular matrix $P \in \mathfrak{Q}_n$ such that

$$P^{-1}AP = J = dg(J_1, J_2) \in \mathscr{C}_n \tag{1.2}$$

is in Jordan canonical form, where J_1 has only real elements and the eigenvalues of J_2 (the diagonal elements) all have positive imaginary parts (**1**). It follows from the results of Lee (**4**) mentioned above that the uniqueness of the eigenvalues of $\phi(A)$ implies the uniqueness of the eigenvalues of A. Thus, the canonical matrix J is uniquely determined up to rearrangements of the diagonal blocks of J_1 and of J_2. The eigenvalues of the complex matrix J will be called the *principal eigenvalues* of A and the characteristic polynomial of J will be called the *principal polynomial* of A. The *characteristic polynomial* of A is defined to be the characteristic polynomial of the $2n \times 2n$ complex matrix $dg(J, \bar{J})$; see (**8**).

2. The induced function. Since portions of the paper are concerned with continuity of functions, limits and neighbourhoods, it is necessary to define a topology on $\mathfrak{Q}_n$. For any $Z = (z_{ij})$ in $\mathfrak{Q}_n$ we define

$$||Z|| = \frac{1}{n} \max_{i,j} |z_{ij}|,$$

where $|z_{ij}| = (z_{ij}\bar{z}_{ij})^{1/2}$. With this definition, $\mathfrak{Q}_n$ becomes a normed ring, and hence a topological space with the topology induced by the norm.

Let F be an intrinsic function on $\mathfrak{Q}_n$ and let A be in the domain of F. From (**1**, Theorem 6) we know that $F(A) = L_A(A)$, where $L_A(x)$ is a uniquely determined real polynomial in x of degree less than the degree of the real minimum polynomial of A. If $\lambda \in \mathscr{C}$ is an eigenvalue of A, there exists an $n \times 1$ matrix $X \neq 0$ (with quaternion elements) such that $AX = X\lambda$ and

$$\begin{aligned} L_A(A)X &= (a_tA^t + a_{t-1}A^{t-1} + \ldots + a_1A + a_0I)X \\ &= a_tA^tX + a_{t-1}A^{t-1}X + \ldots + a_1AX + a_0X \\ &= X(a_t\lambda^t + a_{t-1}\lambda^{t-1} + \ldots + a_1\lambda + a_0) \\ &= X(L_A(\lambda)). \end{aligned}$$

Thus, $L_A(\lambda)$ is an eigenvalue of $L_A(A) = F(A)$. As remarked above, the eigenvalues of the quaternion matrix $A = A_1 + i_2A_2$ are precisely the eigenvalues of the complex matrix

$$\phi(A) = \begin{bmatrix} A_1 & -\bar{A}_2 \\ A_2 & \bar{A}_1 \end{bmatrix},$$

and, since ϕ is an isomorphism, we have that $L_A(\phi(A)) = \phi(L_A(A))$. It now follows from well-known theorems about the eigenvalues of polynomial functions of complex matrices that the $2n$ eigenvalues of $F(A)$ are given by

$$\lambda_i[F(A)] = L_A(\lambda_i[A]), \qquad i = 1, 2, \ldots, 2n, \tag{2.1}$$

where, in general, $\lambda_i[M]$ denotes an eigenvalue of M. The intrinsic function F, thus, induces a mapping of the set of $2n$ eigenvalues of A onto the set of $2n$ eigenvalues of $F(A)$, given by

$$\lambda_i[A] \to \lambda_i[F(A)] = L_A(\lambda_i[A]), \qquad i = 1, 2, \ldots, 2n. \tag{2.2}$$

The set of $2n$ points which forms the image under (2.2) is dependent upon the set of $2n$ eigenvalues of the matrix A and upon the matrix having that set of eigenvalues.

The $2n$ eigenvalues of A (and $F(A)$) occur in n pairs, where each pair consists of a principal eigenvalue and its complex conjugate. If $\lambda_i[A]$ is mapped into $L_A(\lambda_i[A])$, then $\overline{\lambda_i[A]}$ is mapped into $L_A(\overline{\lambda_i[A]}) = \overline{L_A(\lambda_i[A])}$ since $L_A(x)$ is a real polynomial. Thus, the eigenvalue mapping induced by F can be described in terms of a mapping of the n pairs of eigenvalues of A onto the n pairs of eigenvalues of $F(A)$. This mapping is completely determined by the mapping of the n principal eigenvalues of A, although the principal eigenvalues of A do not in general map into the principal eigenvalues of $F(A)$.

For any matrix B which is similar to A ($B = P^{-1}AP$), $F(B)$ is defined and $F(B) = L_B(B)$, where $L_B(x)$ is a real polynomial. Since F is intrinsic we have that

$$L_B(B) = F(B) = F(P^{-1}AP) = P^{-1}F(A)P = P^{-1}L_A(A)P = L_A(P^{-1}AP) = L_A(B).$$

Since $L_A(x)$ and $L_B(x)$ are unique and A and B have the same minimum polynomial, it follows that $L_A(x) = L_B(x)$. From (2.2) we now observe that similar matrices (which of necessity have the same set of eigenvalues) have the same eigenvalue mapping.

If A has distinct principal eigenvalues, any matrix with the same set of principal eigenvalues is similar to A. Thus, in this case, the induced eigenvalue mapping is dependent only upon the set of principal eigenvalues and not upon the matrix with those eigenvalues. If, however, A has repeated principal eigenvalues, then the induced mapping will depend upon the matrix chosen. However, if the domain of F includes a non-derogatory (its Jordan matrix is non-derogatory when considered as a matrix in $\mathscr{C}_n$) matrix B with the same set of principal eigenvalues as A, we define the mapping of this set of n pairs of eigenvalues to be the mapping determined by $F(B)$, i.e.,

$$\lambda_i[A] = \lambda_i[B] \to \lambda_i[F(B)] = L_B(\lambda_i[B]), \qquad i = 1, 2, \ldots, 2n. \tag{2.3}$$

If there exists no non-derogatory matrix B in the domain of F with the same set of principal eigenvalues as A, we shall say that the induced eigenvalue mapping is undefined at A. The necessity of defining the induced eigenvalue mapping in terms of non-derogatory matrices is made apparent later. Since two non-derogatory matrices with the same set of principal eigenvalues are similar, the mapping (2.3) is independent of the choice of the non-derogatory matrix B in the domain of F.

The principal polynomial of a matrix A in $\mathfrak{Q}_n$,

$$\det(xI - J) = x^n - \sigma_1[A]x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}[A]x + (-1)^n\sigma_n[A]$$

(where $P^{-1}AP = J$ is given in (1.2)), determines a unique set of $2n$ points P_i: $(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A])$ in complex n-space, $V_n(\mathscr{C})$, where $\lambda_i[A]$, $i = 1, 2, \ldots, 2n$, are the $2n$ eigenvalues of A. Conversely, any point P: $(\lambda, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ in $V_n(\mathscr{C})$ uniquely determines a corresponding polynomial

$$C(x, P) = x^n - \sigma_1 x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}x + (-1)^n\sigma_n,$$

where σ_n is chosen so that λ or $\bar{\lambda}$, whichever has non-negative imaginary part, is a zero of $C(x, P)$. If there exists in the domain of the intrinsic function F a non-derogatory matrix A with $C(x, P)$ as principal polynomial, then every non-derogatory matrix in $\mathfrak{Q}_n$ with principal polynomial $C(x, P)$ is in the domain of F, and F induces a unique mapping of a subset of $V_n(\mathscr{C})$ into $\mathscr{C}$ defined by

(2.4) $f(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A]) = \lambda_i[F(A)] = L_A(\lambda_i[A]), \quad i = 1, 2, \ldots, 2n.$

This mapping clearly has the following symmetry property for $i = 1, 2, \ldots, 2n$, namely

(2.5) $$f(\overline{\lambda_i[A]}, \sigma_1[A], \ldots, \sigma_{n-1}[A]) = \overline{f(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A])},$$

and is independent of the non-derogatory matrix A with principal polynomial $C(x, P)$. We summarize the above discussion by means of the following theorem which is an extension of (**6**, Theorem 2.2).

THEOREM 2.1. *An intrinsic function F on a domain $\mathfrak{D}$ in $\mathfrak{Q}_n$ induces a single-valued function $f(\lambda, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ mapping a subset of $V_n(\mathscr{C})$ into $\mathscr{C}$. The function f is defined at any point P^0: $(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$ for which there exists a non-derogatory matrix A in $\mathfrak{D}$ with λ^0 as an eigenvalue and with principal polynomial*

$$p(x) = x^n - \sigma_1^0 x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}^0 x + (-1)^n\sigma_n^0.$$

The value of f at P^0 is independent of the choice of the non-derogatory matrix A in $\mathfrak{D}$ and is given by $f(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0) = \lambda^0[F(A)] = L_A(\lambda^0)$, where $L_A(x)$ is the unique real polynomial of lowest degree such that $L_A(A) = F(A)$. If

$$f(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$$

is defined, then $f(\overline{\lambda^0}, \sigma_1^0, \sigma_2^0, \ldots, \sigma_{n-1}^0)$ is also defined and

$$f(\overline{\lambda^0}, \sigma_1^0, \ldots, \sigma_{n-1}^0) = \overline{f(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)}.$$

3. The case of distinct eigenvalues. Let F be an intrinsic function with domain $\mathfrak{D} \subseteq \mathfrak{Q}_n$ and let $A \in \mathfrak{D}$ have distinct principal eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n,$$

where $\lambda_1, \ldots, \lambda_r$ are real. In this case, the canonical matrix (1.2) is

$$J = P^{-1}AP = dg\{\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n\}.$$

Now let $f(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ be the induced function from $V_n(\mathscr{C})$ to $\mathscr{C}$ described in Theorem 2.1 and denote by $f_A(z)$ the function from $\mathscr{C}$ to $\mathscr{C}$ given by

$$f_A(z) = f(z, \sigma_1[A], \ldots, \sigma_{n-1}[A]). \tag{3.1}$$

Let $L_A(x)$ be the unique real polynomial of degree less than $2n - r$, the degree of the real minimum polynomial of A, such that $L_A(A) = F(A)$.

From § 2 we know that the eigenvalues of $F(A)$ are given by

$$\lambda_i[F(A)] = L_A(\lambda_i[A]) = f_A(\lambda_i[A]), \qquad i = 1, 2, \ldots, 2n.$$

Thus, the polynomial $L_A(x)$ is a polynomial of degree less than $2n - r$ which agrees with the function $f_A(z)$ at the $2n - r$ distinct points

$$\lambda_1, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_n, \lambda_{n+1} = \overline{\lambda_{r+1}}, \ldots, \lambda_{2n-r} = \overline{\lambda_n}.$$

This polynomial is unique and is given by the well-known Lagrange interpolation formula

$$L_A(x) = \sum_{j=1}^{2n-r} \left\{ \prod_{i \neq j} \frac{x - \lambda_i}{\lambda_j - \lambda_i} \right\} f_A(\lambda_j). \tag{3.2}$$

Now $L_A(A)$, where $L_A(x)$ is given in (3.2), is precisely the classical definition of the value of the primary function $f_A(Z)$ with stem function $f_A(z)$; see (**4**).

We have established the following theorem.

THEOREM 3.1. *Let F be an intrinsic function on $\mathfrak{D} \subseteq \mathfrak{Q}_n$ and let*

$$f(z, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$$

be the induced function of Theorem 2.1. *Let $A \in \mathfrak{D}$ have distinct principal eigenvalues and let $f_A(z)$ denote the function of z only given by* (3.1). *Then $F(A) = f_A(A)$, where $f_A(Z)$ is the primary function with stem function $f_A(z)$.*

4. The case of repeated eigenvalues. We now seek an extension of Theorem 3.1 to argument matrices with repeated principal eigenvalues. Such an extension will require certain restrictions on the intrinsic function F. First, however, several preliminary results are needed.

Thus far we have been concerned with three mappings involving $\mathfrak{Q}_n$, a subset of $V_n(\mathscr{C})$, and $\mathscr{C}$. We shall have need of two additional mappings. To better illustrate the mappings involved, we include the following diagram.

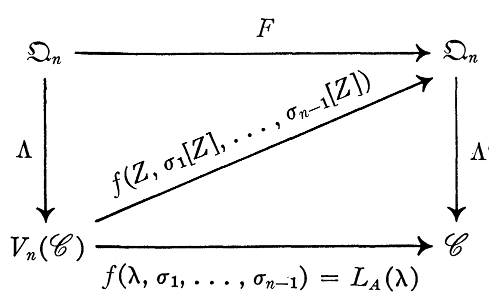

FIGURE 1

In Figure 1, F is an intrinsic function, $f(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ is the induced function of Theorem 2.1, Λ' is the multiple-valued mapping of $\mathfrak{Q}_n$ onto the complex plane mapping each matrix into its $2n$ eigenvalues. The two additional mappings to be defined are Λ and $f(Z, \sigma_1[Z], \ldots, \sigma_{n-1}[Z])$. The characterization of F involves conditions under which the diagram in Figure 1 is commutative.

We now introduce the $2n$-valued mapping Λ of $\mathfrak{Q}_n$ into a subset of $V_n(\mathscr{C})$ defined by

$$(4.1) \qquad \Lambda(A) = (\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A]), \qquad i = 1, 2, \ldots, 2n,$$

where A is in $\mathfrak{Q}_n$, $\lambda_i[A]$ is an eigenvalue of A, and $\sigma_1[A], \ldots, \sigma_{n-1}[A]$ are the first $n - 1$ symmetric functions of the principal eigenvalues of A (the coefficients in the principal polynomial of A).

It is clear that $\Lambda(\mathfrak{Q}_n)$ is a proper subset of $V_n(\mathscr{C})$. In particular,

$$(\lambda, a - bi_1, \sigma_2, \ldots, \sigma_{n-1}),$$

where $b > 0$ is not in $\Lambda(\mathfrak{Q}_n)$.

The subset $\Lambda(\mathfrak{Q}_n)$ of complex n-space $V_n(\mathscr{C})$ can be made into a topological space by defining the open sets of $\Lambda(\mathfrak{Q}_n)$ to be the intersection with $\Lambda(\mathfrak{Q}_n)$ of the open sets in $V_n(\mathscr{C})$ (relative topology). The concepts of neighbourhood and open set are now well-defined in $\Lambda(\mathfrak{Q}_n)$.

THEOREM 4.1. *The $2n$-valued mapping $\Lambda(Z) = (\lambda_i[Z], \sigma_1[Z], \ldots, \sigma_{n-1}[Z])$ is a continuous open map of $\mathfrak{Q}_n$ onto $\Lambda(\mathfrak{Q}_n)$.*

Proof. It follows from the isomorphism (1.1) and from familiar theorems about complex matrices that the eigenvalues of a matrix in $\mathfrak{Q}_n$ are continuous functions of the elements of the matrix, and hence that the symmetric functions $\sigma_1[Z], \sigma_2[Z], \ldots, \sigma_{n-1}[Z]$ are continuous functions of the principal eigenvalues of Z, hence of Z also. Thus, each coordinate $\lambda_i[Z], \sigma_1[Z], \ldots, \sigma_{n-1}[Z]$ is a continuous function of Z; hence, $\Lambda(Z)$ is continuous.

We next show that $\Lambda(Z)$ is an open map. Let S be any open set in $\mathfrak{Q}_n$ and let P_0: $(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ be any point in $\Lambda(S)$, the image of S in $\Lambda(\mathfrak{Q}_n)$. There exists a matrix Z_0 in S such that one image point of Z_0 is P_0. We can assume that Z_0 is non-derogatory since any neighbourhood of Z_0 contains a non-derogatory matrix with the same set of eigenvalues as Z_0. Since S is open, there exists a real number $\delta > 0$ such that the open sphere

$$S_1 = \{Z \in \mathfrak{Q}_n; ||Z - Z_0|| < \delta\}$$

is contained in S.

Since Z_0 is non-derogatory, its Jordan canonical matrix (1.2) is a non-derogatory complex matrix which, by elementary linear algebra, is known to be similar to a companion matrix C_0. It follows that Z_0 is also similar to C_0; therefore, let P be a non-singular matrix in $\mathfrak{Q}_n$ such that

$$P^{-1}Z_0P = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ (-1)^{n-1}\sigma_n & \cdot & \cdot & \cdot & \cdot & \sigma_1 \end{bmatrix} = C_0,$$

where C_0 is the companion matrix similar to Z_0.

Let Z_1 be the matrix defined as follows:

$$P^{-1}Z_1P = \begin{bmatrix} 0 & 1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & 1 \\ (-1)^{n-1}\sigma_n' & \cdot & \cdot & \cdot & \cdot & \sigma_1' \end{bmatrix} = C_1.$$

With Z_1 we associate the points

$$(\lambda_i[Z_1], \sigma_1'[Z_1], \ldots, \sigma_{n-1}'[Z_1]), \qquad i = 1, 2, \ldots, 2n.$$

Then

$$||Z_1 - Z_0|| = ||PC_1P^{-1} - PC_0P^{-1}|| = ||P(C_1 - C_0)P^{-1}|| \leqq$$
$$||P|| \cdot ||C_1 - C_0|| \cdot ||P^{-1}|| = ||P|| \cdot ||P^{-1}|| \, (1/n)\max_i |\sigma_i' - \sigma_i|.$$

Since the nth symmetric function of Z is a continuous function of the principal eigenvalues of Z (hence of Z also), it follows that the matrix Z_1 with image point $(\lambda_i[Z_1], \sigma_1'[Z_1], \ldots, \sigma_{n-1}'[Z])$ can be made to satisfy $||Z_1 - Z_0|| < \delta$ if $(\lambda_i[Z_1], \sigma_1[Z_1], \ldots, \sigma_{n-1}[Z_1])$ is sufficiently close to P_0. Hence, P_0 is an interior point of $\Lambda(\mathfrak{Q}_n)$ and the proof is complete.

COROLLARY 4.1.1. *If a set Γ of points in $\Lambda(\mathfrak{Q}_n)$ is dense at $(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$, then the set of pre-images of Γ under Λ is dense at any Z^0 mapping into $(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$ under Λ.*

COROLLARY 4.1.2. *Let F be an intrinsic function defined on an open set $\mathfrak{D} \subseteq \mathfrak{Q}_n$. Then the induced scalar function $f(\lambda, \sigma_1, \ldots, \sigma_{n-1})$ is defined on an open set $\Lambda(\mathfrak{D})$ in $\Lambda(\mathfrak{Q}_n)$.*

An important result proved in (**6**) is the following theorem.

THEOREM 4.2. *Let $(\lambda^0, \sigma_1^0, \ldots, \sigma_{n-1}^0)$ be a point of $\Lambda(\mathfrak{Q}_n) \subset V_n(\mathscr{C})$ and let σ_n^0 in the equation*

$$x^n - \sigma_1^0 x^{n-1} + \ldots + (-1)^n \sigma_n^0 = 0$$

be so determined that λ^0 (or $\overline{\lambda^0}$, whichever has non-negative imaginary part) is a root. Then there exists a deleted open disk, K, $0 < |\lambda - \lambda^0| < \delta$, of the complex plane, such that for all λ in K the equation

$$x^n - \sigma_1^0 x^{n-1} + \ldots + (-1)^{n-1}\sigma_{n-1}^0 + (-1)^n \sigma_n^0 = 0,$$

with σ_n^0 determined such that λ is a root, has distinct roots.

Our final result before extending Theorem 3.1 (with appropriate additional hypotheses) to matrices with repeated principal eigenvalues is the following theorem.

THEOREM 4.3. *If the intrinsic function F on $\mathfrak{D} \subseteq \mathfrak{Q}_n$ is defined in a neighbourhood of and is continuous at a matrix $A \in \mathfrak{D}$, then the induced function $f(z, \sigma_1, \ldots, \sigma_{n-1})$ is defined in a neighbourhood of and is continuous at each point $(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A])$ of $\Lambda(\mathfrak{Q}_n)$.*

Proof. Let F be an intrinsic function defined in a neighbourhood S of A. Then the induced function $f(z, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ is defined in a neighbourhood Δ_i of P_i: $(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A])$ (Corollary 4.1.2).

If $\{T_{m,j}\}$ is any sequence of points in Δ_j approaching P_j, then there is in S a corresponding sequence $\{Z_{m,j}\}$ of non-derogatory matrices approaching A. For Z in S, $f(\lambda_j[Z], \sigma_1[Z], \ldots, \sigma_{n-1}[Z]) = \lambda_j[F(Z)]$. The eigenvalues of $F(Z)$ are continuous functions of the elements of $F(Z)$, which are, in turn, continuous functions of the elements of Z at $Z = A$. Hence,

$$\lim_{T_{m,j}\to P} f(T_{m,j}) = \lim_{Z_{m,j}\to A} f(\lambda_j[Z_{m,j}], \sigma_1[Z_{m,j}], \ldots, \sigma_{n-1}[Z_{m,j}]) =$$
$$\lim_{Z_{m,j}\to A} \lambda_j[F(Z_{m,j})] = \lambda_j[A] = f(\lambda_j[A], \sigma_1[A], \ldots, \sigma_{n-1}[A]).$$

Thus, $f(z, \sigma_1, \ldots, \sigma_{n-1})$ is continuous at P_j and the proof is complete.

We are now able to generalize Theorem 3.1 to matrices with repeated eigenvalues.

THEOREM 4.4. *Let F be an intrinsic function on $\mathfrak{D} \subseteq \mathfrak{Q}_n$, and let $A \in \mathfrak{D}$. Let $f(z, \sigma_1, \sigma_2, \ldots, \sigma_{n-1})$ be the function from $\Lambda(\mathfrak{Q}_n)$ to $\mathscr{C}$ induced by F. Let $f_A(z)$ be the function of z only, $f_A(z) = f(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$. Then $F(A)$ must be given by the primary function value $f_A(A) = f(A, \sigma_1[A], \ldots, \sigma_{n-1}[A])$* (see **5**) *if either*

Case I. *A has distinct principal eigenvalues;*

Case II. *A has repeated principal eigenvalues, A is an interior point of $\mathfrak{D}$, and $f_A(z)$ is analytic in a z-neighbourhood of the repeated principal eigenvalues of A and their conjugates.*

Proof. Case I is simply Theorem 3.1. In Case II, if A is interior to $\mathfrak{D}$, then by Theorem 4.3 the points

$$P_j: (\lambda_j[A], \sigma_1[A], \ldots, \sigma_{n-1}[A]),$$

$j = 1, 2, \ldots, 2n$, are interior to the domain $\Lambda(\mathfrak{D})$ of $f(z, \sigma_1, \ldots, \sigma_{n-1})$ in $\Lambda(\mathfrak{Q}_n)$. Let $S_1, S_2, \ldots, S_t$ be a collection of spheres in $\Lambda(\mathfrak{Q}_n)$ such that each S_i is in $\Lambda(\mathfrak{D})$ and encloses just one of the t distinct P_j. By Theorem 4.2, the S_i can be taken sufficiently small so that for all points $(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ (with the last $(n-1)$ coordinates fixed as above) which are within S_i, except possibly P_j, the pre-images Z have distinct principal eigenvalues. Let $\mathscr{W}$ be the subset of matrices of $\mathfrak{D}$ which are mapped by Λ into these particular points, $(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ of the S_i. The set of points of the type $(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ is dense at P_j; hence, $\mathscr{W}$ is dense at A in $\mathfrak{Q}_n$ (Corollary 4.1.1). By the continuity of F at A, $\lim_{Z\to A} F(Z)$ exists, and, in particular, $\lim_{Z\to A} F(Z) = \lim_{W\to A} F(W)$, where $W \in \mathscr{W}$.

Since each W has distinct eigenvalues, $F(W) = L_W(W)$, where $L_W(z)$ is given in (**5**) as

$$L_W(z) = \sum_{j=1}^{2n} \left\{ \prod_{i\neq j} \frac{z - \lambda_i[W]}{\lambda_j[W] - \lambda_i[W]} \right\} f_W(\lambda_j[W])$$

which is real since the eigenvalues occur in conjugate pairs. Now, $L_W(z)$ determines the value of $f_A(z) = f(z, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ at $\lambda_1[W], \ldots, \lambda_n[W]$, $\lambda_{n+1}[W] = \overline{\lambda_1[W]}, \ldots, \lambda_{2n}[W] = \overline{\lambda_n[W]}$, where $\lambda_1[W], \ldots, \lambda_n[W]$ are the distinct principal eigenvalues of W (since $\sigma_i[W] = \sigma_i[A], i = 1, 2, \ldots, n-1$). Theorem 4.3 guarantees the continuity of $f_A(z)$ at $z = \lambda_j[A], j = 1, 2, \ldots, 2n$. It is shown in (**7**) that if a function $f_A(z)$ is continuous at $\lambda_j[A], j = 1, 2, \ldots, 2n$, and is analytic in a neighbourhood of the repeated $\lambda_j[A]$, then $L_W(z)$ approaches a unique limiting polynomial $H_A(z)$, as the interpolation points $\lambda_i[W]$ approach the $\lambda_i[A]$ through distinct values. $H_A(z)$ is the Lagrange-Hermite interpolation polynomial,

$$H_A(z) = \sum_{j=1}^{t} \left\{ \prod_{i\neq j} (z-\alpha_i)^{s_i} \left[\sum_{m=0}^{s_j-1} \frac{1}{m!} (z-\alpha_j)^m H_{m,j} \right] \right\}, \tag{4.1}$$

where

$$H_{m,j} = \frac{d^m}{dz^m} \left[f_A(z) \prod_{l\neq j} (z-\alpha_l)^{-s_l} \right]_{z=\alpha_j}$$

and where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the distinct values among the λ_i with respective multiplicities s_i in the real minimum polynomial of A. Since, for each W, $L_W(z)$ is a real polynomial in z and $L_W(z) \to H_A(z)$ as $W \to A$, $H_A(z)$ is also real. This implies that $f^{(k)}(\bar{\alpha}_j) = \overline{f^{(k)}(\alpha_j)}$, where $j = 1, 2, \ldots, t$, $k = 0, 1, 2, \ldots, s_j - 1$.

As $W \in \mathscr{W}$ approaches A, the eigenvalues $\lambda_i[W]$ approach the $\lambda_i[A]$ through distinct values. Hence,

$$\lim_{Z\to A} F(Z) = \lim_{W\to A} F(W) = \lim_{W\to A} L_W(W) = \lim_{W\to A} [L_W(W) - H_A(W)] + \lim_{W\to A} H_A(W).$$

The difference $L_W(W) - H_A(W)$ approaches zero, since

$$\lim_{W\to A}[L_W(z) - H_A(z)] \equiv 0$$

and $H_A(W)$, being a polynomial with fixed coefficients, approaches the limit $H_A(A)$. But $H_A(A)$ is precisely the value $f_A(A)$ of the primary function with stem function $f_A(z)$ (see **4**) and the proof is complete.

5. n-ary functions on $\mathfrak{Q}_n$. Conversely, let $f(z, \sigma_1, \ldots, \sigma_{n-1})$ be a function from $V_n(\mathscr{C})$ to $\mathscr{C}$ with domain $\mathfrak{D}$ such that $(\bar{z}, \sigma_1, \ldots, \sigma_{n-1})$ is in $\mathscr{D}$ if $(z, \sigma_1, \ldots, \sigma_{n-1})$ is in $\mathfrak{D}$. The function $f(z, \sigma_1, \ldots, \sigma_{n-1})$ can be extended to a function on $\mathfrak{Q}_n$ by defining the value $f(A, \sigma_1[A], \ldots, \sigma_{n-1}[A])$ to be the value of the primary function extension of the stem function

$$f_A(z) = f(z, \sigma_1[A], \ldots, \sigma_{n-1}[A]).$$

This value is given by

$$f_A(A) = f(A, \sigma_1[A], \ldots, \sigma_{n-1}[A]) = Pf_A(J)P^{-1}, \tag{5.1}$$

where

$$f_A(J) = \sum_{j=1}^{t} \left\{ \prod_{i \neq j} (J - \alpha_i I)^{s_i} \left[\sum_{k=0}^{s_j - 1} \frac{1}{k!} (J - \alpha_j I)^k H_{k,j} \right] \right\} \tag{5.2}$$

and

$$H_{k,j} = \frac{d^k}{dz^k} \left[f_A(z) \prod_{l \neq j} (z - \alpha_l)^{-s_l} \right]_{z = \alpha_j},$$

where $\alpha_1, \alpha_2, \ldots, \alpha_t$ are the distinct zeros of multiplicity $s_1, s_2, \ldots, s_t$, respectively, in the minimum polynomial of A. It is clear from the definition of a primary function (**4**) that $f_A(A)$ as given in Theorem 4.2 coincides with $f_A(A)$ as given by (5.1). The domain of definition of $f_A(A)$ of (5.1) includes all matrices A such that

1. $\Lambda(A) = \{(\lambda_i[A], \sigma_1[A], \ldots, \sigma_{n-1}[A]), i = 1, 2, \ldots, 2n\}$ is contained in $\mathfrak{D}$;
2. $f_A(z) = f(z, \sigma_1[A], \ldots \sigma_{n-1}[A])$ is analytic at each $\lambda_i[A]$ of multiplicity greater than zero in the minimum polynomial of A and

$$f_A^{(k)}\overline{(\lambda_i[A])} = \overline{f_A^{(k)}(\lambda_i[A])},$$

$i = 1, 2, \ldots, t$; $k = 0, 1, \ldots, s_i - 1$, where $\lambda_i[A]$ is of multiplicity s_i in the minimum polynomial of A.

The function defined by (5.1) subject to conditions (1) and (2) above is called an *n-ary function on* $\mathfrak{Q}_n$ with stem function $f(z, \sigma_1, \ldots, \sigma_{n-1})$. The primary functions are special cases of n-ary functions.

Since the polynomial in J given by (5.2) is real, it follows that an n-ary function on $\mathfrak{Q}_n$ is a poly-function; hence, from (**3**), we have the following theorem.

THEOREM 5.1. *An n-ary function on* $\mathfrak{D} \subseteq \mathfrak{Q}_n$ *is intrinsic.*

As a consequence of Theorem 4.2 and (5.1) we have the following theorem.

THEOREM 5.2. *An intrinsic function on* $\mathfrak{D} \subseteq \mathfrak{Q}_n$, *subject to the conditions of Theorem* 4.2, *is an n-ary function.*

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