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# $n$-PRÜFER DOMAINS

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We introduce $n$-Prüfer domains which are generalisations of Prüfer domains and give several characterisations of them in terms of generalisations of purity, flatness and absolute purity.

## 1. INTRODUCTION

Let $R$ be a commutative domain with 1 and $Q$ its field of quotients. $R$ is called *Prüfer* if every finitely generated ideal of $R$ is projective. Prüfer domains have been characterised in numerous ways. Classical results can be found in Gilmer [**3**].

Here we wish to introduce a generalisation of Prüfer domains: we shall call a domain $R$ *$n$-Prüfer* (for integers $n > 0$ or $n = \infty$) if every finitely generated torsion-free $R$-module of projective dimension $\leqslant n - 1$ is projective. Note that every domain is 1-Prüfer and the $\infty$-Prüfer domains are exactly the Prüfer domains. Trivially, Prüfer domains are $n$-Prüfer for every $0 < n < \infty$, but the converse does not hold.

Examples of such domains $R$ which are $n$-Prüfer for every $0 < n < \infty$, but not Prüfer, are Noetherian domains of Krull dimension 1 which are not integrally closed. The following argument verifies the claim. Let $M$ be a finitely generated, torsion-free $R$-module of projective dimension $\leqslant 1$. Embedding of $M$ into a finitely generated free $R$-module $F$ yields the $R$-module $F/M$. If the projective dimension of $M$ is 1, then the projective dimension of $F/M$ is 2, which contradicts the fact that the Krull dimension of Noetherian domain is equal to its finitistic projective dimension (see, Raynaud and Gruson [**8**]). Hence $M$ is projective, and thus $R$ is 2-Prüfer. Now, induction on the projective dimension of $M$ establishes our claim.

In contrast, Noetherian domains $R$ of Krull dimension $> 1$ are 1-Prüfer but not 2-Prüfer (and thus not $n$-Prüfer for any $n \geqslant 2$). Indeed, let $n > 1$ be the Krull dimension of $R$. By the definition of finitistic projective dimension, there exists an ideal of $R$ of projective dimension $n - 1$. Hence $R$ is not $n$-Prüfer. Now the claim follows from the

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fact that coherent 2-Prüfer domains are $n$-Prüfer every $2 \leqslant n < \infty$, which can be easily proved by induction argument.

In this note, we wish to prove that $n$-Prüfer domains have a number of characterisations, in particular, in terms of generalisations of purity, flatness and absolute purity (see Corollary 2). We also generalise a result on Prüfer domains concerning the coherency of polynomial rings to $n$-Prüfer domains (see Theorem 2). Throughout this note, $n$ will be a positive integer or $\infty$. For unexplained definitions and terminologies, we refer to Fuchs and Salce [**1**, **2**] and Rotman [**7**].

## 2. Preliminaries

Recall that an $R$-module $M$ is *flat* if $\mathrm{Tor}_1^R(M, N) = 0$ holds for all finitely presented $R$-modules $N$. The following generalisation of flatness will be used. An $R$-module $M$ will be called *n-flat* if $\mathrm{Tor}_1^R(M, N) = 0$ holds for all finitely presented $R$-modules $N$ with projective dimension $\leqslant n$. Obviously, direct sums and summands of $n$-flat modules are again $n$-flat. An $R$-submodule $N$ of $M$ is said to be *relatively divisible* (or briefly $RD$) in $M$ if $rN = N \cap rM$ for each $r \in R$. Accordingly, an exact sequence $0 \to N \to M \to L \to 0$ is called an $RD$-exact sequence if the inclusion map embeds $N$ in $M$ as an $RD$-submodule.

We start our discussion with a lemma.

**Lemma 1.** *An $R$-module $M$ is 1-flat if and only if it is torsion-free.*

Proof Let $E$ be an injective cogenerator of the category of $R$-modules, and suppose that the $R$-module $M$ is 1-flat, that is, it satisfies $\mathrm{Tor}_1^R(N, M) = 0$ for all finitely presented $R$-modules $N$ with projective dimension $\leqslant 1$. From the natural isomorphism

$$\mathrm{Ext}_R^1\big(N, \mathrm{Hom}_R(M, E)\big) \cong \mathrm{Hom}_R(\mathrm{Tor}_1^R(N, M), E),$$

it follows that $\mathrm{Ext}_R^1\big(N, \mathrm{Hom}_R(M, E)\big) = 0$ for all finitely presented $R$-modules $N$ with projective dimension $\leqslant 1$. By Fuchs and Salce [**1**, p. 36], $\mathrm{Hom}_R(M, E)$ is then a divisible $R$-module. For the torsion submodule $tM$ of $M$ we have the $RD$-exact sequence $0 \to tM \to M \to M/tM \to 0$ which induces the $RD$-exact sequence $0 \to \mathrm{Hom}_R(M/tM, E) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(tM, E) \to 0$. Here $\mathrm{Hom}_R(tM, E)$ is divisible as an epic image of $\mathrm{Hom}_R(M, E)$ and reduced (because $tM$ is torsion), thus it is 0. Hence $tM$ is 0, and $M$ is torsion-free.

Conversely, if $M$ is a torsion-free $R$-module, then the injection map $M \to Q \bigotimes_R M$ induces an epimorphism $\mathrm{Hom}_R\Big(Q \bigotimes_R M, E\Big) \to \mathrm{Hom}_R(M, E)$. Since $Q \bigotimes_R M$ is torsion-free, the $R$-module $\mathrm{Hom}_R\Big(Q \bigotimes_R M, E\Big)$ is $h$-divisible (that is, an epic image of an injective $R$-module), and therefore so is $\mathrm{Hom}_R(M, E)$. From the exact sequence $0 \to H \to D \to \mathrm{Hom}_R(M, E) \to 0$, where $D$ is a direct sum of copies of $Q$, we obtain $\mathrm{Ext}_R^1\big(N, \mathrm{Hom}_R(M, E)\big) \cong \mathrm{Ext}_R^2(N, H)$. The second Ext is 0 whenever projective $\mathrm{dimension}_R N \leqslant 1$, so the same holds for the first Ext. Hence $\mathrm{Hom}_R\big(\mathrm{Tor}_1^R(N,

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$M), E\big) = 0$ follows. By the choice of $E$ we conclude $\operatorname{Tor}_1^R(N, M) = 0$, showing that $M$ is 1-flat. ∎

Recall that an $R$-module $D$ is said to be *absolutely pure* (or *FP-injective*) if it is a pure submodule in every $R$-module containing it as a submodule. Megibben [**6**] proved that an $R$-module $D$ is absolutely pure if and only if $\operatorname{Ext}_R^1(N, D) = 0$ for all finitely presented $R$-modules $N$. Accordingly, we define an $R$-module $D$ to be *$n$-absolutely pure* if, for all finitely presented $R$-modules $N$ of projective dimension $\leqslant n$, we have $\operatorname{Ext}_R^1(N, D) = 0$. It follows at once that direct sums and summands of $n$-absolutely pure modules are again $n$-absolutely pure. A result similar to Lemma 1 can be obtained.

LEMMA 2. *An $R$-module $D$ is 1-absolutely pure if and only if it is divisible*

PROOF: If $D$ is 1-absolutely pure $R$-module, then $\operatorname{Ext}_R^1(R/L, D) = 0$ for all projective ideals $L$. By Fuchs and Salce [**1**, p. 36], this amounts to the divisibility of $D$. Conversely, suppose $D$ is a divisible $R$-module, and $N$ is a finitely presented $R$-module of projective dimension $\leqslant 1$. Since $N$ has a finite projective resolution, we can apply the natural isomorphism (see for example, Rotman [**7**, p. 257])

$$\operatorname{Tor}_1^R\big(\operatorname{Hom}_R(D, E), N\big) \cong \operatorname{Hom}_R\big(\operatorname{Ext}_R^1(N, D), E\big)$$

for an injective cogenerator $E$. Here $\operatorname{Hom}_R(D, E)$ is a torsion-free $R$-module, so by Lemma 1, it is 1-flat. This implies that the Tor in the last formula vanishes, and consequently, the right side is 0. This leads to the equation $\operatorname{Ext}_R^1(N, D) = 0$, which amounts the 1-absolute purity of $D$. ∎

## 3. $n$-PURITY

Recall that an $R$-module $A$ of $B$ is said to be *pure* if, for all (finitely presented) $R$-modules $N$, the map $N \bigotimes_R A \to N \bigotimes_R B$ induced by the inclusion $A \to B$ is injective. In the same spirit as flatness and absolute purity were generalised, we define a generalisation: an $R$-submodule $A$ of $B$ is called *$n$-pure* if, for all finitely presented $R$-modules $N$ of projective dimension $\leqslant n$, the map $N \bigotimes_R A \to N \bigotimes_R B$ induced by the inclusion $A \to B$ is injective, or equivalently, the map $\operatorname{Hom}_R(N, B) \to \operatorname{Hom}_R(N, B/A)$ induced by the natural map $B \to B/A$ is surjective.

LEMMA 3. *For every $n$ $(0 < n \leqslant \infty)$, the following conditions on an $R$-module $D$ are equivalent:*

(a) *$D$ is $n$-absolutely pure;*

(b) *$D$ is $n$-pure in any (injective) $R$-module $E$ containing $D$ as an $R$-submodule;*

(c) *each $R$-homomorphism $\phi : H \to D$ from a finitely generated $R$-submodule $H$ of projective dimension $\leqslant n - 1$ of a finitely generated free $R$-module $F$ is induced by a map $\gamma : F \to D$.*

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PROOF (a) $\Leftrightarrow$ (b) Consider an exact sequence $0 \to D \to E \to E/D \to 0$ where $E$ is an injective $R$-module containing $D$. Let $N$ be a finitely presented $R$-module of projective dimension $\leqslant n$. The induced sequence $0 \to \mathrm{Hom}_R(N, D) \to \mathrm{Hom}_R(N, E) \to \mathrm{Hom}_R(N, E/D) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^1_R(N, E) = 0$ establishes the result.

(a) $\Leftrightarrow$ (c) The exact sequence $0 \to H \to F \to F/H \to 0$ along with the induced sequence $0 \to \mathrm{Hom}_R(F/H, D) \to \mathrm{Hom}_R(F, D) \to \mathrm{Hom}_R(H, D) \to \mathrm{Ext}^1_R(F/H, D) \to \mathrm{Ext}^1_R(F, D) = 0$ implies the result, since $F/H$ is finitely presented of projective dimension $\leqslant n$. ∎

Note that for an $R$-submodule $H$ of a flat $R$-module $F$, $H$ is pure in $F$ if and only if $F/H$ is flat. This can be generalised easily.

**LEMMA 4.** *Let $H$ be an $R$-submodule of an $n$-flat module $F$. Then $H$ is $n$-pure in $F$ if and only if $F/H$ is $n$-flat.*

In particular, we have

**COROLLARY 1.** *For an $R$-submodule $A$ of a torsion-free $R$-module $B$, the following are equivalent:*

(a) *$A$ is relatively divisible in $B$;*

(b) *$A$ is 1-pure in $B$;*

(c) *$B/A$ is torsion-free.*

Note that $n$-purity can also be characterised in the following way. Consider finite systems of equations over $N$,

$$\sum_{j=1}^{l} r_{ij} x_j = a_i \in N \quad (i = 1, \dots, k), \tag{1}$$

where $r_{ij} \in R$ and $x_1, \dots, x_l$ are unknowns such that the $R$-submodule $H$ of the free $R$-module $F$ on $\{x_1, \dots, x_l\}$ generated by the left members of (1) is of projective dimension $\leqslant n - 1$. $N$ is $n$-pure in $M$ if and only if every such system (1) has a solution in $N$ whenever it has a solution in $M$. Such a system (1) will be called an *$n$-finite* system. Note that an $n$-finite system is an $m$-finite system whenever $n \leqslant m$.

**LEMMA 5.** *Let $L \leqslant N \leqslant M$ be $R$-modules.*

(a) *If $L$ is $n$-pure in $M$, then it is also $n$-pure in $N$.*

(b) *If $L$ is $n$-pure in $N$ and $N$ is $n$-pure in $M$, then $L$ is $n$-pure in $M$.*

(c) *If $N$ is $n$-pure in $M$, then $N/L$ is $n$-pure in $M/L$. The converse holds if $L$ is $n$-pure in $M$.*

PROOF: (a) Any $n$-finite system of equations over $L$ which is solvable in $N$ is trivially solvable in $M$. Hence it has a solution in $L$ by hypothesis.

(b) Any $n$-finite system of equations over $L$ which is solvable in $M$ can be viewed as a system over $N$. Since $N$ is $n$-pure in $M$, it has a solution in $N$. Again, since $L$ is $n$-pure in $N$, it has a solution in $L$.

(c) Suppose $\sum_{j=1}^{l} r_{ij}x_j = a_i + L \in N/L$ is an $n$-finite system of equations over $N/L$ which has a solution $x_j = b_j + L \in M/L$, where $1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l$. Then $\sum_{j=1}^{l} r_{ij}b_j = a_i + p_i$ for some $p_i \in L$. Hence $\sum_{j=1}^{l} r_{ij}x_j = a_i + p_i \in N$ is an $n$-finite system of equations over $N$ which has a solution $x_j = b_j \in M (1 \leqslant j \leqslant l)$. Then $\sum_{j=1}^{l} r_{ij}(b_j + L) = a_i + L$ and $x_j = b_j + L \in M/L$ is a solution of the original $n$-finite system. To prove the converse, let $U$ be a finitely presented $R$-module of projective dimension $\leqslant n$. Then we have a commutative digram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & U \bigotimes_R L & \xrightarrow{f} & U \bigotimes_R N & \longrightarrow & U \bigotimes_R (N/L) & \longrightarrow & 0 \\
 & & \| & & \phi \downarrow & & h \downarrow & & \\
\cdots & \longrightarrow & U \bigotimes_R L & \xrightarrow{g} & U \bigotimes_R M & \longrightarrow & U \bigotimes_R (M/L) & \longrightarrow & 0.
\end{array}$$

Since $L$ is $n$-pure in $M$ and thus $L$ is $n$-pure in $N$ by (a), we have monomorphisms $f, g$. Since $h$ is a monomorphism by hypothesis, $\phi$ is a monomorphism, too. ∎

We take a self-evident definition of an $n$-pure-exact sequence similar to that of an $RD$-exact sequence. By Lemma 5, those elements of $\mathrm{Ext}_R^1(N, M)$ which are represented by $n$-pure-exact sequence form a subgroup, which will be denoted by $n\text{-}\mathrm{Pext}_R^1(N, M)$. It is readily checked that $n\text{-}\mathrm{Pext}_R^1(N, M) = 0$ for all finitely presented modules $N$ of projective dimension $\leqslant n$ and for all $R$-modules $M$.

## 4. $n$-PRÜFER DOMAINS

For $n$-Prüfer domains we can now prove our main result.

**THEOREM 1.** *For a domain $R$ and for every $n$ $(0 < n \leqslant \infty)$, the following conditions are equivalent:*

(a) *$R$ is $n$-Prüfer;*

(b) *1-absolutely pure $R$-modules are $n$-absolutely pure;*

(c) *1-pure $R$-submodules of all $R$-modules are $n$-pure.*

PROOF (c) $\Rightarrow$ (b) Let $D$ be a 1-absolutely pure $R$-module. By Lemma 3, $D$ is 1-pure in its injective hull $E(D)$. By hypothesis, it is $n$-pure, which implies that $D$ is $n$-absolutely pure.

(b) $\Rightarrow$ (a) Let $N$ be a finitely generated torsion-free $R$-module of projective dimension $\leqslant n - 1$. Imbed $N$ into a finitely generated free $R$-module $F$. Then $F/N$ is a finitely presented $R$-module of projective dimension $\leqslant n$. Therefore, by hypothesis, $\mathrm{Ext}_R^1(F/N, D) = 0$ for all divisible $R$-modules $D$. By Lee [5], projective dimension of $F/N$ is $\leqslant 1$, which implies that $N$ is projective.

(a) $\Rightarrow$ (c) Let $A$ be a 1-pure $R$-submodule of $B$ and $N$ a finitely presented $R$-module of projective dimension $\leqslant n$. Consider a presentation $0 \to H \to F \to N \to 0$ of $N$ where $H$ is a finitely generated torsion-free $R$-module of projective dimension $\leqslant n-1$. By hypothesis, $H$ is projective and thus $N$ has projective dimension $\leqslant 1$. Since $A$ is 1-pure, the map $A \bigotimes_R N \to B \bigotimes_R N$ induced by the inclusion $A \to B$ is injective. We conclude that $A$ is $n$-pure. ∎

**Lemma 6.** *1-flat R-modules over n-Prüfer domains R are n-flat.*

Proof: Let $A$ be a 1-flat $R$-module and $N$ be a finitely presented $R$-module of projective dimension $\leqslant n$. In the natural isomorphism

$$\operatorname{Ext}^1_R(N, \operatorname{Hom}_R(A, E)) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_1(N, A), E)$$

where $E$ is an injective $R$-module, Ext is 0 since $\operatorname{Hom}_R(A, E)$ is divisible and thus $n$-sbsolutely pure by hypothesis. Hence the right Hom is 0. Since $E$ was arbitary, $\operatorname{Tor}^R_1(N, A) = 0$, which implies that $A$ is $n$-flat. ∎

In Lee [4], a domain $R$ was called *n-coherent* if every finitely generated torsion-free $R$-module of projective dimension $\leqslant n-1$ is finitely presented. $n$-Prüfer domains are trivially $n$-coherent, and we are going to show that the converse holds when 1-flat modules are $n$-flat.

**Lemma 7.** *For a domain R and for every n ($0 < n \leqslant \infty$), the following are equivalent:*

(a) *R is n-Prüfer;*

(b) *R is n-coherent and all 1-flat R-modules are n-flat.*

Proof: In view of Lemma 6, we have only to prove (b) implies (a). Let $D$ be a 1-absolutely pure $R$-module and $N$ a finitely presented $R$-module of projective dimension $\leqslant n$. Since $R$ is $n$-coherent, $N$ has a finite projective resolution (see Lee [4]). In the natural isomorphism

$$\operatorname{Tor}^R_1(N, \operatorname{Hom}_R(D, E)) \cong \operatorname{Hom}_R(\operatorname{Ext}^1_R(N, D), E),$$

where $E$ is an injective $R$-module, Tor is 0 since $\operatorname{Hom}_R(D, E)$ is torsion-free and thus $n$-flat by hypothesis. Hence $\operatorname{Ext}^1_R(N, D) = 0$ by the choice of $E$. This implies that $D$ is $n$-absolutely pure. By Theorem 1, $R$ is $n$-Prüfer. ∎

It is proved in Fuchs and Salce [2, p. 247] that if a domain $R$ is Prüfer, then the torsion $R$-submodule of every $R$-module is pure. Now we can generalise this result.

**Lemma 8.** *Over n-Prüfer domains R, the torsion R-submodule of every R-module is n-pure.*

Proof. Let $t(M)$ be the torsion $R$-submodule of an $R$-module $M$. Then $M/t(M)$ is torsion-free and thus $n$-flat by Lemma 7. Hence $\operatorname{Tor}^R_1(N, M/t(M)) = 0$ for all finitely

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presented $R$-module $N$ of projective dimension $\leqslant n$. Therefore the map $t(M)\bigotimes_R N \to M \bigotimes_R N$ induced by the inclusion $t(M) \to M$ is injective. This implies that $t(M)$ is $n$-pure in $M$. ∎

We quote a result by Sabbagh [9] which we want to generalise.

LEMMA 9. (Sabbagh [9].) *The ring of polynomials in an arbitrary number of variables over a Prüfer domain is coherent.*

THEOREM 2. *If $R$ is $n$-Prüfer, then $R[x_1, \dots, x_k]$ is $n$-coherent for every integers* $k \geqslant 1$.

PROOF: Let $M$ be a finitely generated torsion-free $R[x_1, \dots, x_k]$-module with projective dimension$_{R[x_1,\dots,x_k]}M \leqslant n-1$. Since $R[x_1, \dots, x_k]$ is a free $R$-module, $M$ is a finitely generated torsion-free $R$-module with projective dimension$_R M \leqslant n-1$. By hypothesis, $M$ is a projective $R$-module and therefore flat. By Raynaud and Gruson [8, Theorem 3.4.6], $M$ is a finitely presented $R[x_1, \dots, x_k]$-module. This implies that $R[x_1, \dots, x_k]$ is $n$-coherent. ∎

We can also verify:

LEMMA 10. *If $R_P$ is $n$-Prüfer for every maximal ideal $P$ of $R$, then $R$ is $n$-Prüfer.*

PROOF: Suppose $M$ is a finitely generated torsion-free $R$-module with projective dimension$_R M \leqslant n-1$. Then $M_P$ is a finitely generated torsion-free $R_P$- module with projective dimension$_{R_P} M_P \leqslant n-1$. By hypothesis, $M_P$ is a projective $R_P$-module. By Fuchs and Salce [2, p. 196], $M$ is a projective $R$-module. Hence $R$ is $n$-Prüfer. ∎

Combining all these, we have

COROLLARY 2 *For a domain $R$ and for every $n$ $(0 < n \leqslant \infty)$, the following implications hold:*

$$\text{(a)} \Rightarrow \text{(b)} \Leftrightarrow \text{(c)} \Leftrightarrow \text{(d)} \Leftrightarrow \text{(e)} \Rightarrow \text{(f)} \Rightarrow \text{(g)},$$

*where*

(a) *$R_P$ is $n$-Prüfer for every ideal $P$ of $R$.*
(b) *$R$ is $n$-Prüfer.*
(c) *1-purity implies $n$-purity.*
(d) *1-absolute purity implies $n$-absolute purity.*
(e) *$R$ is $n$-coherent and 1-flatness implies $n$-flatness.*
(f) *1-flatness implies $n$-flatness.*
(g) *In every $R$-module the torsion submodule is $n$-pure.*

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