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SUBGROUPS OF LOCALLY FINITE PRODUCTS OF LOCALLY NILPOTENT GROUPS

BURKHARD HÖFLING*

Mathematisches Institut, Friedrich-Schiller-Universität, D-07740 Jena, Germany
e-mail: hoefling@mathematik.uni-jena.de

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Abstract. Let the locally finite group G be the product of two locally nilpotent subgroups A and B, and assume that H is a subgroup of G belonging to a group class $\mathfrak{F}$. The question is considered whether there exists a subgroup X of G containing H which belongs to $\mathfrak{F}$ and satisfies $X = (X \cap A)(X \cap B)$. Under various assumptions on G and $\mathfrak{F}$, necessary and sufficient conditions for the existence of such a subgroup X are obtained.

A group G is the product of two subgroups A and B if G equals the set AB, that is, if every element $g \in G$ can be expressed as $g = ab$ with $a \in A$ and $b \in B$. A subgroup H of G will be called *prefactorised* if H is the product of a subgroup of A and a subgroup of B, and in this case, $H = (H \cap A)(H \cap B)$. A prefactorised subgroup H of G is *factorised* if it contains $A \cap B$. If H is any subgroup of $G = AB$, then the *factoriser* X of H is defined as the intersection of all factorised subgroups of G which contain H. By [**1**, Lemma 1.1.2], the subgroup X is a factorised subgroup of G.

Here, we consider the following question. Suppose that the group G is the product of two locally nilpotent subgroups A and B, and let H be a subgroup of G belonging to a group class $\mathfrak{F}$. Does G possess a prefactorised or factorised subgroup which contains H and belongs to the same class of groups $\mathfrak{F}$? For example, under various hypotheses on G, it can be shown that the unique maximal locally nilpotent normal subgroup of G is factorised; see [**1**] and [**3**]. Thus every *normal* locally nilpotent subgroup H of G is contained in a factorised locally nilpotent subgroup.

If the subgroup H in question is not normal (characteristic) in G, one cannot expect that all conjugates (Aut(G)-conjugates) of the $\mathfrak{F}$-group H are contained in prefactorised or factorised $\mathfrak{F}$-group; for example, it is easy to see that every finite product $G = AB$ of two nilpotent subgroups A and B has exactly one prefactorised Sylow p-subgroup for each prime p, namely A_pB_p, where A_p and B_p are the p-components of A and B, respectively. Moreover, these Sylow subgroups form a Sylow basis of G. More generally, if $G = AB$, where A and B are locally nilpotent, and G is a CC-group, satisfies min-p for every prime p or is a $\mathfrak{U}$-group then the sets A_pB_p are Sylow p-subgroups of G which form a Sylow generating basis of G; see [**5**] and [**16**]. Here, $\mathfrak{U}$ denotes the largest subgroup-closed class of locally finite groups G such that for every set π of primes, the maximal π-subgroups of G are conjugate. A group G is a CC-group if $G/C_G(x^G)$ is a Černikov group for every $x \in G$, where x^G denotes the smallest normal subgroup of G which contains x, and the group G satisfies min-p if it has the minimal condition on p-subgroups for the prime p. The class of all periodic locally soluble nilpotent-by-finite groups, which is evidently a subclass of $\mathfrak{U}$, will play an important role in the sequel and will be denoted by $\mathfrak{N}\mathfrak{S}^*$.

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Our main results, which improve the corresponding results about finite groups in [**2**] and also generalise the above-mentioned ones about Sylow p-subgroups, are Theorem 2.3, Theorem 3.2 and Theorem 4.1 below. They can be summarised as follows.

THEOREM. *Let $\mathfrak{D}$ be a class of periodic locally soluble groups which is closed under taking subgroups and factor groups and assume that $\mathfrak{D}$ is a class of CC-groups, of $\mathfrak{N}\mathfrak{S}^*$-groups or groups satisfying min-p for all primes p. We assume that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation of characteristic π. Let $G \in \mathfrak{D}$ be the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G such that $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into H, then H is contained in a prefactorised $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then the factoriser of H is an $\mathfrak{F}$-subgroup of G.*

Note that, if G is an $\mathfrak{N}\mathfrak{S}^*$-group (a CC-group) with Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ and H is a subgroup of G, then it is always possible to find an inner (locally inner) automorphism α such that $\{G_p \mid p \in \mathbb{P}\}$ reduces into H^α. Consequently, in this case every $\mathfrak{F}$-subgroup in the above theorem has a conjugate (local conjugate) which is contained in a prefactorised $\mathfrak{F}$-subgroup of G.

Here a Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ of a group G *reduces* into a subgroup H of G if $\{G_p \cap H \mid p \in \mathbb{P}\}$ is a Sylow generating basis of G. Definitions of Sylow generating bases and locally defined $\mathfrak{D}$-formations can be found in [**7**]; for the latter see also Section 2 below. The characteristic π of a class $\mathfrak{F}$ of groups is the set of primes p such that $\mathfrak{F}$ contains a group of order p.

In particular, if H is $\mathfrak{F}$-maximal in G in the above theorem, that is, if H is not properly contained in another $\mathfrak{F}$-subgroup of G, then H is prefactorised if and only if $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into H (see Corollary 3.3 and Corollary 4.2). As a consequence, G possesses exactly one prefactorised $\mathfrak{F}$-projector if $\mathfrak{D}$ consists of groups satisfying min-p for all primes p, or if $\mathfrak{D}$ is the class of all periodic locally soluble CC-groups (Theorem 3.7 and Theorem 4.7). Recall that if $\mathfrak{F}$ is any class of groups, a subgroup H of G is an $\mathfrak{F}$-projector of G if HN/N is $\mathfrak{F}$-maximal in G/N for every normal subgroup N of G. A similar result about $\mathfrak{F}$-projectors also holds if $\mathfrak{D}$ is contained in the class $\mathfrak{U}$ (see Theorem 5.5).

Most of the above results are based upon the close examination of $\mathfrak{H}$-subgroups of $\mathfrak{N}\mathfrak{S}^*$-groups G which are the product of two locally nilpotent subgroups A and B, where $\mathfrak{H}$ is an $\mathfrak{N}\mathfrak{S}^*$-Schunck class (for a definition see Section 1). The main result about $\mathfrak{H}$-subgroups of nilpotent-by-finite products is Theorem 1.6.

Our notation is standard and follows [**1**] and [**7**]. In particular, if G is a group and π is a set of primes, then a Sylow π-subgroup of G is just a maximal π-subgroup of G.

1. Factorisers of $\mathfrak{H}$-subgroups of $\mathfrak{N}\mathfrak{S}^*$-groups. A class $\mathfrak{H}$ of $\mathfrak{N}\mathfrak{S}^*$-groups is an $\mathfrak{N}\mathfrak{S}^*$-*Schunck class* if $\mathfrak{H}$ consists of all $\mathfrak{N}\mathfrak{S}^*$-groups whose finite primitive image belong to $\mathfrak{H}$ and whose infinite semiprimitive images are the union of finite $\mathfrak{H}$-groups. A group is *semiprimitive* if it is the semidirect product of a finite subgroup M with trivial core and a divisible abelian normal subgroup D all of whose proper M-invariant subgroups are finite. Note that, in view of [**23**], this definition is equivalent with that in [**15**]; see also [**24**].

The following lemma further investigates the structure of certain semiprimitive groups.

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LEMMA 1.1. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups of characteristic π and suppose that $G = M \ltimes D$ is an infinite semiprimitive Černikov group, where D is a radicable abelian p-group for the prime p and M is finite and soluble. If G/D is an $\mathfrak{H}$-group and $p \in \pi$ but G is not an $\mathfrak{H}$-group, then M does not centralise any M-composition factor of D.*

Proof. Since $G \notin \mathfrak{H}$, the $\mathfrak{H}$-subgroup M is an $\mathfrak{H}$-projector of G by [**15**, Lemma 4.1]. Let U/V be an M-composition factor of D which is centralised by M. Then $MU/V = MV/V \times U/V$ and U/V is an elementary abelian p-group and $p \in \pi$. Now the class $\mathfrak{H}^*$ of all finite groups in $\mathfrak{H}$ is a Schunck class of finite groups, hence is closed with respect to finite direct products by [**9**, III, Corollary 6.2]. Therefore MU/V is an $\mathfrak{H}$-group. On the other hand, by [**15**, Corollary 4.7] MV/V is an $\mathfrak{H}$-projector of MU/V. This contradiction shows that M does not centralise any M-composition factor of D.

Next, we deduce a property of groups satisfying the hypotheses of the preceding Lemma 1.1 which will be needed in the sequel.

LEMMA 1.2. *Suppose that G is an infinite semiprimitive Černikov group which is a semidirect product of a radicable abelian normal p-group D and a finite soluble group M. Further, assume that M does not centralise any M-composition factor of D (of a given M-composition series of D). If M is not a p-group, then $N_D(M_{p'}) = 1$ for every Hall p'-subgroup $M_{p'}$ of M.*

Proof. Let

$$1 = D_0 \lhd D_1 \lhd \ldots \lhd D_\alpha = D$$

be an M-composition series of D, where α is an ordinal, whose factors are not centralised by M. Since D does not contain infinite M-invariant subgroups, we have $\alpha \leq \omega$, the least infinite ordinal number. Therefore it suffices to show that $N_{D_n}(M_{p'}) = 1$ for every integer n. We proceed by induction on n, assuming that $n > 0$ and $N_{D_{n-1}}(M_{p'}) = 1$.

Let $H = MD_n$ and $C = C_H(D_n/D_{n-1})$. Put $K = C \cap MD_{n-1} = (C \cap M)D_{n-1}$ and observe that K is a normal subgroup of $H = CM$ because K/D_{n-1} is centralised by C and normalised by M. Since $D_n \cap K = D_{n-1}(D_n \cap M) = D_{n-1}$ by Dedekind's modular law, the factor group D_n/D_{n-1} is H-isomorphic with $D_nK/K = (C \cap M)D_n/K = C/K$. It follows that C/K is a self-centralised minimal normal subgroup of H/K. Therefore $H/K = (MK/K)(C/K)$ is a primitive group by [**9**, **A**, Theorem 15.8 (b)]. Let $R/C = O_{p'}(H/C)$ and $Q = M_{p'} \cap R$, then Q is nontrivial because $C/K = O_p(H/K)$.

Since the p'-group $QK/K = O_{p'}(MK/K)$ cannot be normal in H/K and MK/K is a maximal subgroup of H/K, it follows that $MK = N_H(QK/K)$. Let $g \in N_H(M_{p'}K/K)$, then Q^gK is contained in $M_{p'}^gK \cap R^g = M_{p'}K \cap R = (M_{p'} \cap R)K = QK$ and so $g \in N_H(QK/K) = MK$. It follows that $N_H(M_{p'}) \leq N_H(M_{p'}K/K) \leq MK$. Therefore

$$N_{D_n}(M_{p'}) \leq MK \cap D_n = M(C \cap M)D_{n-1} \cap D_n = MD_{n-1} \cap D_n = D_{n-1},$$

and it follows that $N_{D_n}(M_{p'}) = N_{D_{n-1}}(M_{p'}) = 1$, as required.

By a result of Gross [**12**, Theorem 1], (see also [**1**, Lemma 2.5.2]), a finite primitive group G which is the product of two nilpotent subgroups A and B is either a p-group or A and B are a Sylow p-subgroup and a Hall p'-subgroup of G. The following is a weaker version for semiprimitive groups.

THEOREM 1.3. *Suppose that G is an infinite semiprimitive group with finite residual D and suppose that D is a p-group. If G is the product of two locally nilpotent subgroups A and B, then one of the groups $AO_p(G)/O_p(G)$ and $BO_p(G)/O_p(G)$ is a p-group and the other is a p'-group. In particular, A or B is a p-group.*

Proof. If G is a p-group, the statement is clear. Therefore suppose that G is not a p-group. Since $A_{p'}B_{p'}$ is a Sylow p'-subgroup of G, we must have $A_{p'} \neq 1$ or $B_{p'} \neq 1$. Assume without loss of generality that $B_{p'} \neq 1$. Then $D \cap B_{p'}^G$ is either finite or equals D. Assume first that $D \cap B_{p'}^G$ is finite. Then $D \cap B_{p'}^G$ is contained in $D_n = \{x \in D \mid x^{p^n} = 1\}$ for some integer n and so $D/D_n \cap B_{p'}^G D_n/D_n = 1$, and in particular, $B_{p'}$ is contained in $C_G(D/D_n)$. As in the proof of [**24**, Proposition 2.3 (ii)], there exists an isomorphism $G \to G/D_n$ mapping D to D/D_n, and so we have $D = C_G(D/D_n)$. But then $B_{p'}$ is contained in a p-group, contradicting $B_{p'} \neq 1$. This shows that we must have $D \leq B_{p'}^G$.

Now $O_{p'}(G)$ is contained in $C_G(D) = D$ and so $O_{p'}(G) = 1$. Therefore $[A_{p'}, D] \leq [A_{p'}^G, B_{p'}^G] = 1$ by Lemma 2.7 and Lemma 2.1 of [**10**]. But then $A_{p'}$ is contained in $C_G(D) = D$ and A is a p-group. Thus $B_p^G = B_p^A$ is contained in the Sylow p-subgroup AB_p of G. Consequently B_p^G is contained in $O_p(G)$ and so $BO_p(G)/O_p(G)$ is a p'-group.

Combining Lemma 1.1 and Lemma 1.2 with Theorem 1.3, we obtain a first result about $\mathfrak{H}$-maximal subgroups of infinite semiprimitive Černikov groups which are the product of two locally nilpotent subgroups.

PROPOSITION 1.4. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups and suppose that G is an infinite semiprimitive Černikov group. Further, assume that every finite image of G is an $\mathfrak{H}$-group and that G is not an $\mathfrak{H}$-group. If G is the product of two locally nilpotent subgroups A and B, then G possesses an $\mathfrak{H}$-projector which contains A or B, hence is factorised.*

Proof. Let D denote the finite residual of G, which is a radicable abelian p-group for a prime p. Suppose that B is not a p-group, then by Theorem 1.3, A is a p-group and $B_{p'}$ is a Sylow p'-subgroup of G. Let M be a complement of D in G which contains $B_{p'}$, then by Lemma 1.1 and Lemma 1.2, M contains $B \leq N_G(B_{p'})$ because B is locally nilpotent. Since M is an $\mathfrak{H}$-projector of G by [**15**, Lemma 4.1] and M contains B, it follows that M is factorised.

The following proposition shows that not only semiprimitive Černikov $\mathfrak{H}$-groups are the union of an ascending chain of $\mathfrak{H}$-groups.

PROPOSITION 1.5. *Let $\mathfrak{H}$ be a Schunck class of $\mathfrak{N}\mathfrak{S}^*$-groups. Then a Černikov group G is an $\mathfrak{H}$-group if and only if it is the union of an ascending chain $\{G_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups.*

Proof. If G is the union of an ascending chain $\{G_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups, then $G \in \mathfrak{H}$ by [**15**, Lemma 3.1].

Conversely, suppose that the Černikov group G belongs to the class $\mathfrak{H}$ and let D be the maximal radicable abelian normal subgroup of G and H a finite supplement of D in G. Let L be an $\mathfrak{H}$-projector of H, then $H = L(D \cap H)$ because $H/H \cap D \in \mathfrak{H}$. Therefore $G = LD$ and we may assume without loss of generality that $H \in \mathfrak{H}$.

Assume first that D does not have infinite G-invariant subgroups and let $N = \mathrm{Core}_G(H)$. Then G/N is the union of a chain $\{G_i/N \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups: If $N \leq D$, the group G/N is semiprimitive, and this follows from the definition of a Schunck class. Otherwise, we obtain the same result by induction on $|G : D|$ and the fact that $G/N = (H/N)(DN/N)$. Since H is finite, we may assume without loss of generality that $H \leq G_i$ for every i. Hence $G_i = HD \cap G_i = H(D \cap G_i)$ by the modular law. As N is finite, it suffices to show that every G_i is an $\mathfrak{H}$-group.

Fix an $i \in \mathbb{N}$ and let G_i/K be a finite primitive image of G_i with unique minimal normal subgroup $L/K = F(G_i/K)$. If $N \leq K$, we have $G_i/K \in \mathfrak{H}$, as required. Therefore assume that $L \leq NK$. Then $L = L \cap NK = (L \cap N)K$ by the modular law. Moreover, the abelian normal subgroup $(DK \cap G_i)/K$ of G_i/K is contained in $F(G_i/K) = L/K$. It follows that $G_i = HL = H(L \cap N)K$. Since N is contained in H, we even have $G_i = HK$ and so $G_i/K \cong H/H \cap K \in \mathfrak{H}$.

This shows that every primitive image of G_i is an $\mathfrak{H}$-group, and so $G_i \in \mathfrak{H}$ by the definition of a Schunck class. Thus G is the union of the finite $\mathfrak{H}$-groups $\{G_i \mid i \in \mathbb{N}\}$.

Therefore suppose that D has a proper infinite G-invariant subgroup E. By induction on the rank of a maximal radicable abelian normal subgroup of G, the factor group G/E possesses an ascending chain $\{G_i/E \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups.

We may assume that each G_i contains the subgroup H, so that $G_i = H(G_i \cap D)$ for every $i \in \mathbb{N}$. Let $H_0 = H$, and for each $i \in \mathbb{N}$, let H_i be an $\mathfrak{H}$-maximal supplement of $G_i \cap D$ in G_i containing H_{i-1}. Then the H_i form an ascending chain of $\mathfrak{H}$-groups: let L denote their union. Since H_i is an $\mathfrak{H}$-projector of G_i and $G_i/E \in \mathfrak{H}$, we have $G_i = H_iE$, and so $G = LE = LD$.

Hence $D \cap L$ is normal in G. Now $G/D \cap L = H_iE(D \cap L)/D \cap L$ is isomorphic with $H_iE/H_iE \cap D \cap L = H_iE/(H_i \cap D)(E \cap L)$. Since $G \in \mathfrak{H}$, this shows that $H_iE/(H_i \cap D)(E \cap L)$ also belongs to $\mathfrak{H}$.

Since H_i is an $\mathfrak{H}$-projector of $G_i = H_iE$, it follows that $G_i = H_iE = H_i(H_i \cap D)(E \cap L)$ is contained in L. Now this holds for every $i \in \mathbb{N}$, and therefore $G = L$ is the union of the subgroups $H_i \in \mathfrak{H}$. Thus we may assume without loss of generality that the G_i are $\mathfrak{H}$-groups.

Since the G_i are Černikov groups, by induction on the rank of a maximal radicable abelian normal subgroup, each G_i possesses an ascending chain $\{G_{i,j} \mid j \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. We define an ascending chain $\{G_i^* \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups satisfying $G_i^* \leq G_i$ for every positive integer i: firstly, let $G_1^* = G_{1,1}$. Now let $n > 1$. Since G_n is the union of the subgroups $\{G_{n,j} \mid j \in \mathbb{N}\}$, there exists an integer m such that the $\mathfrak{H}$-group $G_{n,m} = G_n^*$ contains the (finite) subgroups $G_{1,n-1}, G_{2,n-2}, \ldots, G_{n-2,2}, G_{n-1,1}$ and G_{n-1}^* of G_n. By construction, $\{G_n^*\}$ is an ascending chain of $\mathfrak{H}$-groups and $G_{i,j} \leq G_{i+j}^*$ for every $i, j \in \mathbb{N}$. Therefore G is the union of the chain $\{G_n^* \mid n \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups, as required.

Recall that by [**5**] (see also [**16**]) every $\mathfrak{U}$-group G which is the product of two locally nilpotent subgroups A and B possesses a Sylow generating basis of the form $\{A_pB_p \mid p \in \mathbb{P}\}$. Now if G is a $\mathfrak{U}$-group with Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ and H is a subgroup of G, then by [**13**, Lemma 2.1] and [**11**, Theorem 2.10], there exists a $g \in G$ such that $\{G_p \mid p \in \mathbb{P}\}$ reduces into H^g. Thus, with the notation of the following

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theorem, every $\mathfrak{H}$-subgroup possesses a conjugate H into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces.

THEOREM 1.6. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. Further, let H be an $\mathfrak{H}$-subgroup of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces.*

(a) *If π contains $\pi(A) \cap \pi(B)$, then the factoriser of H is an $\mathfrak{H}$-group.*
(b) *If H is a π-group, then the factoriser of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group. Hence H is contained in a prefactorised $\mathfrak{H}$-subgroup of G.*

Proof. (a) Let X denote the factoriser of H. Since the Sylow generating basis

$$\{(X \cap A_p)(X \cap B_p) \mid p \in \mathbb{P}\} = \{X \cap A_pB_p \mid p \in \mathbb{P}\}$$

of X reduces into H, we may assume without loss of generality that $G = X$. Therefore it remains to show that $G \in \mathfrak{H}$. As $\mathfrak{H}$ is a Schunck class and our hypotheses are inherited by factor groups, it suffices to consider the cases when G is a finite primitive group or an infinite semiprimitive Černikov group.

Suppose first that G is finite and primitive. Then by [**12**, Theorem 1], either $G = A = B$ is a cyclic p group for some prime p, or A is a Sylow p-subgroup of G and B is a Hall p'-subgroup of G. In the first case, we have $p \in \pi$ and so $G \in \mathfrak{H}$. Otherwise, the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H, and so $H = (H \cap A_pB_p)(H \cap A_{p'}B_{p'}) = (H \cap A)(H \cap B)$ is factorised. Hence $G = X = H \in \mathfrak{H}$.

If G is an infinite semiprimitive Černikov group, we have $G = M \ltimes D$, where D is a radicable abelian p-group for the prime p and M is finite. Since every primitive image of G/D belongs to $\mathfrak{H}$, we have $M \cong G/D \in \mathfrak{H}$ because $\mathfrak{H}$ is a Schunck class. Now suppose that $G \notin \mathfrak{H}$. Then, by Theorem 1.3 and Proposition 1.4, without loss of generality, A is a p-group containing D and $B \leq M$ is finite. Thus $MD_n = MD_n \cap AB = (MD_n \cap A)B$ is factorised for every $n \in \mathbb{N}$, where $D_n = \{x \in D \mid x^{p^n} = 1\}$. Since $G = \bigcup_{n \in \mathbb{N}} MD_n$, this shows that every finite subgroup U of G is contained in a finite factorised subgroup of G. In particular, the factoriser of every finite subgroup of G is finite.

By Proposition 1.5, the Černikov group H is the union of an ascending chain $\{H_i \mid i \in \mathbb{N}\}$ of finite $\mathfrak{H}$-groups. Since $H \cap D$ has finite index in H, we may assume without loss of generality that $H = H_1(H \cap D)$. Since G is a $\mathfrak{U}$-group, there is a $g \in G$ such that the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H_1^g. Replacing H_i by H_i^g for every $i \in \mathbb{N}$, we may assume that the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H_1. Now $H_i = H_1(H_i \cap D)$ by the modular law, and so by [**15**, Proposition 2.3(d)], the Sylow generating basis $\{A_pB_p\}$ reduces into every H_i. Therefore the factorisers X_i of the H_i are $\mathfrak{H}$-groups by the finite case. The union U of the factorisers of the H_i is clearly a factorised subgroup of G which contains H, and so $G = U$. It follows that $\{X_i \mid i \in \mathbb{N}\}$ is an ascending chain of finite $\mathfrak{H}$-subgroups of the semiprimitive group G, and so $G \in \mathfrak{H}$ by the definition of a Schunck class.

(b) Since H is a π-group and the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H, we have $H \leq A_\pi B_\pi$. Applying (a) to the group $A_\pi B_\pi$, we obtain that the factoriser of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group, as required.

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From this theorem, we derive a necessary and sufficient condition for an $\mathfrak{H}$-maximal subgroup of G to be factorised.

COROLLARY 1.7. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. Let H be an $\mathfrak{H}$-maximal subgroup of G*

(a) *If π contains $\pi(A) \cap \pi(B)$, then H is prefactorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. Thus an $\mathfrak{H}$-maximal subgroup of G is prefactorised if and only if it is factorised.*

(b) *If H is a π-group, then H is prefactorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

Proof. If H is any prefactorised subgroup of G, then by [**16**, Theorem 4.7], the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H. This shows the necessity of our conditions.

Conversely, if π contains $\pi(A) \cap \pi(B)$ and the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H, then the factoriser X of H is an $\mathfrak{H}$-group, by Theorem 1.6. Hence $H = X$ by the $\mathfrak{H}$-maximality of H, and so H is factorised.

As in the proof of Theorem 1.6, statement (b) now follows by considering the Sylow π-subgroup $A_\pi B_\pi$ instead of G.

Since every subgroup of an $\mathfrak{N}\mathfrak{S}^*$-group possesses a conjugate into which a given Sylow generating basis of G reduces, we also have the following result.

COROLLARY 1.8. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B.*

(a) *If π contains $\pi(A) \cap \pi(B)$, then every $\mathfrak{H}$-maximal subgroup of G has a factorised conjugate.*

(b) *Every $\mathfrak{H}$-maximal subgroup of G which is a π-group has a prefactorised conjugate.*

Proof. Let H be an $\mathfrak{H}$-maximal subgroup of G; then, by [**13**, Lemma 2.1], a Sylow generating basis of H can be extended to a Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ of G. Therefore, by [**11**, Theorem 2.10], there exists an element $g \in G$ such that $\{G_p^g \mid p \in \mathbb{P}\} = \{A_pB_p \mid p \in \mathbb{P}\}$. Thus $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces into $H^{g^{-1}}$. The result now follows from Corollary 1.7.

Since $\mathfrak{H}$-projectors are in particular $\mathfrak{H}$-maximal subgroups, we also obtain

COROLLARY 1.9. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class of characteristic π and suppose that the $\mathfrak{N}\mathfrak{S}^*$-group G is the product of two locally nilpotent subgroups A and B. If π contains $\pi(A) \cap \pi(B)$ or an $\mathfrak{H}$-projector of G is a π-group, then G possesses a unique $\mathfrak{H}$-projector H which is prefactorised. If π contains $\pi(A) \cap \pi(B)$, then H is even factorised.*

Proof. By [**15** Corollary 5.2], the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into a unique $\mathfrak{H}$-projector H of G. Therefore the statements follow from Corollary 1.7.

The above results can also be applied to trifactorised groups.

COROLLARY 1.10. *Let $\mathfrak{H}$ be an $\mathfrak{N}\mathfrak{S}^*$-Schunck class and suppose that the group $G \in \mathfrak{N}\mathfrak{S}^*$ has subgroups A, B and C such that $G = AB = AC = BC$, where A and B are locally nilpotent and $C \in \mathfrak{H}$. If $\pi(A) \cap \pi(B)$ is contained in the characteristic of $\mathfrak{H}$, then $G \in \mathfrak{H}$.*

Proof. In view of [**15**, Lemma 3.1], we may assume without loss of generality that C is an $\mathfrak{H}$-maximal subgroup of G. Hence C has a factorised conjugate by Corollary 1.8, and $G = C$ by [**2**, Lemma 1].

2. Factorisers of $\mathfrak{F}$-subgroups of $\mathfrak{N}\mathfrak{S}^*$-groups. The results of the preceding section can also be applied to $\mathfrak{F}$-subgroups, where $\mathfrak{F}$ is locally defined $\mathfrak{D}$-formation for some ϱs-closed subclass $\mathfrak{D}$ of $\mathfrak{N}\mathfrak{S}^*$. In order to accomplish this, we will prove that every locally defined $\mathfrak{D}$-formation $\mathfrak{F}$ can be obtained by intersecting an $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ with $\mathfrak{D}$.

We briefly recall the definition of a locally defined $\mathfrak{D}$-formation given in [**7**]. Let $\mathfrak{D}$ be a class of periodic locally soluble groups which is ϱ-closed and s-closed, that is, closed under taking factor groups and subgroups. Let $\mathfrak{X}$ be a class of groups contained in $\mathfrak{D}$, and for every group $G \in \mathfrak{D}$, let $C_G(\mathfrak{X}, p)$ denote the intersection of the centralisers of all p-principal factors U/V such that $G/C_G(U/V)$ belongs to $\mathfrak{X}$. The subclass $\mathfrak{X}$ of $\mathfrak{D}$ is a $(\mathfrak{D}, p)$-preformation if it is ϱ-closed and for every $G \in \mathfrak{D}$, the factor group $G/C_G(\mathfrak{X}, p)$ belongs to $\mathfrak{X}$.

A class $\mathfrak{F}$ is a $\mathfrak{D}$-formation if there exists a set of primes π and a function f assigning to every $p \in \pi$ a $(\mathfrak{D}, p)$-preformation $f(p)$, such that $G \in \mathfrak{D}$ belongs to $\mathfrak{F}$ if and only if G is a π-group and for every prime p, the group G belongs to the class $\mathfrak{S}_{p'}\mathfrak{S}_p f(p)$. For equivalent definitions, see for example [**7**, Corollary 6.2.5]. Here, $\mathfrak{S}_{p'}\mathfrak{S}_p f(p)$ is the class of all periodic locally soluble groups G having normal subgroups M and N such that $G/M \in f(p)$, M/N is a p-group and N is a p'-group.

LEMMA 2.1. *Let $\mathfrak{D}$ be a ϱ-closed class of finite soluble groups and suppose that $\mathfrak{X}$ is a subclass of $\mathfrak{D}$ such that every $\mathfrak{D}$-group has an $\mathfrak{X}$-projector.*

(a) *There exists an $\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that $\mathfrak{X} = \mathfrak{H} \cap \mathfrak{D}$.*
(b) *Let X be an $\mathfrak{X}$-projector of the $\mathfrak{D}$-group G. Then X is an $\mathfrak{H}$-projector of G if $L \in \mathfrak{D}$ for every subgroup L of G with $X \leq L$ which contains a G-invariant subgroup N such that $L/N \in \mathfrak{H}$.*

Proof. (a) Let $\mathfrak{H}$ be the class of all finite soluble groups whose primitive factor groups belong to $\mathfrak{X}$. Since every $\mathfrak{X}$-group has $\mathfrak{X}$-projectors, $\mathfrak{X}$ is ϱ-closed, and so $\mathfrak{X} \subseteq \mathfrak{H} \cap \mathfrak{D}$. Now let $G \in \mathfrak{H} \cap \mathfrak{D}$ and suppose that X is an $\mathfrak{X}$-projector of G. If $G \notin \mathfrak{X}$, then X is contained in a maximal subgroup M of G. But then $X\,\mathrm{Core}_G(M)/\mathrm{Core}_G(M) \leq M/\mathrm{Core}_G(M)$ is a proper subgroup of the $\mathfrak{X}$-group $G/\mathrm{Core}_G(M)$. This contradiction shows that $\mathfrak{X} = \mathfrak{H} \cap \mathfrak{D}$.

(b) Assume that X is not an $\mathfrak{H}$-projector of G, then there exists a normal subgroup N of G such that XN/N is properly contained in an $\mathfrak{H}$-group L/N. Let M be a maximal subgroup of L containing XN, then $X\,\mathrm{Core}_L(M)/\mathrm{Core}_L(M) \leq M/\mathrm{Core}_L(M)$ is a proper subgroup of $L/\mathrm{Core}_L(M) \in \mathfrak{H}$. But since $L/\mathrm{Core}_L(M)$ is primitive, it belongs to $\mathfrak{X}$. This contradicts the fact that X is an $\mathfrak{X}$-projector of L.

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Note that condition (b) above holds in particular if the class $\mathfrak{D}$ is subgroup-closed, or if the $\mathfrak{X}$-projectors of every $\mathfrak{D}$-group G are $\mathfrak{X}$-covering subgroups of G, that is, if an $\mathfrak{X}$-projector X of G is an $\mathfrak{X}$-projector of L for every subgroup L of G which contains X. Observe also that condition (b) is necessary in the following sense: if X is an $\mathfrak{H}$-projector of G, then by [**9**, III, Theorem 3.21], X is an $\mathfrak{H}$-covering subgroup of G, and since $X \in \mathfrak{X} \subseteq \mathfrak{H}$, it follows that X is also an $\mathfrak{X}$-projector of L for every subgroup L of G which contains X.

However Lemma 2.1 (b) does not hold for $\mathfrak{R}\mathfrak{S}^*$-Schunck classes: let p be a prime, $\mathfrak{X}$ the class of all quasicyclic p-groups and $\mathfrak{D}$ be the class of all cyclic and quasicyclic p-groups. Then every $\mathfrak{D}$-group has an $\mathfrak{X}$-projector. But every $\mathfrak{R}\mathfrak{S}^*$-Schunck class which contains $\mathfrak{X}$ also contains $\mathfrak{D}$. This example even shows that there does not exist an $\mathfrak{R}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that the $\mathfrak{X}$-projectors of a $\mathfrak{D}$-group coincide with its $\mathfrak{H}$-projectors. However, locally defined $\mathfrak{D}$-formations are still induced from $\mathfrak{R}\mathfrak{S}^*$-Schunck classes.

Proposition 2.2. *Let $\mathfrak{D}$ be a ϱs-closed subclass of $\mathfrak{R}\mathfrak{S}^*$ and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation. Then there exists a $\mathfrak{R}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that $\mathfrak{F} = \mathfrak{D} \cap \mathfrak{H}$.*

Proof. Let $\mathfrak{H}_0$ be the class of all finite soluble groups whose primitive images belong to $\mathfrak{F}$ and let $\mathfrak{H}$ denote the class of all $\mathfrak{R}\mathfrak{S}^*$-groups G such that every finite primitive and every infinite semiprimitive factor group is the union of an ascending chain of $\mathfrak{H}_0$-groups. By [**15**, Proposition 3.3], $\mathfrak{H}$ is a $\mathfrak{R}\mathfrak{S}^*$-Schunck class satisfying $\mathfrak{H}^* = \mathfrak{H}_0$.

Assume that f is a preformation function for $\mathfrak{F}$ and let $G \in \mathfrak{F}$. In order to show that $G \in \mathfrak{H}$, it clearly suffices to show that every infinite semiprimitive image of G is the union of finite $\mathfrak{F}$-groups. Thus we may assume that G is an infinite semiprimitive group. In particular, $G = M \ltimes D$, where M is finite, $\mathrm{Core}_G(M) = 1$ and D is an abelian p-group. Therefore $O_{p'}(G) = 1$ and hence $G \in \mathfrak{S}_p f(p)$. It follows that $M \in \mathfrak{S}_p f(p)$, and so also $MD_n \in \mathfrak{S}_p f(p)$ for every $n \in \mathbb{N}$, where $D_n = \{x \in D \mid x^{p^n} = 1\}$. Similarly, if q is a prime $\neq p$, then $G \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$, and so also M and the subgroups MD_n $(n \in \mathbb{N})$ belong to $\mathfrak{S}_{q'}\mathfrak{S}_q f(q)$. Thus $MD_n \in \mathfrak{F}^* \subseteq \mathfrak{H}_0$ for every $n \in \mathbb{N}$, and so $G = \bigcup_{n \in \mathbb{N}} MD_n \in \mathfrak{H}$. Thus $\mathfrak{F} \subseteq \mathfrak{H}$.

Now let $G \in \mathfrak{H} \cap \mathfrak{D}$ and let F be an $\mathfrak{F}$-projector of G. If $G \notin \mathfrak{F}$, then F is contained in a major subgroup M of G. By [**23**], $G/\mathrm{Core}_G(M)$ is either finite and primitive or infinite and semiprimitive, and $M/\mathrm{Core}_G(M)$ is finite. Thus in both cases $F\,\mathrm{Core}_G(M)/\mathrm{Core}_G(M)$ is a proper finite subgroup of $G/\mathrm{Core}_G(M)$. Since the latter is the union of an ascending chain of finite $\mathfrak{H}$-groups and $\mathfrak{F}^* = \mathfrak{H}^* \cap \mathfrak{D}$, the subgroup $F\,\mathrm{Core}_G(M)/\mathrm{Core}_G(M)$ cannot be $\mathfrak{F}$-maximal in $G/\mathrm{Core}_G(M)$. This contradiction shows that $G \in \mathfrak{F}$.

If $\mathfrak{F}$ is a locally defined formation of characteristic π, then every $\mathfrak{F}$-group is a π-group. This shows that the hypothesis of Theorem 1.6.(b) is always satisfied if $\mathfrak{H} = \mathfrak{F}$ is a locally defined formation. Thus we obtain:

Theorem 2.3. *Let $\mathfrak{D}$ be a ϱs-closed class of $\mathfrak{R}\mathfrak{S}^*$-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the $\mathfrak{D}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorised*

$\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then H is even contained in a factorised $\mathfrak{F}$-subgroup of G.

Proof. By the definition of $\mathfrak{F}$, the $\mathfrak{F}$-group H is a π-group. Hence H is contained in the Sylow π-subgroup $A_\pi B_\pi$ of G. Moreover, by Proposition 2.2, there exists an $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that $\mathfrak{F} = \mathfrak{D} \cap \mathfrak{H}$. Now it follows from Theorem 1.6 that the factoriser X of H in $A_\pi B_\pi$ is an $\mathfrak{H}$-group, hence an $\mathfrak{F}$-group. Since $A_\pi B_\pi$ is a prefactorised subgroup of G, the subgroup X is the required prefactorised subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then $A \cap B$ is a π-group and so $A \cap B = A_\pi \cap B_\pi$ is contained in X. Hence X is a factorised subgroup of G.

Note that Corollary 1.7, Corollary 1.8, Corollary 1.9 and Corollary 1.10 can also be formulated in terms of locally defined $\mathfrak{D}$-formations, where $\mathfrak{D}$ is a ϱs-closed subclass of $\mathfrak{N}\mathfrak{S}^*$.

3. Factorisers of $\mathfrak{F}$-subgroups of *FC*- and *CC*-groups. Since the concept of Schunck classes has not yet been extended to the class of all periodic locally soluble *CC*-groups, we formulate our theorems for locally defined $\mathfrak{D}$-formations of periodic locally soluble *CC*-groups only. Recall that a group G is an *FC*-group (a *CC*-group), if $G/C_G(x^G)$ is finite (is a Černikov group). Note also that *FC*-groups are *CC*-groups, so that our results hold in particular for locally defined formations of periodic locally soluble *FC*-groups.

First, we show that, as in the case of $\mathfrak{N}\mathfrak{S}^*$-groups, every $\mathfrak{F}$-subgroup of a *CC*-group G is contained in an $\mathfrak{F}$-maximal subgroup of G.

LEMMA 3.1. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Moreover, let G be a $\mathfrak{D}$-group.*

(a) *The group G is an $\mathfrak{F}$-group if and only if G is a π-group and $G/C_G(x^G) \in \mathfrak{F}$ for every $x \in G$.*
(b) *The class $\mathfrak{F}$ is closed with respect to unions of chains of subgroups.*

Proof. (a) If G is an $\mathfrak{F}$-group, then clearly every factor group of G belongs to $\mathfrak{F}$. Conversely, suppose that $G/C_G(x^G) \in \mathfrak{F}$ for every $x \in G$. Since $Z(G) = \bigcap_{x \in G} C_G(x^G)$, we have $G/Z(G) \in \mathfrak{F}$ by [**7**, Lemma 6.2.8], and so it follows from the definition of a locally defined formation that also $G \in \mathfrak{F}$.

(b) Let $\{G_i\}$ be a chain of $\mathfrak{F}$-subgroups of the $\mathfrak{D}$-group G and assume without loss of generality that $G = \bigcup G_i$. If $x \in G$, then $G/C_G(x^G)$ is a Černikov group. Since $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$ is obviously a locally defined $(\mathfrak{D} \cap \mathfrak{N}\mathfrak{S}^*)$-formation, by Proposition 2.2, there exists an $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^* = \mathfrak{H} \cap \mathfrak{D}$. Hence it follows from [**15**, Lemma 3.1] that the factor groups $G/C_G(x^G)$ are $\mathfrak{H}$-groups for every $x \in G$. Since every factor group of G is a $\mathfrak{D}$-group, it follows that $G \in \mathfrak{F}$ by (a).

Now we can prove an analogue of Theorem 2.3 for periodic *CC*-groups which are the product of two locally nilpotent subgroups.

THEOREM 3.2. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the*

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$\mathfrak{D}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorised $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then the factoriser of H is an $\mathfrak{F}$-subgroup of G.

Proof. Suppose first that $\pi(A) \cap \pi(B) \subseteq \pi$ and let X denote the factoriser of H in G. By [**16**, Theorem 4.7], the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into X. Therefore we may assume without loss of generality that $G = X$. Hence it remains to show that $G \in \mathfrak{F}$.

Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group. Moreover, the Sylow generating basis

$$\{A_pB_pC_G(x^G)/C_G(x^G) \mid p \in \mathbb{P}\}$$

of $G/C_G(x^G)$ reduces into the group $HC_G(x^G)/C_G(x^G)$. Since $\mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$ is a locally defined $\mathfrak{N}\mathfrak{S}^*$-formation, by Theorem 2.3 the factoriser $Y/C_G(x^G)$ of the $\mathfrak{F}$-group $HC_G(x^G)/C_G(x^G)$ is also an $\mathfrak{F}$-group. Now Y is a factorised subgroup of G containing H, and so $G = Y$ and $G/C_G(x^G) \in \mathfrak{F}$. Therefore $G \in \mathfrak{F}$ by Lemma 3.1 (a).

As in the case of Theorem 1.6, we deduce a number of useful consequences, whose proofs are similar to the corresponding results about nilpotent-by-finite groups. First, we derive a necessary and sufficient condition for an $\mathfrak{F}$-maximal subgroup of G to be factorised.

COROLLARY 3.3. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the $\mathfrak{D}$-group G is the product of two locally nilpotent subgroups A and B and let H be an $\mathfrak{F}$-maximal subgroup of G.*

(a) *The subgroup H is prefactorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*
(b) *If π contains $\pi(A) \cap \pi(B)$, then the subgroup H is factorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

Since the Sylow bases of a periodic locally soluble CC-groups are locally conjugate by [**19**, Theorem 4.3], the following lemma shows that in Theorem 3.2, every $\mathfrak{F}$-subgroup H has a local conjugate into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces. Recall that an automorphism α of a group G is called *locally inner* if, for every finite subset X of G, there exists an element $g \in G$ such that $x^\alpha = x^g$ for every $x \in X$. Two Sylow bases $\{G_p \mid p \in \mathbb{P}\}$ and $\{H_p \mid p \in \mathbb{P}\}$ of G are locally conjugate if there exists a locally inner automorphism α such that $G_p^\alpha = H_p$ for every $p \in \mathbb{P}$.

LEMMA 3.4. *Let G be a periodic locally soluble CC-group and H a subgroup of G. Then every Sylow generating basis of H can be extended to a Sylow generating basis of G.*

Proof. Let $\{H_p \mid p \in \mathbb{P}\}$ be a Sylow generating basis of H. For every prime p, put

$$H_{p'} = \langle H_q \mid q \in \mathbb{P}, q \neq p\rangle.$$

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Moreover, let $G_{p'}$ be a Sylow p'-subgroup of G which contains $H_{p'}$. Define

$$G_p = \bigcap_{q \in \mathbb{P}, q \neq p} G_{q'},$$

then $\{G_p \mid p \in \mathbb{P}\}$ is a Sylow generating basis of G by [**19**, Lemma 4.2]. Since H_p is contained in G_p for every prime p, the Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ reduces into H.

For $\mathfrak{F}$-maximal subgroups, this has the following consequence.

THEOREM 3.5. *Let $\mathfrak{D}$ be a Qs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Suppose that the CC-group G is the product of two locally nilpotent subgroups A and B. Then:*

(a) *Every $\mathfrak{F}$-maximal subgroup of G is locally conjugate to a prefactorised $\mathfrak{F}$-maximal subgroup of G.*
(b) *If π contains $\pi(A) \cap \pi(B)$, then every $\mathfrak{F}$-maximal subgroup of G is locally conjugate to a factorised $\mathfrak{F}$-maximal subgroup of G.*

To prove that a periodic locally soluble CC-group which is the product of two locally nilpotent subgroups has at most one prefactorised $\mathfrak{F}$-projector, we need the following result.

PROPOSITION 3.6. *Let $\mathfrak{D}$ be a Qs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. If the $\mathfrak{D}$-group G possesses an $\mathfrak{F}$-projector, then every Sylow generating basis of G reduces into a unique $\mathfrak{F}$-projector of G. Thus the $\mathfrak{F}$-projectors of G are locally conjugate.*

Proof. Let H be an $\mathfrak{F}$-projector of G, then by Lemma 3.4, there exists a Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ of G which reduces into H. Now assume that L is another $\mathfrak{F}$-projector into which $\{G_p \mid p \in \mathbb{P}\}$ reduces. Let $x \in G$, then the Sylow generating basis

$$\{G_p C_G(x^G)/C_G(x^G) \mid p \in \mathbb{P}\}$$

of $G/C_G(x^G)$ reduces into both $HC_G(x^G)/C_G(x^G)$ and $LC_G(x^G)/C_G(x^G)$. Therefore

$$HC_G(x^G)/C_G(x^G) = LC_G(x^G)/C_G(x^G)$$

by [**15**, Corollary 5.2]. Put $H^* = \bigcap_{x \in G} HC_G(x^G)$, then

$$H^* C_G(x^G)/C_G(x^G) = HC_G(x^G)/C_G(x^G) \in \mathfrak{F}.$$

Thus by [**7**, Lemma 6.2.8], $H^*/Z(G) \in \mathfrak{F}$, and it follows from the definition of a locally defined formation that $H^* \in \mathfrak{F}$. Since H^* contains both H and L, it follows from the $\mathfrak{F}$-maximality of H and L that $H = H^* = L$.

Now let H and H^* be arbitrary $\mathfrak{F}$-projectors of G and suppose that $\{G_p \mid p \in \mathbb{P}\}$ and $\{G_p^* \mid p \in \mathbb{P}\}$ are Sylow bases of G reducing into H and H^*, respectively. Since the Sylow bases of G are locally conjugate by [**19**, Theorem 4.3], there exists a locally inner automorphism ϕ of G such that $G_p^\phi = G_p^*$ for every $p \in \mathbb{P}$. The Sylow generating basis $\{G_p^* \mid p \in \mathbb{P}\}$ reduces into H^ϕ and H^*, and so we have $H^* = H^\phi$ by the first part.

The next theorem shows that a result similar to Corollary 1.9 holds for ϱs-closed classes of periodic locally soluble CC-groups if they admit projectors. Note that $\mathfrak{F}$-projectors exist (and are locally conjugate) if $\mathfrak{D}$ is the class of all periodic locally soluble CC-groups [**20**], or if $\mathfrak{D}$ is a ϱs-closed class of periodic locally soluble FC-groups [**21**]; see also [**22**].

Theorem 3.7. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble CC-groups. Suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation of characteristic π, where $\mathfrak{D}$ is either the class of all periodic locally soluble CC-groups, or a ϱs-closed class of FC-groups. If G is a $\mathfrak{D}$-group which is the product of two locally nilpotent subgroups A and B, then G has at most one prefactorised $\mathfrak{F}$-projector. If G has $\mathfrak{F}$-projectors, then G possesses a unique $\mathfrak{F}$-projector which is prefactorised. If π contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorised.*

Despite the fact the Sylow bases of a periodic locally soluble CC-group need not be conjugate, also a result similar to Corollary 1.10 can be obtained.

Theorem 3.8. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble CC-groups and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Moreover, suppose that the $\mathfrak{D}$-group G has subgroups A, B and C such that $G = AB = AC = BC$. If A and B are locally nilpotent, $C \in \mathfrak{F}$ and $\pi(A) \cap \pi(B)$ is contained in π, then $G \in \mathfrak{F}$.*

Proof. Let $x \in G$, then $G/C_G(x^G)$ is a Černikov group. By Proposition 2.2, there exists an $\mathfrak{N}\mathfrak{S}^*$-Schunck class $\mathfrak{H}$ such that $\mathfrak{H} \cap \mathfrak{D} = \mathfrak{F} \cap \mathfrak{N}\mathfrak{S}^*$. Therefore $G/C_G(x^G) \in \mathfrak{H} \cap \mathfrak{D} \subseteq \mathfrak{F}$ by Corollary 1.10 and $G \in \mathfrak{F}$ by Lemma 3.1 (a).

4. Factorisers of $\mathfrak{F}$-subgroups of groups with min-p for all primes p. Since periodic locally soluble groups satisfying the minimal condition on p-subgroups for every prime p are residually Černikov groups by [**18**, Theorem 3.17], the methods applied to periodic CC-groups which are the product of two locally nilpotent subgroups yield essentially the same results for periodic locally soluble groups satisfying min-p for every prime p. The main difficulties are due to the fact that Sylow bases of the latter class of groups are not so well-behaved as in the case of CC-groups.

Theorem 4.1. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the $\mathfrak{D}$-group G is the product of two locally nilpotent groups A and B. If H is an $\mathfrak{F}$-subgroup of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, then H is contained in a prefactorised $\mathfrak{F}$-subgroup of G. If $\pi(A) \cap \pi(B) \subseteq \pi$, then H is even contained in a factorised $\mathfrak{F}$-subgroup of G.*

Proof. Let X denote the factoriser of H in $A_\pi B_\pi$, then we may assume without loss of generality that $G = X$. Since by [**18**, Theorem 3.17], the factor group

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$G/O_{\sigma'}(G)$ is a Černikov group for every finite set σ of primes, an argument similar to that in the proof of Theorem 3.2 shows that $G/O_{\sigma'}(G) \in \mathfrak{F}$. Now the intersection of all subgroups $O_{\sigma'}(G)$, where σ is a finite set of primes, is trivial, and we have $G \in \mathfrak{F}$ by [**7**, Lemma 6.2.8].

For $\mathfrak{F}$-maximal subgroups, this has the following consequence.

Corollary 4.2. *Let $\mathfrak{D}$ be a QS-closed class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the $\mathfrak{D}$-group G is the product of two locally nilpotent subgroups A and B and let H be an $\mathfrak{F}$-maximal subgroup of G.*

(a) *The subgroup H is prefactorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*
(b) *If π contains $\pi(A) \cap \pi(B)$, then the subgroup H is factorised if and only if the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces into H.*

The proof of the next theorem does not use the above results about periodic locally soluble products satisfying min-*p*. Instead, it relies on the nilpotent-by-finite case.

Theorem 4.3. *Let $\mathfrak{D}$ be a QS-closed class of periodic locally soluble groups satisfying min-p for all primes p and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Further, suppose that the $\mathfrak{D}$-group G has a triple factorisation $G = AB = AC = BC$ by three subgroups A, B and C, where A and B are locally nilpotent and $C \in \mathfrak{F}$. If $\pi(A) \cap \pi(B) \subseteq \pi$, then $G \in \mathfrak{F}$.*

Proof. Let σ be a finite set of primes. By [**18**, Theorem 3.17], the factor group $G/O_{\sigma'}(G)$ is a Černikov group and so by Corollary 1.10, we have $G/O_{\sigma'}(G) \in \mathfrak{F}$ for every finite set σ of primes. Since the intersection of the subgroups $O_{\sigma'}(G)$, where σ is a finite set of primes, is trivial, we have $G \in \mathfrak{F}$ by [**7**, Lemma 6.2.8].

The following result is probably well known.

Lemma 4.4. *Let $\mathfrak{D}$ be a QS-closed class of periodic locally soluble groups and assume that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation. Further, let H be an $\mathfrak{F}$-maximal subgroup of the $\mathfrak{D}$-group G and assume that $\mathcal{N}$ is a set of normal subgroups of G such that the intersection of all $N \in \mathcal{N}$ is trivial. Then*

$$H = \bigcap_{N \in \mathcal{N}} HN.$$

Proof. Let $L = \bigcap_{N \in \mathcal{N}} HN$, then $LN = HN$ for every $N \in \mathcal{N}$. This shows that $L/L \cap N \cong LN/N = HN/N \in \mathfrak{F}$. Therefore by [**7**, Lemma 6.2.8], we have $L \in \mathfrak{F}$. Since H is contained in L and H is $\mathfrak{F}$-maximal, we have $H = L$, as required.

The next proposition will be used to show that a periodic locally soluble group satisfying min-*p* for every prime *p* which is the product of two locally nilpotent subgroups has at most one prefactorised $\mathfrak{F}$-projector.

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PROPOSITION 4.5. *Let $\mathfrak{D}$ be a* QS*-closed class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation. If the $\mathfrak{D}$-group G has an $\mathfrak{F}$-projector, then every Sylow generating basis of G reduces into at most one $\mathfrak{F}$-projector of G.*

Proof. Let H and L be $\mathfrak{F}$-projectors of G into which the Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ of G reduces. Let $p \in \mathbb{P}$, then the Sylow generating basis

$$\{G_q O_{p'}(G)/O_{p'}(G) \mid q \in \mathbb{P}\}$$

of $G/O_{p'}(G)$ reduces into $HO_{p'}(G)/O_{p'}(G)$ and $LO_{p'}(G)/O_{p'}(G)$. Thus by [**15**, Corollary 5.2], we have $HO_{p'}(G) = LO_{p'}(G)$. Since $\bigcap_{p\in\mathbb{P}} O_{p'}(G) = 1$, it follows from Lemma 4.4 that $H = L$.

Although the Sylow bases of a periodic locally soluble group G satisfying min-p for every prime p are locally conjugate by [**8**], G may have $L\mathfrak{N}$-projectors into which no Sylow generating basis reduces [**6**, Section 5], even if G is countable. Therefore our next result might also be of independent interest. Recall that a group G is *co-Hopfian* if it does not contain a proper subgroup isomorphic with G. In particular, every periodic radical group satisfying min-p is co-hopfian, see [**4**].

PROPOSITION 4.6. *Let $\mathfrak{D}$ be a* QS*-closed class of countable locally finite-soluble group satisfying min-p for all primes p. If $G \in \mathfrak{D}$ and the locally defined $\mathfrak{D}$-formation $\mathfrak{F}$ is a class of co-Hopfian groups, then every Sylow generating basis of G reduces into a unique $\mathfrak{F}$-projector of G.*

Proof. Let $\{G_p \mid p \in \mathbb{P}\}$ be a Sylow generating basis of G and let $\{p_1, p_2, \ldots\}$ denote the set of all primes in their natural order. Set $N_i = O_{\{p_{i+1}, p_{i+2},\ldots\}}$ for every $i \in \mathbb{N}$, then G/N_i is a Černikov group by [**18**, Theorem 3.17]. Hence it has an $\mathfrak{F}$-projector H_i/N_i into which the Sylow generating basis $\{G_pN_i/N_i \mid p \in \mathbb{P}\}$ of G/N_i reduces. Let $H = \bigcap_{n\in\mathbb{N}} H_i$, then by [**15**, Proposition 2.3 (a)], the Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$ also reduces into H. Continuing as in the proof of [**6**, Theorem 3.4], H is an $\mathfrak{F}$-projector of G. The uniqueness statement now follows from Proposition 4.5.

Thus we obtain the following result about projectors of groups which are the product of two locally nilpotent subgroups and satisfy min-p for every prime p.

THEOREM 4.7. *Let $\mathfrak{D}$ be a* QS*-closed class of periodic locally soluble groups satisfying min-p for every prime p and $\mathfrak{F}$ a locally defined $\mathfrak{D}$-formation of characteristic π. Then every $\mathfrak{D}$-group G which is the product of two locally nilpotent subgroups A and B has at most one prefactorised $\mathfrak{F}$-projector. If $\mathfrak{F}$ is a class of co-Hopfian groups, then G possesses a unique $\mathfrak{F}$-projector which is prefactorised. If, in addition, π contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorised.*

Proof. Suppose that H is a prefactorised $\mathfrak{F}$-projector of G. By [**16**, Theorem 5.7], the group G is countable with Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ which reduces into H. Thus by Proposition 4.5, G possesses at most one prefactorised $\mathfrak{F}$-projector.

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Now assume that $\mathfrak{F}$ is a class of co-hopfian groups. Since G is countable, we may clearly suppose that $\mathfrak{D}$ and $\mathfrak{F}$ consist of countable groups. Thus by Proposition 4.6, G possesses an $\mathfrak{F}$-projector L into which $\{A_pB_p \mid p \in \mathbb{P}\}$ reduces. Thus the second statement of the theorem follows from Corollary 4.2.

5. Projectors in soluble and hypoabelian $\mathfrak{U}$-groups. Let $\mathfrak{F}$ be a locally defined $\mathfrak{U}$-formation. Although we have not been able to prove the existence of prefactorised $\mathfrak{F}$-maximal subgroups of a $\mathfrak{U}$-group G which is the product of two locally nilpotent subgroups, we have nevertheless obtained positive results for the most important class of $\mathfrak{F}$-maximal subgroups of G, namely for $\mathfrak{F}$-projectors of G. As a first step, we consider periodic locally soluble groups which are the extension of a p-group by an $\mathfrak{F}$-group.

Let G be a group and suppose that $\mathfrak{F}$ is any class of groups. Then $G^{\mathfrak{F}}$ denotes the intersection of all normal subgroups N of G such that $G/N \in \mathfrak{F}$. Observe that if $\mathfrak{F}$ is a $\mathfrak{D}$-formation for some ϱ-closed class $\mathfrak{D}$ of groups, then $G/G^{\mathfrak{F}} \in \mathfrak{F}$.

Proposition 5.1. *Suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation of characteristic π for some ϱs-closed class $\mathfrak{D}$ of locally finite groups. Let G be a $\mathfrak{D}$-group such that $G^{\mathfrak{F}}$ is a p-group for some $p \in \pi$ and suppose that H is an $\mathfrak{F}$-maximal subgroup of G which satisfies $G = HG^{\mathfrak{F}}$. Then:*

(a) $H = N_G(O_{p'}(H))$.
(b) *If the Sylow p'-subgroups of every subgroup S of G are conjugate in S, then every Sylow p'-subgroup of G reduces into at most one conjugate of H.*
(c) *If $G^{\mathfrak{F}}$ is abelian, then H complements $G^{\mathfrak{F}}$.*
(d) *If $G^{\mathfrak{F}}$ is abelian, then every Sylow p'-subgroup of G reduces into at most one complement of $G^{\mathfrak{F}}$.*

Proof. (a) Let $Q = O_{p'}(H)$ and set $L = N_G(Q)$, then clearly, $H \leq L$. We will show that $L \in \mathfrak{F}$. Then the desired result will follow from the $\mathfrak{F}$-maximality of H. If $q \neq p$ is a prime, then $G/N \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ by hypothesis, where $N = G^{\mathfrak{F}}$, and so also $L/L \cap N$ belongs to that class. Since N is a q'-group, this shows that $L \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ for every prime $q \neq p$.

Now $L = L \cap HN = H(L \cap N)$ and $(H \cap N) \cap Q(L \cap N) = Q(H \cap N)$ by the modular law, and so

$$L/Q(L \cap N) = H(L \cap N)/Q(L \cap N) \cong H/Q(H \cap N) \in \mathfrak{S}_p f(p)$$

because $H/Q \in \mathfrak{S}_p f(p)$. Therefore also $L/Q \in \mathfrak{S}_p f(p)$ and consequently $L \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$. Since G is a π-group contained in $\mathfrak{D}$, the same is true for L, and we have $L \in \mathfrak{F}$ by the definition of a locally defined $\mathfrak{D}$-formation. Therefore $H = L = N_G(O_{p'}(H))$.

(b) Suppose that the Sylow p'-subgroup $G_{p'}$ reduces into H and H^g. Then $G_{p'}^{g^{-1}}$ reduces into H. Let H_p be a Sylow p-subgroup of H, then $H = (H \cap G_{p'})H_p$ by [**11**, Lemma 2.1]. Therefore $G_{p'} = G_{p'} \cap HN = G_{p'} \cap (H \cap G_{p'})H_pN = (H \cap G_{p'})(G_{p'} \cap H_pN) = (H \cap G_{p'})$ is a Sylow p'-subgroup of H, and by the same argument, also $G_{p'}^{g^{-1}}$ is a Sylow p'-subgroups of H. Since H is a $\mathfrak{U}$-group, it follows that $G_{p'}^{g^{-1}} = G_{p'}^h$ for some $h \in H$. Therefore $gh \in N_G(G_{p'})$. Since $G_{p'}$ is contained in H, we clearly have

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$N_G(G_{p'}) \leq N_G(O_{p'}(H))$ and so $gh \in H$ by (a). This shows that $g \in H$, proving that $H = H^g$.

(c) Put $N = G^{\mathfrak{F}}$ and $Q = O_{p'}(H)$ and observe that NQ is a normal subgroup of G. Therefore also $K = [N, Q] = [N, NQ]$ is normal in G.

First, we show that $G/K \in \mathfrak{F}$. Since N/K is a p-group, we have $G/K \in \mathfrak{S}_{q'}\mathfrak{S}_q f(q)$ for every prime $q \neq p$. Now $G/NQ \in \mathfrak{S}_p f(p)$ as in the proof of (a). Since $Q^N = Q[Q, N] = QK$ and $Q \trianglelefteq H$, the subgroup QK is normalised by $NH = G$ and so QK is a normal subgroup of G. Moreover, QN/QK is a p-group, and so also $G/QK \in \mathfrak{S}_p f(p)$. But then $G/K \in \mathfrak{S}_{p'}\mathfrak{S}_p f(p)$, and so $G/K \in \mathfrak{F}$. Therefore we have $N = G^{\mathfrak{F}} \leq K$ and so $N = [N, Q]$.

Next, we show that $C_N(Q) = 1$. Let $x \in C_N(Q)$. Since $x \in N$, we have $x = \prod_{i=1}^{n}[y_i, q_i]$, where $y_i \in N$ and $q_i \in Q$. Let $Q_0 = \langle q_1, \ldots, q_n \rangle \leq Q$ which is a finitely generated subgroup of Q, hence is finite, and so also $Y = \langle x, y_1, \ldots y_n \rangle^{Q_0} \leq N$ is finite. Applying [**17**, III.13.4] to the finite group $Q_0 Y$, we obtain that $Y = [Y, Q_0] \times C_Y(Q_0)$. In particular, we have $x \in [Y, Q_0] \cap C_Y(Q_0) = 1$ and so $C_N(Q) = 1$.

Now the normal p-subgroup $H \cap N$ of H centralises $Q = O_{p'}(H)$ and so $H \cap N = 1$, as required.

(d) Suppose that the Sylow p'-subgroup $G_{p'}$ of G reduces into H and H^*. Since both H and H^* complement $N = G^{\mathfrak{F}}$ by (c), we have $O_{p'}(H)N/N = O_{p'}(G/N) = O_{p'}(H^*)N/N$. So $O_{p'}(H^*) = G_{p'} \cap NO_{p'}(H) = O_{p'}(H)$ and thus $H = H^*$ by (a).

Our next lemma is the key to finding prefactorised $\mathfrak{F}$-projectors.

LEMMA 5.2. *Let π be a set of primes and suppose that the group G is the product of two subgroups A and B. Further, assume that A and B have Sylow subgroups A_π, $A_{\pi'}$, B_π and $B_{\pi'}$ respectively such that $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. If $A_\pi B_\pi$ is a Sylow π-subgroup of G and N is a normal π'-subgroup of G such that $L/N = O_\pi(G/N)$ is a prefactorised subgroup of G/N, then $L \cap A_\pi B_\pi$ is a prefactorised Sylow π-subgroup of L.*

Proof. By hypothesis, we have $L/N = (L/N \cap AN/N)(L/N \cap BN/N)$ and so

$$L = (L \cap AN)(L \cap BN) = (L \cap A)N(L \cap B)$$

by the modular law. Since L/N is a π-group, it follows that $A_{\pi'} \cap L \leq N$ and $B_{\pi'} \cap L \leq N$. Since $A = A_\pi \times A_{\pi'}$, we have $L \cap A = (L \cap A_\pi) \times (L \cap A_{\pi'})$, and hence we obtain $L = (L \cap A_\pi)(L \cap B_\pi)N$. Now the set $(L \cap A_\pi)(L \cap B_\pi)$ is clearly contained in $L \cap A_\pi B_\pi$ which is a π-group. Put $A^* = (L \cap A_\pi)N$ and $B^* = (L \cap B_\pi)N$, then [**5**, Lemma 2], applied to $L = A^* B^*$, shows that $(L \cap A_\pi)(L \cap B_\pi)$ is a Sylow π-subgroup of L, and so $L \cap A_\pi B_\pi = (L \cap A_\pi)(L \cap B_\pi)$, as required.

Recall that a group is hypoabelian if it has a descending series with abelian factors. Hence every soluble group is hypoabelian. Note also that the following theorem does not claim that $\mathfrak{F}$-projectors or Sylow generating bases do exist in the group G or, in case they exist, that any Sylow generating basis of G reduces into an $\mathfrak{F}$-projector of G.

THEOREM 5.3. *Let $\mathfrak{D}$ be a ϱs-closed class of periodic locally soluble groups and suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation. Assume that $G \in \mathfrak{D}$ and that H is an*

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$\mathfrak{F}$-projector of G. If G is hypoabelian or a $\mathfrak{U}$-group, then every Sylow generating basis of G reduces into at most one $\mathfrak{F}$-projector of G.

Proof. Suppose that $\{G_p \mid p \in \mathbb{P}\}$ is a Sylow generating basis of G and that H and L are $\mathfrak{F}$-projectors of G into which $\{G_p \mid p \in \mathbb{P}\}$ reduces.

Since G is hypoabelian or a $\mathfrak{U}$-group, there exists an ordinal α such that G possesses a descending series

$$G = N_0 \rhd N_1 \rhd \cdots \rhd N_\alpha = 1$$

whose factors $N_\beta/N_{\beta+1}$ are p-groups for some prime p depending on $\beta < \alpha$. In case G is hypoabelian, we may also assume that every factor $N_\beta/N_{\beta+1}$ is abelian. Let $\beta < \alpha$, then by [**15**, Proposition 2.3 (b)] the Sylow generating basis $\{G_pN_\beta/N_\beta \mid p \in \mathbb{P}\}$ reduces into the $\mathfrak{F}$-projectors HN_β/N_β and LN_β/N_β of G/N_β, and so by transfinite induction, we have $HN_\beta = LN_\beta$ for all $\beta < \alpha$. Thus if α is a limit ordinal, then we have

$$H = \bigcap_{\beta<\alpha} HN_\beta = \bigcap_{\beta<\alpha} LN_\beta = L,$$

by Lemma 4.4.

Otherwise, α has a predecessor $\alpha - 1$. Then $N_{\alpha-1}$ is a p-group for a prime p, and $HN_{\alpha-1} = LN_{\alpha-1}$. Now H and L are $\mathfrak{F}$-maximal subgroups of $HN_{\alpha-1}$ and $\{G_p \mid p \in \mathbb{P}\}$ reduces into $HN_{\alpha-1}$ by [**15**, Proposition 2.3 (d)]. In particular, if $G_{p'} = \langle G_q \mid q \in \mathbb{P}, q \neq p\rangle$, then $G_{p'}$ reduces into $HN_{\alpha-1}$, H and L. The result now follows from Proposition 5.1 (b) if $G \in \mathfrak{U}$ and from Proposition 5.1 (d) if G is hypoabelian.

Since every $\mathfrak{U}$-group G possesses $\mathfrak{F}$-projectors by [**11**] and by [**13**, Lemma 2.1], there exists a Sylow generating basis of G reducing into a given subgroup of G, we have:

Corollary 5.4. *Let $\mathfrak{D}$ be a Qs-closed class of $\mathfrak{U}$-groups and suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation. If $G \in \mathfrak{D}$, then every Sylow generating basis of G reduces into exactly one $\mathfrak{F}$-projector of G.*

Now we are ready to prove the main theorem of this section.

Theorem 5.5. *Let $\mathfrak{D}$ be a Qs-closed class of $\mathfrak{U}$-groups and suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation of characteristic π. Moreover, let the $\mathfrak{D}$-group G be the product of two locally nilpotent subgroups A and B. If G has a normal subgroup N such that $G/N \in \mathfrak{F}$ and N has a hypoabelian Sylow π-subgroup, then G has a unique prefactorised $\mathfrak{F}$-projector H, and this $\mathfrak{F}$-projector contains $A_\pi \cap B_\pi$. Thus if the characteristic π of $\mathfrak{F}$ contains $\pi(A) \cap \pi(B)$, then H is factorised.*

Proof. By Corollary 5.4, there exists a unique $\mathfrak{F}$-projector H of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces, and by [**16**, Theorem 4.7], this is the only $\mathfrak{F}$-projector of G which may be prefactorised.

Since every $\mathfrak{F}$-group is a π-group, H is contained in the Sylow π-subgroup $A_\pi B_\pi$ of G. Since H is also an $\mathfrak{F}$-projector of $A_\pi B_\pi$ by [**11**, Theorem 5.4], it will suffice to show that H is a factorised subgroup of $A_\pi B_\pi$. Since $N \cap A_\pi B_\pi$ is hypoabelian, we may assume without loss of generality that $G = A_\pi B_\pi$ and that N is hypoabelian.

Now let

$$N = N_1 \rhd N_2 \rhd \ldots \rhd N_\alpha = 1$$

be a descending normal series of N with abelian factors which are p-groups for suitable primes p. Clearly, we may assume that $\alpha > 1$. Let $\beta < \alpha$, then the Sylow generating basis

$$\{A_p B_p N_\beta / N_\beta \mid p \in \mathbb{P}\}$$

of G/N_β reduces into the $\mathfrak{F}$-projector HN_β/N_β of G/N_β and hence by induction on α, the subgroup HN_β is factorised for all $\beta < \alpha$. If α is a limit ordinal, then by Lemma 4.4,

$$H = \bigcap_{\beta<\alpha} HN_\beta$$

and so H is factorised. Therefore assume that α has a predecessor. Now the Sylow generating basis $\{A_p B_p \mid p \in \mathbb{P}\}$ of G reduces into the factorised subgroup $HN_{\alpha-1}$, and consequently it suffices to consider the case when $G = HN_{\alpha-1}$ and $N = N_{\alpha-1}$. Since $G/N \in \mathfrak{F}$ and N is an abelian p-group, also the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is an abelian p-group. Thus we may assume without loss of generality that $N = G^{\mathfrak{F}}$. Then H complements N by Proposition 5.1 (c), and so $O_{p'}(G/N) = O_{p'}(H)N/N$. Since $O_{p'}(G/N)$ is a prefactorised subgroup of G/N by [**16**, Theorem 5.3], it follows from Lemma 5.2 that $O_{p'}(H) = A_{p'}B_{p'} \cap O_{p'}(H)N$ is prefactorised. Moreover, $A_{p'} \cap O_{p'}(H)N = A_{p'} \cap O_{p'}(H)$ is a normal subgroup of $A_{p'}$, hence of A, and similarly, $B_{p'} \cap O_{p'}(H)$ is a normal subgroup of B. Therefore by [**26**, *Hilfssatz* 7] (see also [**1**, Lemma 1.2.2]), the normaliser $N_G(O_{p'}(H))$ of $O_{p'}(H) = (A_{p'} \cap O_{p'}(H))(B_{p'} \cap O_{p'}(H))$ is factorised. Since we have $H = N_G(O_{p'}(H))$ by Proposition 5.1 (a), it follows that H is factorised.

Since by [**25**, Theorem A1], every periodic locally soluble linear group is a soluble $\mathfrak{U}$-group, we also have:

Corollary 5.6. *Let $\mathfrak{D}$ be a* Qs*-closed class of periodic locally soluble linear groups and suppose that $\mathfrak{F}$ is a locally defined $\mathfrak{D}$-formation of characteristic π. Moreover, let the $\mathfrak{D}$-group G be the product of two locally nilpotent subgroups A and B. Then G has a unique prefactorised $\mathfrak{F}$-projector, and this $\mathfrak{F}$-projector contains $A_\pi \cap B_\pi$. Thus if the characteristic π of $\mathfrak{F}$ contains $\pi(A) \cap \pi(B)$, then this $\mathfrak{F}$-projector is factorised.*

6. System normalisers and Carter subgroups of $\mathfrak{U}$-groups. Let G be a $\mathfrak{U}$-group which is the product of two locally nilpotent subgroups. If G is not hypoabelian, the techniques used in the last section to prove the existence of a prefactorised $\mathfrak{F}$-projector of G cannot be applied any more. This is mainly due to the fact that then Proposition 5.1 (c) does not hold if $G^{\mathfrak{F}}$ is a nonabelian p-group. However, we have a positive result about Carter subgroups of $\mathfrak{U}$-groups. Recall that a Carter subgroup is

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simply an $\mathit{L}\mathfrak{N}$-projector, where $\mathit{L}\mathfrak{N}$ denotes the class of all locally nilpotent groups. Also, if G is a group with Sylow generating basis $\{G_p \mid p \in \mathbb{P}\}$, then the subgroup $H = \bigcap_{p\in\mathbb{P}} N_G(G_p)$ is the *system normaliser of G associated with the Sylow generating basis* $\{G_p \mid p \in \mathbb{P}\}$.

PROPOSITION 6.1. *Suppose that the* $\mathfrak{U}$*-group G is the product of two locally nilpotent subgroups. Then G has a factorised system normaliser.*

Proof. Let $\{A_pB_p \mid p \in \mathbb{P}\}$ be the Sylow generating basis of G consisting of prefactorised Sylow subgroups of G. Then for each $p \in \mathbb{P}$, A_p and B_p are normal subgroups of A and B, respectively, and so by [**26**, *Hilfssatz* 7], $N_G(A_pB_p)$ is factorised. Therefore also the system normaliser $D = \bigcap_{p\in\mathbb{P}} N_G(A_pB_p)$ is factorised.

The preceding result about system normalisers can now be used to prove the existence of a unique factorised Carter subgroup.

THEOREM 6.2. *Suppose that the* $\mathfrak{U}$*-group G is the product of two locally nilpotent subgroups. Then G has a unique prefactorised Carter subgroup, and this Carter subgroup is factorised.*

Proof. By Corollary 5.4, there exists a unique Carter subgroup C of G into which the Sylow generating basis $\{A_pB_p \mid p \in \mathbb{P}\}$ of G reduces. Therefore by [**16**, Theorem 4.7], this is the only Carter subgroup of G which may be prefactorised.

Let n denote the length of the Hirsch-Plotkin series of G. If $n \leq 2$, the Carter subgroups of G coincide with its system normalisers [**11**, Theorem 5.1]. So in this case, the result follows from Proposition 6.1. Therefore assume that $n \geq 3$ and let R denote the Hirsch-Plotkin radical of G. Then CR/R is a Carter subgroup of G/R into which the Sylow generating basis $\{A_pB_pR/R \mid p \in \mathbb{P}\}$ of G/R reduces. Thus by induction on n, the subgroup CR of G is factorised. Since C is also a Carter subgroup of CR and $n(CR) = 2 < n$, the subgroup C is factorised in CR, hence in G.

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