J. Fluid Mech. (2023), *vol.* 961, A3, doi:10.1017/jfm.2023.128

Six wave interaction equations in finite-depth gravity waves with surface tension

Mark J. Ablowitz[1], **Xu-Dan Luo**[2] **and Ziad H. Musslimani**[3],†

[1]Department of Applied Mathematics, University of Colorado, Campus Box 526, Boulder, CO 80309-0526, USA

[2]Key Laboratory of Mathematics Mechanization, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China

[3]Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510, USA

(Received 8 August 2022; revised 22 December 2022; accepted 2 February 2023)

Three wave resonant triad interactions in two space and one time dimensions form a well-known system of first-order quadratically nonlinear evolution equations that arise in many areas of physics. In deep water waves, they were first derived by Simmons in 1969 and later shown to be exactly solvable by Ablowitz & Haberman in 1975. Specifically, integrability was established by introducing a system of six wave interactions whose symmetry reduction leads to the well-known three wave equations. Here, it is shown that the six wave interaction and classical three wave equations satisfying triad resonance conditions in finite-depth gravity waves can be derived from the non-local integro-differential formulation of the free surface gravity wave equation with surface tension. These quadratically nonlinear six wave interaction equations and their reductions to the classical and non-local complex as well as real reverse space–time three wave interaction equations are integrable. Limits to infinite and shallow water depth are also discussed.

Key words: Hamiltonian theory

1. Introduction

Waves occur widely in real world phenomena. This includes waves in fluids, optics, acoustics and elasticity among many others. Linear waves find many important applications across various scientific disciplines. As such, their mathematical and physical properties have been extensively investigated dating back to the pioneers of dynamics. In comparison with their linear counterparts, nonlinear waves have been relatively less studied and, in certain cases, even their underlying processes are not so well understood.

† Email address for correspondence: musslimani@math.fsu.edu

However, much more is known about weakly interacting nonlinear waves for which researchers often consider the dominant approximation to be a collection of Fourier modes.

In weakly nonlinear deep water waves, Stokes (1847) found a relationship between the frequency and amplitude of the dominant Fourier mode. More than a hundred years later Benjamin & Feir (1967) showed that this wave was unstable. Soon afterwards, by allowing the envelope of the wave to vary slowly in space and time Zakharov (1968) (see also Zakharov 1998) showed that the complex amplitude of the envelope satisfies a two space, one time dimensional nonlinear Schrödinger (NLS) equation. Remarkably, the one space and one time dimensional NLS equation was shown to be an integrable equation (Zakharov & Shabat 1972). The integrability of the NLS equation followed after another well-known one-dimensional equation, the Korteweg–de Vries (KdV) equation was found to be an integrable system (Gardner *et al.* 1967); see also Ablowitz & Segur (1981), Novikov *et al.* (1984) and Ablowitz (2011).

The KdV equation arises in the study of long gravity waves. Its two-dimensional extension, the so-called Kadomtsev–Petviashvili equation (Kadomtsev & Petviashvili 1970) is also obtained for long waves and was also found to be integrable, see e.g. Ablowitz & Clarkson (1991) and Ablowitz & Segur (1979).

When we consider the leading order to be a sum of harmonics (Fourier modes) the theory of resonant wave interactions is fundamental. This is often termed resonant interaction theory. Many key ideas were discovered by O.M. Phillips in the study of deep water waves (Phillips 1960); see also Phillips (1966). Suppose we consider the elevation of the water wave η to be approximated by a sum of M harmonics:

$$\eta(\boldsymbol{r}, t) = \sum_{j=1}^{M} \mathcal{A}_j \exp(\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{r} - \omega_j t)) + \text{c.c.}, \tag{1.1}$$

where $\boldsymbol{r}$ is the two-dimensional transverse spatial coordinate, $\boldsymbol{k}_j$ are the wavenumbers, $\omega_j = \omega(\boldsymbol{k}_j)$ the corresponding wave frequencies and c.c. stands for complex conjugation. Furthermore, the amplitudes $\mathcal{A}_j$ are assumed to vary slowly in space and time. In deep water and when surface tension is neglected, the dispersion relation of a single Fourier mode is given by

$$\omega^2 = gk, \quad k = |\boldsymbol{k}|. \tag{1.2}$$

In this scenario, wave interactions become significant when $M = 4$ with the wavenumbers $\boldsymbol{k}_j$ and their corresponding frequencies $\omega_j = \omega(\boldsymbol{k}_j), j = 1, 2, 3, 4$ satisfy the following resonance criteria:

$$\left.\begin{aligned} \boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 \pm \boldsymbol{k}_4 &= 0, \\ \omega_1 \pm \omega_2 \pm \omega_3 \pm \omega_4 &= 0. \end{aligned}\right\} \tag{1.3}$$

These relations are sometimes referred to as quartet resonance conditions. By allowing $\mathcal{A}_j$ to vary slowly in time (only) and using multiple scale methods, Benney (1962) derived nonlinear equations that governed the slow dynamics of quartet wave resonance. If ϵ denotes the size of the nonlinearity (which is assumed to be asymptotically small; ϵ is typically taken to be proportional to the (small) slope of the wave elevation) then these equations are valid for time scales of the order of $1/\epsilon^2$. Later, Benney & Newell (1967) found the wave amplitude satisfies a more general four wave resonant interaction equations in space and time that obey the resonance condition (1.3). Contrary to the deep water case without surface tension (for which resonant interaction requires four waves), resonant three modes interaction can, however, occur when surface tension is included. With surface tension the infinite depth dispersion relation is given by

$$\omega^2 = gk + \sigma k^3, \quad k = |\boldsymbol{k}|, \tag{1.4}$$

where σ denotes the ratio between the surface tension and fluid density. In this case, the so-called triad resonances satisfy the wavenumber–frequency condition

$$\left.\begin{aligned} \boldsymbol{k}_1 \pm \boldsymbol{k}_2 \pm \boldsymbol{k}_3 = 0, \\ \omega_1 \pm \omega_2 \pm \omega_3 = 0. \end{aligned}\right\} \tag{1.5}$$

In 1964, Bretherton (1964) provided a model partial differential equation that had such triad resonance. Bretherton also found and solved the equations for the underlying slowly varying temporal amplitudes valid on a $1/\epsilon$ time scale, an order of magnitude faster than quartet resonances. Soon afterward McGoldrick (1965) showed that gravity waves with surface tension exhibited such triad resonance phenomena. McGoldrick also found the equations describing the slowly varying temporal equations that describe the wave interactions on a $1/\epsilon$ time scale. Subsequently experiments were carried out that further elucidated the phenomena (McGoldrick 1970; Henderson & Hammack 1987; Perlin, Henderson & Hammack 1990; Hammack & Henderson 1993).

In 1969, Simmons (1969) gave a geometric argument that explained why the triad resonance occurs in deep gravity waves with surface tension and found the slowly varying envelope equations in both space and time. To simplify the analysis, which was tedious, Simmons derived a (2+1)-dimensional three wave system by employing a Lagrangian approach which had been recently pioneered by Whitham (1974). We also remark that the general form of the interaction equations in conservative systems was found by Hasselmann (1967); later, a perturbation approach was formulated by Case & Chiu (1977). A detailed discussion of the above and additional references can be found in Craik (1988), (see also Dyachenko, Zakharov & Kuznetsov 1996; Dyachenko, Korotkevich & Zakharov 2003; Korotkevich, Dyachenko & Zakharov 2016). Notably, the three wave equations are solvable by the inverse scattering transform in both one spatial dimension (Ablowitz & Haberman 1975*b*; Zakharov & Manakov 1975; Kaup 1976) and in two spatial dimensions (Ablowitz & Haberman 1975*a*; Kaup 1981).

In this paper, a six wave triad resonant interaction system corresponding to both infinite and finite-depth gravity waves and in the presence of surface tension is obtained from the so-called Ablowitz–Musslimani–Fokas non-local formulation of gravity waves (Ablowitz, Fokas & Musslimani 2006) (which has the advantage of only depending on the surface variables). Using a space–time multiscale asymptotic expansion, a hierarchy of equations governing the evolution of the surface variable at each order in the perturbation parameter ϵ is derived. The leading-order solution is expressed as a superposition of six wavepackets corresponding to wavenumbers $\pm\boldsymbol{k}_j$, frequencies $\pm\omega(\boldsymbol{k}_j)$ and distinct amplitudes $A_j(\boldsymbol{k}), B_j(\boldsymbol{k}), j = 1, 2, 3$, where in general $A_j(\boldsymbol{k}) \neq B_j^*(\boldsymbol{k})$. The desired six wave evolution equations governing the space–time slow dynamics of these amplitudes are obtained from a secularity condition at order ϵ. Furthermore, using the integrable symmetry relations between the amplitudes, we obtain an integrable system of three wave interactions that include in it: (i) classical, (ii) non-local complex reverse space–time and (iii) non-local real reverse space–time systems (Ablowitz & Musslimani 2017). These six wave interaction equations reduce back to the well-known result in infinite depth.

The solutions of the non-local three wave systems have been recently investigated in one spatial dimension (Ablowitz, Luo & Musslimani 2023). The six wave system corresponds to complex reductions of gravity waves. Complex reductions of physical systems have been widely investigated; e.g. self-dual reductions of Yang–Mills and Einstein equations, Kadomtsev–Petviashvili and Davey–Stewartson equations etc.; they exhibit interesting and novel properties, see e.g. Ward (1977), Gibbons, Page & Pope (1990), Mason & Woodhouse (1996) and Fokas (2006).

The complex nature of gravity waves is itself an active subject, see e.g. Dyachenko, Lushnikov & Korotkevich (2016), Lushnikov (2016), Dyachenko *et al.* (2021) and references therein. To the best of our knowledge, the classical three wave system in finite depth has not been previously obtained. We also make some remarks about the shallow-depth reduction of the six wave system which also appears to be a new classical and non-local integrable system. As mentioned above, starting from the classical water wave equations in finite depth, written in non-local form, we derive the well-known system of (2+1)-dimensional first-order quadratically nonlinear six wave interaction equations which was introduced and shown to be integrable by Ablowitz & Haberman (1975*a*). In doing so, we connect the mathematical theory of integrable interaction equations and the physics of classical fluids. Consider the following two-dimensional space–time system of first-order quadratically nonlinear evolution equations:

$$\frac{\partial N_{lj}}{\partial t} + \alpha_{lj}\frac{\partial N_{lj}}{\partial x} + \beta_{lj}\frac{\partial N_{lj}}{\partial y} = \sum_{m=1}^{3}(\alpha_{lm} - \alpha_{mj})N_{lm}N_{mj}, \quad l, j = 1, 2, 3, \tag{1.6}$$

where $\boldsymbol{N} \equiv (N_{lj}), l, j = 1, 2, 3$ is an 3×3 complex matrix whose elements are functions of $\boldsymbol{r} = (x, y)$ and time t satisfying $N_{jj} = 0, j = 1, 2, 3$. Furthermore, the constant matrices (α_{lj}) and (β_{lj}) are assumed to be real and symmetric with $\alpha_{jj} = \beta_{jj} = 0$. System (1.6) arises from a compatibility condition between two linear pairs (see Ablowitz & Clarkson (1991) for further details regarding their derivation and soliton solutions in both one and two space dimensions). Thus, by construction, the matrix $\boldsymbol{N}$ has six complex components which we denote by

$$\boldsymbol{N} = \begin{pmatrix} 0 & A_3 & A_2 \\ B_3 & 0 & B_1 \\ B_2 & A_1 & 0 \end{pmatrix}. \tag{1.7}$$

Substituting (1.7) into (1.6) leads to the following six wave interaction equations:

$$\left.\begin{aligned} (\partial_T + C_{1,x}\partial_x + C_{1,y}\partial_y)A_1 - \mathrm{i}\sigma_1 B_2 A_3 = 0, \\ (\partial_T + C_{2,x}\partial_x + C_{2,y}\partial_y)A_2 - \mathrm{i}\sigma_2 B_1 A_3 = 0, \\ (\partial_T + C_{3,x}\partial_x + C_{3,y}\partial_y)A_3 - \mathrm{i}\sigma_3 A_1 A_2 = 0, \\ (\partial_T + C_{1,x}\partial_x + C_{1,y}\partial_y)B_1 + \mathrm{i}\sigma_1 A_2 B_3 = 0, \\ (\partial_T + C_{2,x}\partial_x + C_{2,y}\partial_y)B_2 + \mathrm{i}\sigma_2 A_1 B_3 = 0, \\ (\partial_T + C_{3,x}\partial_x + C_{3,y}\partial_y)B_3 + \mathrm{i}\sigma_3 B_1 B_2 = 0, \end{aligned}\right\} \tag{1.8}$$

where we identify the constants $C_{j,x}, C_{j,y}, \sigma_j, j = 1, 2, 3$ with the coefficients $\alpha_{lj}, \beta_{lj}, l, j = 1, 2, 3$ in (1.6). Therefore, the A, B system is equivalent to the sixth-order wave system (1.6). This sixth-order wave system has reductions to the integrable classical three wave equations (e.g. when $B_j = A_j^*$),$j = 1, 2, 3$ and to the non-local complex and real three wave systems (i.e. when $B_j(x, t) = A_j^*(-x, -t), j = 1, 2, 3$ or $B_j(x, t) = A_j(-x, -t), j = 1, 2, 3$, respectively (Ablowitz & Musslimani 2017) (see also Ablowitz & Musslimani 2013, 2016, 2019).

In §§ 4–5 we establish that these non-local reductions are asymptotic limits of gravity waves with surface tension in both infinite and finite depth, respectively. In this regard, $C_{j,x}, C_{j,y}, j = 1, 2, 3$ correspond to the respective group velocities. In infinite depth the well-known result of Simmons (1969) is obtained. The shallow-depth limit to our knowledge also leads to a new system of equations even in the classical three wave interaction case.

A summary of the new results is listed below:

(i) The derivation of six wave interaction equation in infinite depth starting from the non-local formulation of classical gravity waves with surface tension (§ 4).
(ii) The derivation of six wave equation for finite and shallow-depth cases (§ 5).
(iii) The derivation of classical three wave interaction for finite depth. The shallow water reduction follows as a special case (§ 5).

It is remarkable that the system (1.6) stands out as a 'complete asymptotic reduction' of the gravity wave equations with surface tension in both infinite and finite depth. By complete asymptotic reduction we mean that in the limit when the asymptotic parameter ϵ vanishes, the equations after transformation are exactly the integrable equations found in 1975 (Ablowitz & Haberman 1975*a*). These equations are given by (1.6) or more explicitly system (1.8). The small parameter ϵ is associated with small wave elevation; see § 2 below. This six wave system in one or two dimensions can be viewed as the multi wave analogue of the NLS/Davey–Stewartson (DS) equations and the KdV/Kadomtsev–Petviashvili (KP) equations. The NLS/DS/KdV/KP equations are also complete asymptotic reductions of the gravity wave equations with surface tension. Other interacting wave systems such as those interactions satisfying non-trivial quartet resonant interactions in gravity waves have not been shown to be integrable.

2. Formulation

The free surface gravity wave equations with a flat bottom of depth h are given by

Euler ideal flow

$$\nabla^2\phi = 0, \quad -h < z < \eta(\boldsymbol{r}, t), \tag{2.1}$$

where ϕ is the velocity potential.

No flow through bottom

$$\frac{\partial\phi}{\partial z} = 0, \quad \text{on } z = -h. \tag{2.2}$$

Bernoulli or pressure equation

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = \sigma\nabla\cdot\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right), \quad \text{on } z = \eta(\boldsymbol{r}, t). \tag{2.3}$$

Nonlinear kinematic boundary condition

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta, \quad \text{on } z = \eta(\boldsymbol{r}, t), \tag{2.4}$$

where ∇ is the horizontal gradient operator. As mentioned earlier, σ is proportional to surface tension. These four equations constitute the classical equations for gravity waves. Here, the unknowns are as follows: $\phi(\boldsymbol{r}, z, t)$ the velocity potential; $\eta(\boldsymbol{r}, t)$ the elevation, $\boldsymbol{r} = (x, y)$ is the horizontal coordinate, z the vertical coordinate and t is time. This is a free-boundary problem for the unknowns $\phi(\boldsymbol{r}, z, t)$ and $\eta(\boldsymbol{r}, t)$. In Ablowitz *et al.* (2006), the above wave problem was reformulated as a system of two non-local differential integral equations for two surface unknowns: η and $q = q(\boldsymbol{r}, t) = \phi(\boldsymbol{r}, \eta(\boldsymbol{r}, t))$. The equations are given by the following system:

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r}} \left(\mathrm{i}\eta_t \cosh[k(\eta+h)] - \frac{\boldsymbol{k}\boldsymbol{\cdot}\nabla q}{k} \sinh[k(\eta+h)] \right) = 0, \tag{2.5}$$

$$q_t + \frac{1}{2}|\nabla q|^2 + g\eta - \frac{(\eta_t + \nabla q \boldsymbol{\cdot} \nabla\eta)^2}{2(1+|\nabla\eta|^2)} = \sigma\nabla\boldsymbol{\cdot}\left(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\right). \tag{2.6}$$

It is assumed that the envelopes associated with the wave elevation η and derivatives of the velocity potential q (i.e. $\nabla q, q_t$) decay rapidly to zero at infinity. Equation (2.6) is Bernoulli's equation on the free surface. Once q is obtained then we solve Laplace's equation (2.1) for $\phi(\boldsymbol{r}, z, t)$. The non-local formulation is particularly useful for asymptotic calculations. In the infinite-depth limit ($h \to \infty$) (2.5) reduces to

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r}} \left(\mathrm{i}\eta_t - \frac{\boldsymbol{k}}{k}\boldsymbol{\cdot}\nabla q \right) = 0. \tag{2.7}$$

We consider the weakly nonlinear waves case for which it is convenient to let $\eta \to \epsilon\eta, q \to \epsilon q$ and assume $|\epsilon|$ to be small; we assume ϵ is proportional to the size of the slope of the wave elevation. Doing so and expanding to order ϵ we find: (i) for finite depth

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r}} \left(\mathrm{i}\eta_t(1 + \epsilon k \tanh(kh)\eta) - \frac{\boldsymbol{k}}{k}\boldsymbol{\cdot}\nabla q(\tanh(kh) + \epsilon k\eta) \right) = 0, \tag{2.8}$$

and (ii) for infinite depth ($h \to \infty$)

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{r}} \left((\mathrm{i}\eta_t - \frac{\boldsymbol{k}}{k}\boldsymbol{\cdot}\nabla q)(1 + \epsilon k\eta) + \cdots \right) = 0. \tag{2.9}$$

Note that these two equations differ only by simple factors. The free surface Bernoulli equation (2.6) is unchanged regardless of finite or infinite depth; this equation to order ϵ reads

$$q_t = -g\eta + \sigma\nabla^2\eta + \frac{\epsilon}{2}(\eta_t^2 - |\nabla q|^2). \tag{2.10}$$

The linear dispersion relation for two-dimensional gravity waves with finite depth and surface tension is

$$\omega^2 = (gk + \sigma k^3)\tanh(kh), \tag{2.11}$$

while for the infinite-depth case is given by (1.4). In Simmons (1969), Simmons showed that triad resonance in the form given by (1.5) occurs in infinite-depth water waves. In this paper, we show that such triad resonance also holds in finite depth as well as shallow water.

3. Triad resonances in infinite, finite and shallow depth

The purpose of this section is to numerically establish the existence of a triad resonance in the form

$$\left.\begin{aligned} \boldsymbol{k}_3 &= \boldsymbol{k}_1 + \boldsymbol{k}_2, \\ \omega(\boldsymbol{k}_3) &= \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2), \end{aligned}\right\} \tag{3.1}$$

with the gravity wave dispersion relation $\omega(\boldsymbol{k})$ given by

$$\omega(\boldsymbol{k}) = \sqrt{(g|\boldsymbol{k}| + \sigma|\boldsymbol{k}|^3)\tanh(h|\boldsymbol{k}|)}. \tag{3.2}$$

Of particular interest are the cases corresponding to finite and shallow depth. We note in infinite depth investigations, suitable parameter values of depth, surface tension and density were used by Henderson and Hammack in their experimental studies of ripple instabilities with resonant triads (Henderson & Hammack 1987). In our study below, we consider only some parameter values in finite, shallow and infinite depth. It is outside the scope of this paper to do an intensive numerical parameter analysis. Here, the wavenumbers $\boldsymbol{k}$ are measured in units of cm^{-1}; water depth h in cm; frequency (dispersion) ω in s^{-1}; surface tension coefficient $\sigma \equiv$ surface tension /density in $\mathrm{cm}^3\ \mathrm{s}^{-2}$; and gravity $g = 980\ \mathrm{cm\ s}^{-2}$. We identify three regimes: (i) shallow-depth case that corresponds to small kh; (ii) finite depth that occur for moderate values of kh and (iii) infinite depth valid for large kh. To this end, let us consider the auxiliary function $\varOmega$ defined as

$$\varOmega(\boldsymbol{k}_1, \boldsymbol{k}_2) \equiv \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) - \omega(\boldsymbol{k}_1 + \boldsymbol{k}_2). \tag{3.3}$$

Thus, the zero level set of the function $\varOmega$ corresponds to a resonant triad. We numerically solve the equation $\varOmega(\boldsymbol{k}_1, \boldsymbol{k}_2) = 0$ to identify all resonant points in the wavenumber space. In doing so, we use the following set of parameters to produce all figures.

(i) One-dimensional case: illustrations.

- (*a*) Shallow depth: surface tension = 81 dyn cm^{-1}; density = 1 gram cm^{-3}; water depth $h = 0.5$ cm. This leads to surface tension coefficient $\sigma \equiv$ surface tension/density equals to 81 $\mathrm{cm}^3\ \mathrm{s}^{-2}$. Typical values for the wavenumbers are $k_1 \approx 0.3\ \mathrm{cm}^{-1}$; $k_2 \approx 0.3\ \mathrm{cm}^{-1}$ (marked as a full circle in figure 1*a*) and $k_3 = k_1 + k_2 \approx 0.6\ \mathrm{cm}^{-1}$. With these values at hand, we find $k_1 h \approx 0.15$; $k_2 h \approx 0.15$ and $k_3 h \approx 0.3$ which represent a shallow-depth limit.
- (*b*) Finite depth: surface tension = 81 dyn cm^{-1}; density = 1 gram cm^{-3}; water depth $h = 0.6$ cm. This results in $\sigma = 81\ \mathrm{cm}^3\ \mathrm{s}^{-2}$. In this case, we choose the following set of wavenumbers: $k_1 \approx 1.5\ \mathrm{cm}^{-1}$; $k_2 \approx 2\ \mathrm{cm}^{-1}$ (indicated by a full circle in figure 1*b*) and $k_3 = k_1 + k_2 \approx 2.5\ \mathrm{cm}^{-1}$. Thus, we have $k_1 h \approx 0.9$; $k_2 h \approx 1.2$ and $k_3 h \approx 1.5$ which is a finite-depth regime.
- (*c*) Infinite depth: surface tension = 73 dyn cm^{-1}; density = 1 gram cm^{-3}; water depth $h = 2$ cm and $\sigma = 73\ \mathrm{cm}^{-3}\ \mathrm{s}^{-2}$. Here, prototypical values for the wavenumbers are as follows: $k_1 \approx 2.5\ \mathrm{cm}^{-1}$; $k_2 \approx 2.6\ \mathrm{cm}^{-1}$ (shown by a full circle in figure 1*c*) and $k_3 = k_1 + k_2 \approx 5.1\ \mathrm{cm}^{-1}$. In this case we find, $k_1 h \approx 5$; $k_2 h \approx 5.2$ and $k_3 h \approx 10.2$ which fall into the limit of an infinite depth.

(ii) Two-dimensional case: illustrations.

- (*a*) Shallow depth: surface tension = 73 dyn cm^{-1}; density = 1 gram cm^{-3}; water depth $h = 0.35$ cm and surface tension coefficient σ equals to 73 $\mathrm{cm}^3\ \mathrm{s}^{-2}$.

Typical values for the wavenumbers are $k_1 = |\boldsymbol{k}_1| \approx 1\,\text{cm}^{-1}$; $k_2 = |\boldsymbol{k}_2| \approx 0.54\,\text{cm}^{-1}$ (marked as a full circle in figure 2*a*) and $k_3 = |\boldsymbol{k}_1 + \boldsymbol{k}_2| \approx 1.54\,\text{cm}^{-1}$. With these values at hand, we find $k_1 h \approx 0.35$; $k_2 h \approx 0.19$ and $k_3 h \approx 0.5$ which represent a shallow water limit.

(*b*) Finite depth: surface tension $= 73$ dyn cm^{-1}; density $= 1$ gram/cm^3; water depth $h = 0.6\,\text{cm}$ and $\sigma = 73$ cm^3 s^{-2}. Some prototypical values of wavenumbers are as follows: $k_1 = |\boldsymbol{k}_1| \approx 2.1$ cm^{-1}; $k_2 = |\boldsymbol{k}_2| \approx 1.9$ cm^{-1} (marked as a full circle in figure 2*b*) and $k_3 = |\boldsymbol{k}_1 + \boldsymbol{k}_2| \approx 4$ cm^{-1}. With these values at hand, we find $k_1 h \approx 1.3$; $k_2 h \approx 1.1$ and $k_3 h \approx 2.4$ which falls into the finite-depth fluid regime.

(*c*) Infinite depth: surface tension $= 73$ dyn cm^{-1}; density $= 1$ gram/cm^3; water depth $h = 2$ cm and $\sigma = 73$ cm^3 s^{-2}. Example values of wavenumbers are as follows: $k_1 = |\boldsymbol{k}_1| \approx 2.8$ cm^{-1}; $k_2 = |\boldsymbol{k}_2| \approx 2.3$ cm^{-1} (marked as a full circle in figure 2*c*) and $k_3 = |\boldsymbol{k}_1 + \boldsymbol{k}_2| \approx 5\,\text{cm}^{-1}$. With these values at hand, we find $k_1 h \approx 5.6$; $k_2 h \approx 4.6$ and $k_3 h \approx 10$ which falls into the infinite-depth case.

First, we will discuss typical values in the one-dimensional case. Figure 1 shows the existence of a numerical resonant triad for various values of depth: (*a*) shallow depth $h = 0.5\,\text{cm}$ with $\sigma = 81$ cm^3 s^{-2}, (*b*) finite depth $h = 0.6\,\text{cm}$, with $\sigma = 81$ cm^3 s^{-2} and (*c*) deep depth with $h = 2\,\text{cm}$ and $\sigma = 73$ cm^3 s^{-2}. For convenience, we show in figure 1(*d*) a plot of the tanh function that helps identify various water depth limits. For completeness, in figure 2 we also show two more cases corresponding to shallow and finite-depth water limits with surface tension $\sigma = 73$ cm^3 s^{-2}.

Next, we discuss the two-dimensional case. Here, in all cases we take the surface tension coefficient to be $\sigma = 73$ cm^3 s^{-2}. Figure 3 illustrates numerical resonant triad for various values of depth: (*a*) shallow depth $h = 0.35\,\text{cm}$, (*b*) finite depth $h = 0.6\,\text{cm}$ and (*c*) deep case with $h = 2\,\text{cm}$. To help identify each limiting case we show in figure 2(*d*) a graph of the tanh as a function of the dimensionless parameter kh.

Gravity waves in fluids with different surface tension than pure water are included in our formulation. High surface tension fluids such as mercury in finite depth have been used in experiments to exhibit remarkable surface wave fluid phenomena. In Falcon, Laroche & Fauve (2002), solitary waves on an air interface with mercury in shallow depth exhibit solitary waves of depression as opposed to solitary waves of elevation.

4. Six wave interaction in infinite depth

In this section we outline the derivation of the six wave equations in infinite depth starting from the non-local formulation of gravity waves as given by (2.9) and (2.10) up to order ϵ. It is convenient to take the gradient of (2.10) and to define $\nabla q \equiv \boldsymbol{Q} = (Q_1, Q_2)$. With this definition, we find the following equations to order ϵ:

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} (1 + \epsilon k \eta) \left(\mathrm{i}\eta_t - \frac{\boldsymbol{k}\cdot\boldsymbol{Q}}{k} \right) = 0, \tag{4.1}$$

$$\boldsymbol{Q}_t + g\nabla\eta + \frac{\epsilon}{2}\nabla\left(|\boldsymbol{Q}|^2 - \eta_t^2\right) = \sigma\nabla^2\nabla\eta. \tag{4.2}$$

We can eliminate the $\boldsymbol{Q}$ variable from (4.1) by taking the time derivative of (4.1) and then substituting the expression for $\boldsymbol{Q}_t$ from (4.2) back into the resulting system. This yields (to

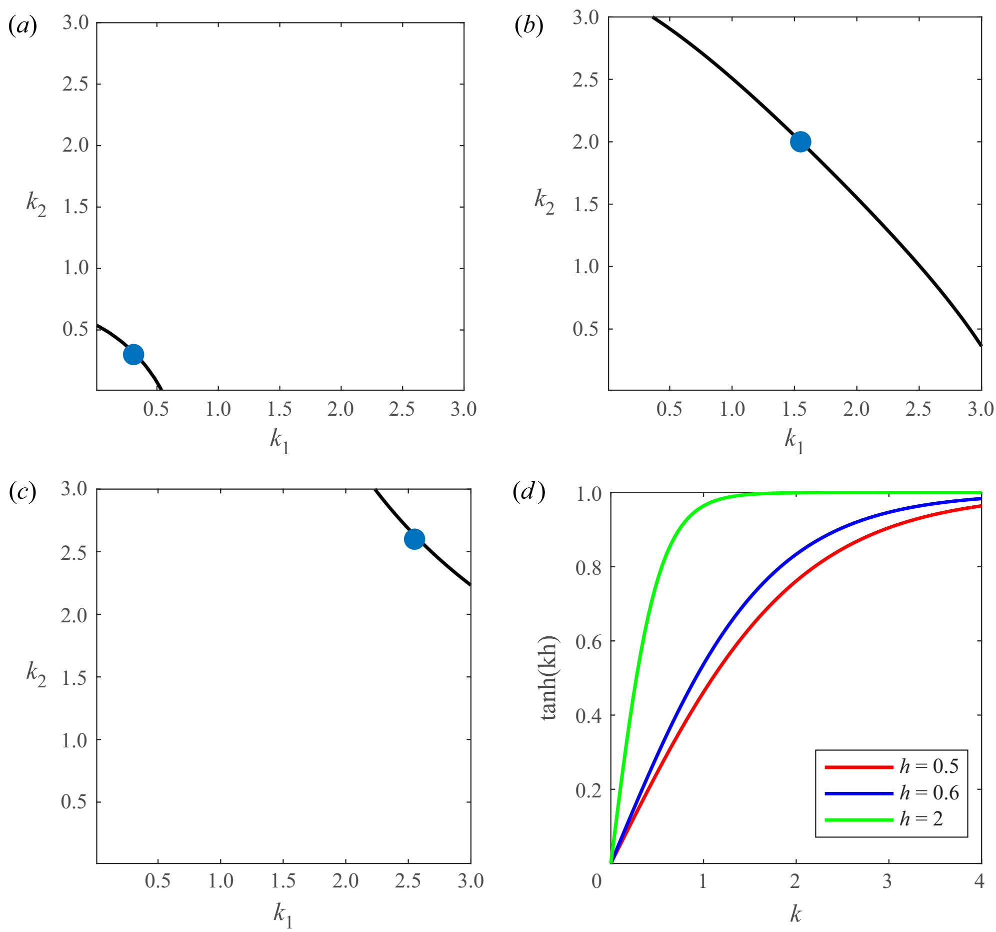

Figure 1. One-dimensional resonant triad curves satisfying the wavenumber–frequency resonance condition $k_3 = k_1 + k_2$ and $\omega_3 = \omega_1 + \omega_2$. All wavenumbers are measured in units of cm^{-1} and depth h in cm. The numerical values for the depth h and surface tension used to produce the figure are respectively given by (*a*) $h = 0.5$ cm; $\sigma = 81\ \mathrm{cm}^3\ \mathrm{s}^{-2}$, (*b*) $h = 0.6$ cm; $\sigma = 81\ \mathrm{cm}^3\ \mathrm{s}^{-2}$ and (*c*) $h = 2$ cm; $\sigma = 73\ \mathrm{cm}^3\ \mathrm{s}^{-2}$. Shown in (*d*) is the graph of the tanh function for various values of depth. The full circles indicate typical wavenumber values leading to (*a*) shallow-depth limit with $k_1 h \approx 0.15$; $k_2 h \approx 0.15$ and $k_3 h \approx 0.3$; (*b*) finite depth with $k_1 h \approx 0.9$; $k_2 h \approx 1.2$ and $k_3 h \approx 1.5$; and (*d*) infinite-depth limit when $k_1 h \approx 5$; $k_2 h \approx 5.2$ and $k_3 h \approx 10.2$.

first order in ϵ)

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \left\{ \mathrm{i}\eta_{tt} + \frac{\boldsymbol{k}}{k} \cdot (g\nabla\eta - \sigma\nabla^2\nabla\eta) \right. \\ \left. + \epsilon \left(\frac{\boldsymbol{k}}{2k} \cdot \nabla(|\boldsymbol{Q}|^2 - \eta_t^2) + \mathrm{i}k(\eta\eta_t)_t - (\boldsymbol{k}\cdot\boldsymbol{Q}\eta)_t \right) \right\} = 0. \tag{4.3}$$

Here, we will use a space–time multiple scale perturbation procedure to find the six wave interaction equations. Before doing so, it is important to mention that most applications of multiple scale asymptotic expansion is done within the framework of ordinary or partial differential equations. This is in sharp contrast to our case where now multiple scale method is being applied on system of equations of an integral type which further

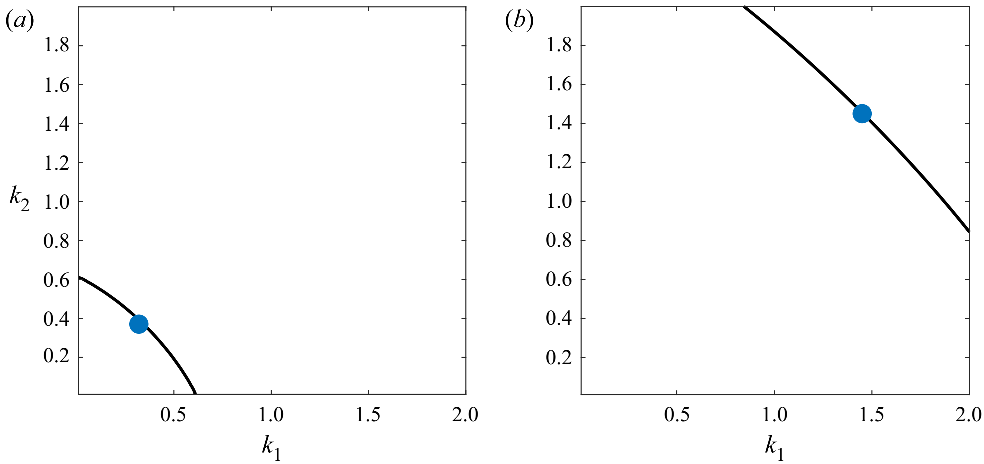

Figure 2. One-dimensional resonant triad curves satisfying the wavenumber–frequency resonance condition $k_3 = k_1 + k_2$ and $\omega_3 = \omega_1 + \omega_2$. All wavenumbers are measured in units of cm^{-1} and depth h in cm. The numerical values for the depth h and surface tension used to produce the figure are respectively given by (*a*) $h = 0.475$ cm; $\sigma = 73\ \mathrm{cm}^3\ \mathrm{s}^{-2}$, (*b*) $h = 0.52$ cm; $\sigma = 73\ \mathrm{cm}^3\ \mathrm{s}^{-2}$. The full circles indicate typical wavenumber values leading to (*a*) shallow-depth limit with $k_1 h \approx 0.152$; $k_2 h \approx 0.175$ and $k_3 h \approx 0.33$; (*b*) finite depth with $k_1 h \approx 0.75$; $k_2 h \approx 0.75$ and $k_3 h \approx 1.5$.

complicates the process of scale separation. As such, a new approach should be adapted. In fact, similar situation has been previously encountered by Ablowitz & Haut Terry (2009) in their study of two fluid layers. To this end, we introduce a slow space–time scales $\boldsymbol{R} = \epsilon \boldsymbol{r}$, $T = \epsilon t$ and assume that the wave elevation and velocity potential to depend on both the fast and slow variables, i.e. $\eta = \eta(\boldsymbol{r}, t, \boldsymbol{R}, T; \epsilon)$ and $Q = Q(\boldsymbol{r}, t, \boldsymbol{R}, T; \epsilon)$. Substituting the asymptotic expansion $\eta = \eta^{(0)} + \epsilon \eta^{(1)} + \cdots$, $Q = Q^{(0)} + \epsilon Q^{(1)} + \cdots$ and the transformation $\nabla \to \nabla_{\boldsymbol{r}} + \epsilon \nabla_{\boldsymbol{R}}$; $\partial_t \to \partial_t + \epsilon \partial_T$ into (4.3) leads to a hierarchy of equations at each order in ϵ. For example, the 'leading'-order equation (that has no apparent order-ϵ terms in it) takes the form

$$L_{\boldsymbol{k}}[\eta^{(0)}] \equiv \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}} \left[\mathrm{i}\eta_{tt}^{(0)} + \frac{\boldsymbol{k}}{k} \cdot \left(g \nabla_{\boldsymbol{r}} \eta^{(0)} - \sigma \nabla_{\boldsymbol{r}}^2 \nabla_{\boldsymbol{r}} \eta^{(0)} \right) \right] = 0. \tag{4.4}$$

For the type of wave elevation $\eta^{(0)}$ considered in this paper (such as those that have a slowly varying amplitude and a carrier wave which are of the form given by (1.1)), this 'leading'-order contributions, i.e. (4.4), could include in it 'hidden' higher-order contributions in powers of ϵ. This is the case due to the presence of an integral in (4.4) and, hence, lack of full separation of scales. This can be clearly seen by examining a typical term in the hierarchy that looks like

$$I \equiv \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}' \mathrm{e}^{-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}'} G(\boldsymbol{k}) f(\boldsymbol{R}', T) \exp(\mathrm{i}s(\boldsymbol{k}_0 \cdot \boldsymbol{r}' - \omega_0 t)), \tag{4.5}$$

where $\boldsymbol{k}_0, \omega_0$ belongs to the set of resonant wavenumbers and frequencies; $\boldsymbol{R}' \equiv \epsilon \boldsymbol{r}'$ and $s = \pm 1$. Multiplying (4.5) by $\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r})$ and integrating over the $\boldsymbol{k}$ variable gives

$$\tilde{I} = \mathrm{e}^{-\mathrm{i}s\omega_0 t} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}' \, \mathrm{d}\boldsymbol{k} \mathrm{e}^{\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}} G(\boldsymbol{k}) f(\boldsymbol{R}', T) \exp(-\mathrm{i}(\boldsymbol{k} - s\boldsymbol{k}_0) \cdot \boldsymbol{r}'). \tag{4.6}$$

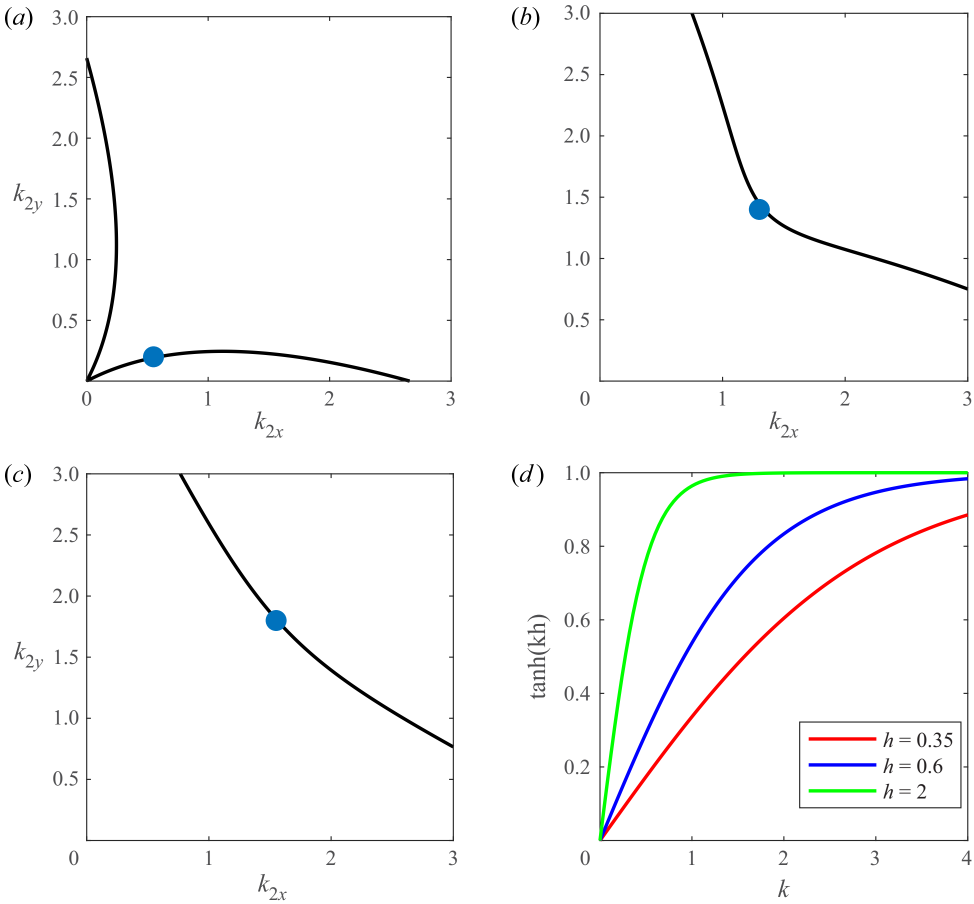

Figure 3. Two-dimensional resonant triad curves satisfying the wavenumber–frequency resonance condition $\boldsymbol{k}_3 = \boldsymbol{k}_1 + \boldsymbol{k}_2$ and $\omega_3 = \omega_1 + \omega_2$. All wavenumbers are measured in units of cm^{-1} and depth in cm. In all cases, the surface tension coefficient is taken to be $\sigma = 73\ \text{cm}^3\ \text{s}^{-2}$. The wavenumbers and depth are as follows: (*a*) $k_{1x} = k_{1y} = 0.7\ \text{cm}^{-1}$ and depth $h = 0.35$ cm, (*b*) $k_{1x} = k_{1y} = 1.5\ \text{cm}^{-1}$ and depth $h = 0.6$ cm, (*c*) $k_{1x} = k_{1y} = 2\ \text{cm}^{-1}$ and depth $h = 2$ cm. Furthermore, we show in (*d*) the graph of tanh function for various values of depth h that help identify the three limiting regimes of shallow, finite and deep depth. As is the case with figure 1, here, full circles indicate typical wavenumber values leading to (*a*) shallow-depth limit with $k_1h \approx 0.35$; $k_2h \approx 0.19$ and $k_3h \approx 0.5$; (*b*) finite depth with $k_1h \approx 1.3$; $k_2h \approx 1.1$ and $k_3h \approx 2.4$ and (*d*) infinite-depth limit when $k_1h \approx 5.6$; $k_2h \approx 4.6$ and $k_3h \approx 10$.

Next, introduce the change of variables: $\boldsymbol{k} - s\boldsymbol{k}_0 \equiv \epsilon \boldsymbol{K}$ and $\boldsymbol{R} \equiv \epsilon \boldsymbol{r}$. With this, (4.6) becomes

$$\begin{aligned}\tilde{I} &= \exp(\mathrm{i}s(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{R}'\, \mathrm{d}\boldsymbol{K} \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R}} G(s\boldsymbol{k}_0 + \epsilon \boldsymbol{K}) f(\boldsymbol{R}', T) \mathrm{e}^{-\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R}'} \\ &= \exp(\mathrm{i}s(\boldsymbol{k}_0 \cdot \boldsymbol{r} - \omega_0 t)) \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{K} G(s\boldsymbol{k}_0 + \epsilon \boldsymbol{K}) \hat{f}(\boldsymbol{K}, T) \mathrm{e}^{\mathrm{i}\boldsymbol{K}\cdot\boldsymbol{R}}. \end{aligned} \tag{4.7}$$

In the above, $\hat{f}$ denotes the two-dimensional Fourier transform of f defined by

$$\hat{f}(\boldsymbol{k}) = F[f] \equiv \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} f(\boldsymbol{r}). \tag{4.8}$$

Finally, replacing $G(s\boldsymbol{k}_0+\epsilon \boldsymbol{K})$ in (4.7) by its first-order approximation $G(s\boldsymbol{k}_0)+\epsilon \boldsymbol{K}\cdot\nabla_{\boldsymbol{k}}G(s\boldsymbol{k}_0)$ and $\boldsymbol{K}f(\boldsymbol{r})$ by $-\mathrm{i}\nabla_{\boldsymbol{R}}f$ we find

$$\tilde{I}=2\pi\exp(\mathrm{i}s(\boldsymbol{k}_0\cdot\boldsymbol{r}-\omega_0 t))\left(G(s\boldsymbol{k}_0)f(\boldsymbol{R},T)-\mathrm{i}\epsilon\nabla_{\boldsymbol{k}}G(s\boldsymbol{k}_0)\cdot\nabla_{\boldsymbol{R}}f+O(\epsilon^2)\right). \tag{4.9}$$

The consequences of (4.9) are twofold: (i) it allows one to identify any 'hidden' contributions in powers of ϵ that otherwise are absent and (ii) it provides a mechanism to get rid of the integral that appears in (4.5) by replacing it with its equivalent differential form. The latter is vital in deriving the six wave interaction system.

The three resonant wave ansatze for the wave elevation considered throughout this paper is given by

$$\eta^{(0)}(\boldsymbol{r},t;\boldsymbol{R},T)=\sum_{j=1}^{3}\left(A_j(\boldsymbol{R},T)\mathrm{e}^{-\mathrm{i}\theta_j}+B_j(\boldsymbol{R},T)\mathrm{e}^{\mathrm{i}\theta_j}\right), \tag{4.10}$$

where $\theta_j\equiv \boldsymbol{k}_j\cdot\boldsymbol{r}-\omega_j t$, $\omega_j=\omega(k_j)$, $k_j=|\boldsymbol{k}_j|$. Then

$$L_{\boldsymbol{k}}[G(\boldsymbol{k})\eta^{(0)}]=0. \tag{4.11}$$

Furthermore, for the ansatz given in (4.10) we find $G(\boldsymbol{k})=\mathrm{i}[-\omega_j^2+s\boldsymbol{k}\cdot\boldsymbol{k}_j(g+\sigma|\boldsymbol{k}_j|^2)/k]$, $s=\pm1$ and $k=|\boldsymbol{k}|$. Note that $\nabla_{\boldsymbol{k}}G(s\boldsymbol{k}_j)=0$ hence, no higher order in powers of ϵ are present in (4.4). Furthermore, $G(s\boldsymbol{k}_j)=0$ whenever, the frequency ω satisfies the infinite-depth water wave dispersion given in (1.4). The leading-order solution for $\boldsymbol{Q}$ is obtained from linearizing (4.2). This yields

$$\boldsymbol{Q}^{(0)}(\boldsymbol{r},t,\boldsymbol{R},T)=\sum_{j=1}^{3}\frac{\omega_j\boldsymbol{k}_j}{k_j}\left(A_j(\boldsymbol{R},T)\mathrm{e}^{-\mathrm{i}\theta_j}+B_j(\boldsymbol{R},T)\mathrm{e}^{\mathrm{i}\theta_j}\right). \tag{4.12}$$

Next, we analyse the order-ϵ equation given by

$$\begin{aligned}-L_{\boldsymbol{k}}[\eta^{(1)}]=&\int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\Bigg[2\mathrm{i}\eta_{tT}^{(0)}+\frac{\boldsymbol{k}}{k}\cdot\Big(g\nabla_{\boldsymbol{R}}\eta^{(0)}-\sigma\nabla_{\boldsymbol{r}}^2\nabla_{\boldsymbol{R}}\eta^{(0)}\\&-2\sigma(\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{R}})\nabla_{\boldsymbol{r}}\eta^{(0)}\Big)\Bigg]\\&+\int_{\mathbb{R}^2}\mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\left(\frac{\boldsymbol{k}}{2k}\cdot\nabla_{\boldsymbol{r}}(|\boldsymbol{Q}^{(0)}|^2-(\eta_t^{(0)})^2)+\mathrm{i}k\left(\eta_t^{(0)}\eta^{(0)}\right)_t\right.\\&\left.-\boldsymbol{k}\cdot\left(\boldsymbol{Q}^{(0)}\eta^{(0)}\right)_t\right).\end{aligned} \tag{4.13}$$

Substituting (4.10) and (4.12) gives rise to a linear non-homogeneous equation that exhibits secular terms proportional to $\exp(\mathrm{i}s\theta_j)$ with wavenumbers $\boldsymbol{k}_j$ and corresponding frequencies $\omega_j=\omega(k_j)$, $j=1,2,3$ satisfying the resonance triad condition (3.1). In terms of the phases θ_j the resonance relation takes the alternative form

$$\theta_3 = \theta_1 + \theta_2. \tag{4.14}$$

The conditions that guarantee removal of such terms leads to the following six wave interaction equations (valid for infinite water depth):

$$2v^I_{p_1}\left(\partial_T + \boldsymbol{C}^I_1 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) A_1 - \mathrm{i}\gamma^I B_2 A_3 = 0, \tag{4.15}$$

$$2v^I_{p_2}\left(\partial_T + \boldsymbol{C}^I_2 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) A_2 - \mathrm{i}\gamma^I B_1 A_3 = 0, \tag{4.16}$$

$$2v^I_{p_3}\left(\partial_T + \boldsymbol{C}^I_3 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) A_3 - \mathrm{i}\gamma^I A_1 A_2 = 0, \tag{4.17}$$

$$2v^I_{p_1}\left(\partial_T + \boldsymbol{C}^I_1 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) B_1 + \mathrm{i}\gamma^I B_3 A_2 = 0, \tag{4.18}$$

$$2v^I_{p_2}\left(\partial_T + \boldsymbol{C}^I_2 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) B_2 + \mathrm{i}\gamma^I B_3 A_1 = 0, \tag{4.19}$$

$$2v^I_{p_3}\left(\partial_T + \boldsymbol{C}^I_3 \cdot \boldsymbol{\nabla}_{\boldsymbol{R}}\right) B_3 + \mathrm{i}\gamma^I B_1 B_2 = 0, \tag{4.20}$$

where $\boldsymbol{C}^I_j \equiv \boldsymbol{\nabla}_{\boldsymbol{k}}\omega(k_j)$ and $v^I_{p_j} \equiv \omega(k_j)/k_j$ are the group and phase velocities, respectively; γ^I denotes the nonlinear coefficient corresponding to the infinite-depth case (see the Appendix)

$$\gamma^I = \omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3, \quad \boldsymbol{e}_j \equiv \boldsymbol{k}_j/k_j. \tag{4.21}$$

For the one-dimensional case for which $\boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 1, \boldsymbol{e}_1\cdot\boldsymbol{e}_3 = 1$ and $\boldsymbol{e}_2\cdot\boldsymbol{e}_3 = 1$ we find $\gamma^I = 2\omega_1\omega_2$ When $B_j = A_j^*$ the resulting equations agree with those given by Simmons (1969).

5. Six wave interaction in finite depth

In this section we turn our attention to the study of the six wave resonant interaction system corresponding to the finite-depth wave limit. The mathematical derivation closely follows the infinite-depth case with the exception that the model coefficients are now dependent on the depth h. We begin by taking the time derivative of the non-local (2.8) for finite depth that contains in it the $O(1)$ and $O(\epsilon)$ terms. This yields (recall that $\boldsymbol{Q} \equiv \boldsymbol{\nabla} q$)

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\left\{\mathrm{i}\eta_{tt} - \frac{\tanh(kh)}{k}\boldsymbol{k}\cdot\boldsymbol{Q}_t + \epsilon\left[\mathrm{i}k\tanh(kh)(\eta\eta_t)_t - (\boldsymbol{k}\cdot\boldsymbol{Q}\eta)_t\right]\right\} + O(\epsilon^2) = 0. \tag{5.1}$$

We substitute the free surface Bernoulli equation (4.2) for $\boldsymbol{Q}_t$ (which is valid for the finite-depth case as well) back into (5.1) to find

$$\int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r}\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\left\{\mathrm{i}\eta_{tt} + \frac{\tanh(kh)}{k}\boldsymbol{k}\cdot(g\boldsymbol{\nabla}\eta - \sigma\nabla^2\boldsymbol{\nabla}\eta)\right.$$
$$\left. + \ \epsilon\left(\frac{\tanh(kh)}{2k}\boldsymbol{k}\cdot\boldsymbol{\nabla}(|\boldsymbol{Q}|^2 - \eta_t^2) + \mathrm{i}k\tanh(kh)(\eta\eta_t)_t - (\boldsymbol{k}\cdot\boldsymbol{Q}\eta)_t\right)\right\} = 0. \tag{5.2}$$

Next, we employ a space–time multiple scale perturbative expansion to isolate the 'leading' and order-ϵ equations for the wave amplitude and velocity potential. In this regard, the integral-to-differential formulation presented in (4.5)–(4.9) is used to identify any 'hidden' order-ϵ contributions that could be embedded in the 'leading'-order equation. With this at hand, the 'leading'-order equation reads

$$L_{\boldsymbol{k}}^{h}[\eta^{(0)}] \equiv \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \left[\mathrm{i}\eta_{tt}^{(0)} + \frac{\tanh(kh)}{k} \boldsymbol{k} \cdot \left(g\nabla_{\boldsymbol{r}}\eta^{(0)} - \sigma \nabla_{\boldsymbol{r}}^2 \nabla_{\boldsymbol{r}}\eta^{(0)} \right) \right] = 0. \tag{5.3}$$

The three wave form (4.10) would transform (5.3) into the integral representation given by (4.5) with $G(\boldsymbol{k}) = \mathrm{i}[-\omega_j^2 + s\tanh(kh)(g + \sigma k_j^2)\boldsymbol{k}\cdot\boldsymbol{k}_j/k]$, $s = \pm 1$ and $k = |\boldsymbol{k}|$. Additionally, we find

$$\nabla_{\boldsymbol{k}} G(s\boldsymbol{k}_j) = \mathrm{i}(gk_j + \sigma k_j^3)\nabla_{\boldsymbol{k}} \tanh(kh)|_{\boldsymbol{k}=s\boldsymbol{k}_j} \neq 0. \tag{5.4}$$

There are two important implications emanating from (5.4) especially in conjunction with (4.9). The first is concerned with the identification of the genuine leading-order contribution that takes the form $G(s\boldsymbol{k}_j) = 0$. The latter is always satisfied so long the frequency ω_j follows the finite-depth wave dispersion relation (2.11). The associated eigenfunction (wave elevation) remains unchanged while the velocity potential assumes the new form

$$\boldsymbol{Q}^{(0)}(\boldsymbol{r}, t, \boldsymbol{R}, T) = \sum_{j=1}^{3} \frac{\omega_j \boldsymbol{k}_j}{k_j \tanh(k_j h)} \left(A_j(\boldsymbol{R}, T)\mathrm{e}^{-\mathrm{i}\theta_j} + B_j(\boldsymbol{R}, T)\mathrm{e}^{\mathrm{i}\theta_j} \right). \tag{5.5}$$

Secondly, as a consequence of (5.4) the order-ϵ equation for $\eta^{(1)}$ now acquires a new term which is proportional to $\nabla_{\boldsymbol{k}} G(s\boldsymbol{k}_j)$

$$\begin{aligned} -L_{\boldsymbol{k}}^{h}[\eta^{(1)}] = \int_{\mathbb{R}^2} \mathrm{d}\boldsymbol{r} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \Bigg[& 2\mathrm{i}\eta_{tT}^{(0)} + \frac{\tanh(kh)}{k} \boldsymbol{k} \cdot \Big(g\nabla_{\boldsymbol{R}}\eta^{(0)} - \sigma \nabla_{\boldsymbol{r}}^2 \nabla_{\boldsymbol{R}}\eta^{(0)} \\ & - 2\sigma(\nabla_{\boldsymbol{r}} \cdot \nabla_{\boldsymbol{R}})\nabla_{\boldsymbol{r}}\eta^{(0)} \Big) \\ & - \mathrm{i}\nabla_{\boldsymbol{k}} G \cdot \nabla_{\boldsymbol{R}}\eta^{(0)} + \frac{\tanh(kh)}{2k} \boldsymbol{k} \cdot \nabla_{\boldsymbol{r}} \left(|\boldsymbol{Q}^{(0)}|^2 - (\eta_t^{(0)})^2 \right) \\ & + \mathrm{i}k\tanh(kh) \left(\eta_t^{(0)}\eta^{(0)} \right)_t - \boldsymbol{k} \cdot \left(\boldsymbol{Q}^{(0)}\eta^{(0)} \right)_t \Bigg]. \end{aligned} \tag{5.6}$$

Substituting (4.10) and (5.5) into (4.13) gives rise to a non-homogeneous integral equation whose solvability condition (boundedness of the associated eigenfunctions) determines the six wave interactions equations. They are given by

$$2v_1^F \left(\partial_T + \boldsymbol{C}_1^F \cdot \nabla_{\boldsymbol{R}} \right) A_1 - \mathrm{i}\gamma^F B_2 A_3 = 0, \tag{5.7}$$

$$2v_2^F \left(\partial_T + \boldsymbol{C}_2^F \cdot \nabla_{\boldsymbol{R}} \right) A_2 - \mathrm{i}\gamma^F B_1 A_3 = 0, \tag{5.8}$$

$$2v_3^F \left(\partial_T + \boldsymbol{C}_3^F \cdot \nabla_{\boldsymbol{R}} \right) A_3 - \mathrm{i}\gamma^F A_1 A_2 = 0, \tag{5.9}$$

$$2v_1^F \left(\partial_T + \boldsymbol{C}_1^F \cdot \nabla_{\boldsymbol{R}} \right) B_1 + \mathrm{i}\gamma^F B_3 A_2 = 0, \tag{5.10}$$

$$2v_2^F \left(\partial_T + \boldsymbol{C}_2^F \cdot \nabla_{\boldsymbol{R}} \right) B_2 + \mathrm{i}\gamma^F B_3 A_1 = 0, \tag{5.11}$$

$$2v_3^F \left(\partial_T + \boldsymbol{C}_3^F \cdot \nabla_{\boldsymbol{R}} \right) B_3 + \mathrm{i}\gamma^F B_1 B_2 = 0, \tag{5.12}$$

where $\boldsymbol{C}_j^F = \nabla_{\boldsymbol{k}}\omega(\boldsymbol{k} = \boldsymbol{k}_j)$ is the group velocity; $v_j^F \equiv \omega(\boldsymbol{k}_j)/[k_j \tanh(k_j h)]$ and γ^F is the nonlinear coupling coefficient respectively given by

$$\boldsymbol{C}_j^F = \frac{(g + 3\sigma k_j^2)\tanh(k_j h)\boldsymbol{k}_j}{2k_j\omega_j} + \frac{h(g + \sigma k_j^2)\operatorname{sech}^2(k_j h)\boldsymbol{k}_j}{2\omega_j}, \tag{5.13}$$

$$\gamma^F = \frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1 h)\tanh(k_2 h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1 h)\tanh(k_3 h)} + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2 h)\tanh(k_3 h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3. \tag{5.14}$$

The above equations are equivalent to the six wave equations (1.8) by taking $\sigma_j = \gamma^F/(2v_j^F), j = 1, 2, 3$. We next discuss various important limiting cases such as the infinite and shallow depth.

Infinite depth
This amounts to taking the $h \to \infty$ in (5.14) recovering the result given in (4.21).

Shallow-depth limit
This is an interesting and important case that also apparently has not been studied in the literature. This amount to 'small' depth h such that $\tanh(k_j h) \approx hk_j$. In this limit the six wave interaction shallow-depth equations are given by

$$\frac{2\omega_1}{k_1^2 h}\left(\partial_T + \boldsymbol{C}_1^S\cdot\nabla_{\boldsymbol{R}}\right)A_1 - \mathrm{i}\gamma^S B_2 A_3 = 0, \tag{5.15}$$

$$\frac{2\omega_2}{k_2^2 h}\left(\partial_T + \boldsymbol{C}_2^S\cdot\nabla_{\boldsymbol{R}}\right)A_2 - \mathrm{i}\gamma^S B_1 A_3 = 0, \tag{5.16}$$

$$\frac{2\omega_3}{k_3^2 h}\left(\partial_T + \boldsymbol{C}_3^S\cdot\nabla_{\boldsymbol{R}}\right)A_3 - \mathrm{i}\gamma^S A_1 A_2 = 0, \tag{5.17}$$

$$\frac{2\omega_1}{k_1^2 h}\left(\partial_T + \boldsymbol{C}_1^S\cdot\nabla_{\boldsymbol{R}}\right)B_1 + \mathrm{i}\gamma^S B_3 A_2 = 0, \tag{5.18}$$

$$\frac{2\omega_2}{k_2^2 h}\left(\partial_T + \boldsymbol{C}_2^S\cdot\nabla_{\boldsymbol{R}}\right)B_2 + \mathrm{i}\gamma^S B_3 A_1 = 0, \tag{5.19}$$

$$\frac{2\omega_3}{k_3^2 h}\left(\partial_T + \boldsymbol{C}_3^S\cdot\nabla_{\boldsymbol{R}}\right)B_3 + \mathrm{i}\gamma^S B_1 B_2 = 0, \tag{5.20}$$

where, as $h \to 0$, the dispersion relation is given by

$$\omega^2 = (gk^2 + \sigma k^4)h. \tag{5.21}$$

With this at hand, the group velocity and nonlinear coefficient are given by

$$\boldsymbol{C}_j^S = \frac{h(g + 2\sigma k_j^2)\boldsymbol{k}_j}{\omega_j}, \tag{5.22}$$

and

$$\gamma^S = \frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{(k_1 h)(k_2 h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{(k_1 h)(k_3 h)} + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{(k_2 h)(k_3 h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3. \tag{5.23}$$

Equations (5.15)–(5.20) comprise the shallow-depth six wave interaction equations. While written in this form they have the depth h in them, they can be rescaled to eliminate this dependence.

6. Conclusion

Multi wave parametric resonant interactions are ubiquitous in nonlinear sciences. They occur in a wide range of physical settings beyond gravity waves; e.g. optics, plasma physics and other areas of fluid mechanics to name a few. Generally speaking, they form when two or more waves with wavenumbers k_j and corresponding frequencies ω_j 'add up' to generate a new wave form with wavenumber k_n and frequency ω_n that satisfy the resonant condition: $\omega_n = \sum_j \omega_j$ and $k_n = \sum_j k_j$. We also remark that such parametric processes find application in nonlinear optics where second and third harmonic generations are used for frequency conversion (Buryak *et al.* 2002). In fluid mechanics on the other hand, they are sometimes utilized to explain formation of rogue waves (Yang & Yang 2022, 2021).

Of particular interest are the three wave resonant triad interactions which occur on an order $1/\epsilon$ time scale where ϵ is a measure of nonlinearity. Mathematically speaking, they form a coupled system of three first-order quadratically nonlinear evolution equations in two space one time dimension. As mentioned in the introduction, they were first derived by Simmons (1969) in the context of deep-depth waves. Later, they were shown to be exactly solvable in 1975 by Ablowitz & Haberman (1975*a*) using inverse scattering transform methods. Importantly, their integrability was established by introducing a system of six wave interactions whose integrable symmetry reduction leads to the same three wave equations as Simmons found.

In this paper we show that the six wave interaction system found by Ablowitz & Haberman, in the context of integrability theory, can be derived from the equations of classical gravity waves. Rather than employing the depth-dependent equations of water waves ((2.1)–(2.4)) we use the Ablowitz–Fokas–Musslimani (Ablowitz *et al.* 2006) (2+1)-dimensional non-local reformulation and solve for the free surface variables of gravity waves; see ((2.5)–(2.6)). This has the advantage of reducing some of the tedious algebra. From this this formulation the six wave interaction equations satisfying triad resonance conditions are obtained in both finite and infinite-depth cases. The shallow-depth limit is also considered. The derivation has several steps: (i) introduce a small amplitude wave elevation and velocity potential via a small parameter $0 < \epsilon \ll 1$ and recast the non-local equations in terms of the new variables. (ii) Expand the resulting system and keep terms up to order ϵ. (iii) Implement a space–time multiple scale perturbation expansion. Since the equations are non-local one must carefully consider how to apply multiscale methods. Equations ((4.5)–(4.9)) are used here to transform non-local equations to differential equations and hence achieve full scale separation. (iv) Make a three wave ansatz for the wave amplitude and velocity potential. (v) Remove all secular terms at the order-ϵ equation and obtain the desired system of six waves equations. As a closing remark, we point out that the six wave system admits a generalized integrable non-local symmetry reductions in the form $r(x, t) = \sigma q(x_0 - x, t_0 - t)$ with arbitrary real parameters x_0, t_0 (Ablowitz & Musslimani 2021). This in turn leads to a new system of interacting shifted three wave system whose study is left for a future work.

Funding. This work was partially supported by NSF under grant DMS-2005343 and NSFC under grant number 12101590.

Declaration of interests. The authors report no conflict of interest.

Author ORCIDs.
Xu-Dan Luo https://orcid.org/0000-0002-3833-5530;
Ziad H. Musslimani https://orcid.org/0000-0002-5846-1219.

Appendix A

Here, we provide a detailed account of the derivation of the six wave equations for infinite depth in particular deriving the linear and nonlinear coefficients that appear in (4.15)–(4.20). To start with, note that the right-hand side of (4.13) has two contributing factors: (i) linear and (ii) nonlinear. Substituting (4.10) into (4.13) and using the integral-differential identity discussed in (4.5)–(4.9) (that allows one to transform the integral to differential form) we find

$$\begin{aligned}
\text{linear terms} &\equiv 2\mathrm{i}\eta_{tT}^{(0)} + \frac{\boldsymbol{k}}{k}\cdot\left(g\nabla_{\boldsymbol{R}}\eta^{(0)} - \sigma\nabla_{\boldsymbol{r}}^2\nabla_{\boldsymbol{R}}\eta^{(0)} - 2\sigma(\nabla_{\boldsymbol{r}}\cdot\nabla_{\boldsymbol{R}})\nabla_{\boldsymbol{r}}\eta^{(0)}\right)\\
&= \sum_{j=1}^{3}\left(-2\omega_j\frac{\partial A_j}{\partial T}\mathrm{e}^{-\mathrm{i}\theta_j} + 2\omega_j\frac{\partial B_j}{\partial T}\mathrm{e}^{\mathrm{i}\theta_j}\right)\\
&\quad - \sum_{j=1}^{3}\frac{\boldsymbol{k}_j}{k_j}\cdot\left(g\nabla_{\boldsymbol{R}}A_j + \sigma k_j^2\nabla_{\boldsymbol{R}}A_j + 2\sigma\boldsymbol{k}_j(\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}})A_j\right)\mathrm{e}^{-\mathrm{i}\theta_j}\\
&\quad + \sum_{j=1}^{3}\frac{\boldsymbol{k}_j}{k_j}\cdot\left(g\nabla_{\boldsymbol{R}}B_j + \sigma k_j^2\nabla_{\boldsymbol{R}}B_j + 2\sigma\boldsymbol{k}_j(\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}})B_j\right)\mathrm{e}^{\mathrm{i}\theta_j}\\
&= -\sum_{j=1}^{3}\left[2\omega_j\frac{\partial A_j}{\partial T} + \frac{(g+3\sigma k_j^2)}{k_j}\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}}A_j\right]\mathrm{e}^{-\mathrm{i}\theta_j}\\
&\quad + \sum_{j=1}^{3}\left[2\omega_j\frac{\partial B_j}{\partial T} + \frac{(g+3\sigma k_j^2)}{k_j}\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}}B_j\right]\mathrm{e}^{\mathrm{i}\theta_j}\\
&= -\sum_{j=1}^{3}2\omega_j\left[\frac{\partial A_j}{\partial T} + \nabla_{\boldsymbol{k}}\omega_j\cdot\nabla_{\boldsymbol{R}}A_j\right]\mathrm{e}^{-\mathrm{i}\theta_j} + \sum_{j=1}^{3}2\omega_j\left[\frac{\partial B_j}{\partial T} + \nabla_{\boldsymbol{k}}\omega_j\cdot\nabla_{\boldsymbol{R}}B_j\right]\mathrm{e}^{\mathrm{i}.\theta_j}
\end{aligned} \tag{A1}$$

Next, we turn our attention to the nonlinear terms which we write as $R_1 + R_2$ with (here $\boldsymbol{Q} = (Q_1, Q_2)$)

$$R_1 = \frac{k_x}{k}\left[Q_1Q_{1x} + Q_2Q_{2x} - \eta_t\eta_{tx}\right] + \frac{k_y}{k}\left[Q_1Q_{1y} + Q_2Q_{2y} - \eta_t\eta_{ty}\right], \tag{A2}$$

$$R_2 = \mathrm{i}kF\left[(\eta\eta_t)_t\right] - k_xF\left[(Q_1\eta)_t\right] - k_yF\left[(Q_2\eta)_t\right]. \tag{A3}$$

Substituting the expressions for the wave elevation $\eta^{(0)}$ and velocity potential $\boldsymbol{Q}^{(0)}$ into (A2) we obtain

$$R_1 = -\mathrm{i}\sum_{j,\ell=1}^{3}\left(\frac{\omega_j\omega_\ell(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell} + \frac{\omega_\ell\omega_j}{k}(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)\right)A_jA_\ell\mathrm{e}^{-\mathrm{i}(\theta_\ell+\theta_j)}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\left(\frac{\omega_j\omega_\ell(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}+\frac{\omega_\ell\omega_j}{k}(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)\right)B_jB_\ell\mathrm{e}^{\mathrm{i}(\theta_j+\theta_\ell)}$$

$$-\,\mathrm{i}\sum_{j,\ell=1}^{3}\left(\frac{\omega_j\omega_\ell(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}-\frac{\omega_\ell\omega_j}{k}(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)\right)B_jA_\ell\mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\left(\frac{\omega_j\omega_\ell(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}-\frac{\omega_\ell\omega_j}{k}(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)\right)A_jB_\ell\mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)}. \tag{A4}$$

Next, we compute the quantity R_2. We find

$$R_2=-\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j+\omega_\ell)\left(k\omega_\ell+\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)A_jA_\ell\mathrm{e}^{-\mathrm{i}(\theta_\ell+\theta_j)}$$

$$-\,\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j+\omega_\ell)\left(k\omega_\ell-\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)B_jB_\ell\mathrm{e}^{\mathrm{i}(\theta_\ell+\theta_j)}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j-\omega_\ell)\left(k\omega_\ell+\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)B_jA_\ell\mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j-\omega_\ell)\left(k\omega_\ell-\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)A_jB_\ell\mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)}. \tag{A5}$$

Thus, we have

$$R_1+R_2=$$

$$-\,\mathrm{i}\sum_{j,\ell=1}^{3}\omega_j\omega_\ell\left[\frac{(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}+\frac{\boldsymbol{k}\cdot\boldsymbol{k}_\ell}{k}+\frac{(\omega_j+\omega_\ell)}{\omega_j\omega_\ell}\left(k\omega_\ell+\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)\right]A_jA_\ell\mathrm{e}^{-\mathrm{i}(\theta_j+\theta_\ell)} \tag{A6}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\omega_j\omega_\ell\left(\frac{(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}+\frac{\boldsymbol{k}\cdot\boldsymbol{k}_\ell}{k}-\frac{(\omega_j+\omega_\ell)}{\omega_j\omega_\ell}\left(k\omega_\ell-\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)\right)B_jB_\ell\mathrm{e}^{\mathrm{i}(\theta_j+\theta_\ell)} \tag{A7}$$

$$-\,\mathrm{i}\sum_{j,\ell=1}^{3}\omega_j\omega_\ell\left(\frac{(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}-\frac{\boldsymbol{k}\cdot\boldsymbol{k}_\ell}{k}-\frac{(\omega_j-\omega_\ell)}{\omega_j\omega_\ell}\left(k\omega_\ell+\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)\right)B_jA_\ell\mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)} \tag{A8}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\omega_j\omega_\ell\left(\frac{(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell}-\frac{\boldsymbol{k}\cdot\boldsymbol{k}_\ell}{k}+\frac{(\omega_j-\omega_\ell)}{\omega_j\omega_\ell}\left(k\omega_\ell-\frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j}\right)\right)A_jB_\ell\mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)} \tag{A9}$$

In what follows, we explicitly compute the nonlinear coefficients that lead to resonant three wave triad, i.e. those that satisfy $\theta_1 + \theta_2 = \theta_3$. The imaginary number i is already accounted for in the six waves (5.7)–(5.12).

Coefficient of $e^{-i\theta_1}$.
Here, we have contribution from resonant terms with $j = 2, \ell = 3$ (coming from (A8)) as well as resonant terms from (A9) with $j = 3, \ell = 2$ and $\boldsymbol{k} = -\boldsymbol{k}_1$. Thus

$$\begin{aligned}\gamma_1^I &= \frac{\omega_2\omega_3(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)}{k_1k_2k_3} - \frac{\omega_2\omega_3}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_3) + (\omega_2-\omega_3)\left(k_1\omega_3 - \frac{\omega_2\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_2}\right)\\ &\quad - \frac{\omega_3\omega_2(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1k_3k_2} + \frac{\omega_3\omega_2}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) - (\omega_3-\omega_2)\left(-k_1\omega_2 - \frac{\omega_3\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3}\right)\\ &= k_1\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).\end{aligned} \tag{A10}$$

Coefficient of $e^{-i\theta_2}$.
Here, we have contribution from resonant terms with $j = 1, \ell = 3$ (from (A8)) and contribution from resonant terms with $j = 3, \ell = 1$ (from (A9)) with $\boldsymbol{k} = -\boldsymbol{k}_2$. Thus,

$$\begin{aligned}\gamma_2^I &= \frac{\omega_1\omega_3(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)}{k_2k_1k_3} - \frac{\omega_1\omega_3}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3) + (\omega_1-\omega_3)\left(k_2\omega_3 - \frac{\omega_1\boldsymbol{k}_2\cdot\boldsymbol{k}_1}{k_1}\right)\\ &\quad - \frac{\omega_3\omega_1(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2k_3k_1} + \frac{\omega_3\omega_1}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_1) - (\omega_3-\omega_1)\left(-k_2\omega_1 - \frac{\omega_3\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3}\right)\\ &= k_2\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).\end{aligned} \tag{A11}$$

Coefficient of $e^{-i\theta_3}$.
Here, we have contribution from resonant terms with $j = 1, \ell = 2$ and $j = 2, \ell = 1$ (from (A6)) with $\boldsymbol{k} = -\boldsymbol{k}_3$. This gives

$$\begin{aligned}\gamma_3 &= -\left(-\frac{\omega_1\omega_2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)}{k_3k_1k_2} - \frac{\omega_1\omega_2}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_2) + (\omega_1+\omega_2)\left(k_3\omega_2 - \frac{\omega_1\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{k_1}\right)\right)\\ &\quad - \left(-\frac{\omega_2\omega_1(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)}{k_3k_2k_1} - \frac{\omega_2\omega_1}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_1) + (\omega_2+\omega_1)\left(k_3\omega_1 - \frac{\omega_2\boldsymbol{k}_3\cdot\boldsymbol{k}_2}{k_2}\right)\right)\\ &= k_3\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).\end{aligned} \tag{A12}$$

Coefficient of $e^{i\theta_1}$.
Here, we have contribution from resonant terms with $j = 3, \ell = 2$ (from (A8)) and contribution from resonant terms with $j = 2, \ell = 3$ (from (A9)) with $\boldsymbol{k} = \boldsymbol{k}_1$. Thus

$$\begin{aligned}\gamma_1 &= -\frac{\omega_3\omega_2(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1k_3k_2} + \frac{\omega_3\omega_2}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2) + (\omega_3-\omega_2)\left(k_1\omega_2 + \frac{\omega_3\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3}\right)\\ &\quad + \frac{\omega_2\omega_3(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)}{k_1k_2k_3} - \frac{\omega_2\omega_3}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_3) - (\omega_2-\omega_3)\left(-k_1\omega_3 + \frac{\omega_2\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_2}\right)\\ &= k_1\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).\end{aligned} \tag{A13}$$

Coefficient of $e^{i\theta_2}$.
Here, we have contribution from resonant terms with $j=3, \ell=1$ (from (A8)) and contribution from resonant terms with $j=1, \ell=3$ (from (A9)) with $\boldsymbol{k}=\boldsymbol{k}_2$. Thus,

$$\gamma_2 = -\frac{\omega_3\omega_1(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2k_3k_1} + \frac{\omega_3\omega_1}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_1) + (\omega_3-\omega_1)\left(k_2\omega_1 + \frac{\omega_3\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3}\right) \tag{A14}$$

$$+\frac{\omega_1\omega_3(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)}{k_2k_1k_3} - \frac{\omega_1\omega_3}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3) - (\omega_1-\omega_3)\left(-k_2\omega_3 + \frac{\omega_1\boldsymbol{k}_2\cdot\boldsymbol{k}_1}{k_1}\right)$$

$$= k_2\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right). \tag{A15}$$

Coefficient of $e^{i\theta_3}$.
Here, we have contribution from resonant terms with $j=1, \ell=2$ and $j=2, \ell=1$ (from (A7)) with $\boldsymbol{k}=\boldsymbol{k}_3$. This gives

$$\gamma_3 = \frac{\omega_1\omega_2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)}{k_3k_1k_2} + \frac{\omega_1\omega_2}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_2) + (\omega_1+\omega_2)\left(-k_3\omega_2 + \frac{\omega_1\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{k_1}\right)$$

$$+\frac{\omega_2\omega_1(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)}{k_3k_2k_1} + \frac{\omega_2\omega_1}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_1) + (\omega_2+\omega_1)\left(-k_3\omega_1 + \frac{\omega_2\boldsymbol{k}_3\cdot\boldsymbol{k}_2}{k_2}\right)$$

$$= k_3\left(\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3 + \omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3 + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right). \tag{A16}$$

Appendix B

The purpose of this appendix is to derive the linear and nonlinear coefficients associated with the six wave equations in finite depth given by (5.7)–(5.12). Specifically, we compute the group velocity and nonlinear coupling coefficients. A frequent use of the integral-to-differential identity shown in (4.5)–(4.9) will be made that allows one to determine all secular terms without having to deal with any integral. Our starting point is the right-hand side of (5.6) which comprises of linear and nonlinear terms. Upon substituting (4.10) and (5.5) into (5.6) the linear terms read

$$\text{linear terms} = \sum_{j=1}^{3}\left(-2\omega_j\frac{\partial A_j}{\partial T}e^{-i\theta_j} + 2\omega_j\frac{\partial B_j}{\partial T}e^{i\theta_j}\right)$$

$$-\sum_{j=1}^{3}\frac{\tanh(k_jh)}{k_j}\boldsymbol{k}_j\cdot(g\nabla_{\boldsymbol{R}}A_j + \sigma k_j^2\nabla_{\boldsymbol{R}}A_j + 2\sigma\boldsymbol{k}_j(\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}})A_j)e^{-i\theta_j}$$

$$+\sum_{j=1}^{3}\frac{\tanh(k_jh)}{k_j}\boldsymbol{k}_j\cdot(g\nabla_{\boldsymbol{R}}B_j + \sigma k_j^2\nabla_{\boldsymbol{R}}B_j + 2\sigma\boldsymbol{k}_j(\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}})B_j)e^{i\theta_j}$$

$$-h(gk_j + \sigma k_j^3)\text{sech}^2(k_jh)\boldsymbol{k}_j\cdot\nabla_{\boldsymbol{R}}A_je^{-i\theta_j}$$

$$+ h(gk_j + \sigma k_j^3)\mathrm{sech}^2(k_jh)\boldsymbol{k}_j \cdot \nabla_{\boldsymbol{R}} B_j \mathrm{e}^{\mathrm{i}\theta_j}$$

$$= -\sum_{j=1}^{3} 2\omega_j \left[\frac{\partial A_j}{\partial T} + \nabla_{\boldsymbol{k}}\omega_j \cdot \nabla_{\boldsymbol{R}} A_j\right] \mathrm{e}^{-\mathrm{i}\theta_j} + \sum_{j=1}^{3} 2\omega_j \left[\frac{\partial B_j}{\partial T} + \nabla_{\boldsymbol{k}}\omega_j \cdot \nabla_{\boldsymbol{R}} B_j\right] \mathrm{e}^{\mathrm{i}.\theta_j} \tag{B1}$$

Next, we compute the nonlinear terms which can be express as a sum of two contributions: $\tanh(kh)R_1 + R_2$ with (recall that $\boldsymbol{Q} = (Q_1, Q_2)$)

$$R_1 = \frac{k_x}{k}[Q_1Q_{1x} + Q_2Q_{2x} - \eta_t\eta_{tx}] + \frac{k_y}{k}F[Q_1Q_{1y} + Q_2Q_{2y} - \eta_t\eta_{ty}], \tag{B2}$$

$$R_2 = \mathrm{i}k\tanh(kh)(\eta\eta_t)_t - k_x(Q_1\eta)_t - k_y(Q_2\eta)_t. \tag{B3}$$

Substituting (4.10) and (5.5) into (B2) one gets

$$R_1 = -\mathrm{i}\sum_{j,\ell=1}^{3}\left(\varGamma_{h,j\ell}^{(1)} + \varGamma_{j\ell}^{(2)}\right)A_jA_\ell \mathrm{e}^{-\mathrm{i}(\theta_\ell+\theta_j)} + \mathrm{i}\sum_{j,\ell=1}^{3}\left(\varGamma_{h,j\ell}^{(1)} + \varGamma_{j\ell}^{(2)}\right)B_jB_\ell \mathrm{e}^{\mathrm{i}(\theta_j+\theta_\ell)}$$
$$-\mathrm{i}\sum_{j,\ell=1}^{3}\left(\varGamma_{h,j\ell}^{(1)} - \varGamma_{j\ell}^{(2)}\right)B_jA_\ell \mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)} + \mathrm{i}\sum_{j,\ell=1}^{3}\left(\varGamma_{h,j\ell}^{(1)} - \varGamma_{j\ell}^{(2)}\right)A_jB_\ell \mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)}, \tag{B4}$$

where

$$\varGamma_{h,j\ell}^{(1)} = \frac{\omega_j\omega_\ell(\boldsymbol{k}_j\cdot\boldsymbol{k}_\ell)(\boldsymbol{k}\cdot\boldsymbol{k}_\ell)}{kk_jk_\ell\tanh(k_jh)\tanh(k_\ell h)}, \quad \varGamma_{j\ell}^{(2)} = \frac{\omega_j\omega_\ell}{k}(\boldsymbol{k}\cdot\boldsymbol{k}_\ell). \tag{B5a,b}$$

Next, we compute the quantity R_2. We find

$$R_2 = -\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j+\omega_\ell)\left(k\tanh(kh)\omega_\ell + \varGamma_{h,j\ell}^{(3)}\right)A_jA_\ell \mathrm{e}^{-\mathrm{i}(\theta_\ell+\theta_j)}$$
$$-\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j+\omega_\ell)\left(k\tanh(kh)\omega_\ell - \varGamma_{h,j\ell}^{(3)}\right)B_jB_\ell \mathrm{e}^{\mathrm{i}(\theta_\ell+\theta_j)}$$
$$+\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j-\omega_\ell)\left(k\tanh(kh)\omega_\ell + \varGamma_{h,j\ell}^{(3)}\right)B_jA_\ell \mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)}$$
$$+\mathrm{i}\sum_{j,\ell=1}^{3}(\omega_j-\omega_\ell)\left(k\tanh(kh)\omega_\ell - \varGamma_{h,j\ell}^{(3)}\right)A_jB_\ell \mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)}, \tag{B6}$$

where

$$\varGamma_{h,j\ell}^{(3)} = \frac{\omega_j\boldsymbol{k}\cdot\boldsymbol{k}_j}{k_j\tanh(k_jh)}. \tag{B7}$$

With this at hand, all nonlinear terms that appear on the right-hand side of (5.6) are given by

$$\tanh(kh)R_1 + R_2$$

$$= -\mathrm{i}\sum_{j,\ell=1}^{3}\left\{\tanh(kh)\left(\Gamma_{h,j\ell}^{(1)} + \Gamma_{j\ell}^{(2)}\right) + (\omega_j+\omega_\ell)\left(k\tanh(kh)\omega_\ell + \Gamma_{h,j\ell}^{(3)}\right)\right\}A_jA_\ell \mathrm{e}^{-\mathrm{i}(\theta_\ell+\theta_j)} \tag{B8}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\left\{\tanh(kh)\left(\Gamma_{h,j\ell}^{(1)} + \Gamma_{j\ell}^{(2)}\right) - (\omega_j+\omega_\ell)\left(k\tanh(kh)\omega_\ell - \Gamma_{h,j\ell}^{(3)}\right)\right\}B_jB_\ell \mathrm{e}^{\mathrm{i}(\theta_j+\theta_\ell)} \tag{B9}$$

$$-\,\mathrm{i}\sum_{j,\ell=1}^{3}\left\{\tanh(kh)\left(\Gamma_{h,j\ell}^{(1)} - \Gamma_{j\ell}^{(2)}\right) - (\omega_j-\omega_\ell)\left(k\tanh(kh)\omega_\ell + \Gamma_{h,j\ell}^{(3)}\right)\right\}B_jA_\ell \mathrm{e}^{\mathrm{i}(\theta_j-\theta_\ell)} \tag{B10}$$

$$+\,\mathrm{i}\sum_{j,\ell=1}^{3}\left\{\tanh(kh)\left(\Gamma_{h,j\ell}^{(1)} - \Gamma_{j\ell}^{(2)}\right) + (\omega_j-\omega_\ell)\left(k\tanh(kh)\omega_\ell - \Gamma_{h,j\ell}^{(3)}\right)\right\}A_jB_\ell \mathrm{e}^{-\mathrm{i}(\theta_j-\theta_\ell)}. \tag{B11}$$

Next, we are ready now to identify all secular terms that satisfy the triad resonant condition $\theta_1+\theta_2=\theta_3$.

Coefficient of $\mathrm{e}^{-\mathrm{i}\theta_1}$.
Here, we have contribution from resonant terms with $j=2$, $\ell=3$ coming from (B10) and $j=3$, $\ell=2$ arising from (B11). In both cases, $\boldsymbol{k}=-\boldsymbol{k}_1$. Thus

$$\begin{aligned}
&-\left\{\tanh(k_1h)\left(\Gamma_{h,23}^{(1)} - \Gamma_{23}^{(2)}\right) - (\omega_2-\omega_3)\left(k_1\tanh(k_1h)\omega_3 + \Gamma_{h,23}^{(3)}\right)\right\}\\
&\quad + \tanh(k_1h)\left(\Gamma_{h,32}^{(1)} - \Gamma_{32}^{(2)}\right) + (\omega_3-\omega_2)\left(k_1\tanh(k_1h)\omega_2 - \Gamma_{h,32}^{(3)}\right)\\
&= \tanh(k_1h)\left(\frac{\omega_2\omega_3(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)}{k_1k_2k_3\tanh(k_2h)\tanh(k_3h)} - \frac{\omega_2\omega_3}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)\right.\\
&\quad\left. - k_1\omega_1\omega_3 + \frac{\omega_1\omega_2\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_2\tanh(k_2h)\tanh(k_1h)}\right)\\
&\quad + \tanh(k_1h)\left(-\frac{\omega_3\omega_2(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1k_3k_2\tanh(k_3h)\tanh(k_2h)} + \frac{\omega_3\omega_2}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\right.\\
&\quad\left. + k_1\omega_1\omega_2 + \frac{\omega_1\omega_3\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3\tanh(k_3h)\tanh(k_1h)}\right).
\end{aligned} \tag{B12}$$

After some algebra, we get

$$\begin{aligned}
\gamma_1^F &= k_1\tanh(k_1h)\left(\frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)}\right.\\
&\quad\left. + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).
\end{aligned} \tag{B13}$$

Coefficient of $\mathrm{e}^{-\mathrm{i}\theta_2}$.
In this case, the only terms that contribute to resonance are those with $j = 1$, $\ell = 3$ (B10) and $j = 3$, $\ell = 1$ (B11). In addition, we find $\boldsymbol{k} = -\boldsymbol{k}_2$ and

$$
\begin{aligned}
&-\left\{\tanh(k_2h)\left(\Gamma^{(1)}_{h,13} - \Gamma^{(2)}_{13}\right) - (\omega_1 - \omega_3)\left(k_2\tanh(k_2h)\omega_3 + \Gamma^{(3)}_{h,13}\right)\right\} \\
&\quad + \tanh(k_2h)\left(\Gamma^{(1)}_{h,31} - \Gamma^{(2)}_{31}\right) + (\omega_3 - \omega_1)\left(k_2\tanh(k_2h)\omega_1 - \Gamma^{(3)}_{h,31}\right) \\
&= \tanh(k_2h)\left(\frac{\omega_1\omega_3(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)}{k_2k_1k_3\tanh(k_1h)\tanh(k_3h)} - \frac{\omega_1\omega_3}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\right. \\
&\quad \left. - k_2\omega_2\omega_3 + \frac{\omega_2\omega_1\boldsymbol{k}_2\cdot\boldsymbol{k}_1}{k_1\tanh(k_1h)\tanh(k_2h)}\right) \\
&\quad + \tanh(k_2h)\left(-\frac{\omega_3\omega_1(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2k_3k_1\tanh(k_3h)\tanh(k_1h)} + \frac{\omega_3\omega_1}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)\right. \\
&\quad \left. + k_2\omega_1\omega_2 + \frac{\omega_2\omega_3\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3\tanh(k_3h)\tanh(k_2h)}\right).
\end{aligned} \tag{B14}
$$

Thus, the nonlinear coefficient γ_2^F reads

$$
\begin{aligned}
\gamma_2^F &= k_2\tanh(k_2h)\left(\frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)}\right. \\
&\quad \left. + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right).
\end{aligned} \tag{B15}
$$

Coefficient of $\mathrm{e}^{-\mathrm{i}\theta_3}$.
All contributing terms to the resonant condition $\theta_1 + \theta_2 = \theta_3$ arise from (B8) with $j = 1$, $\ell = 2$ as well as $j = 2$, $\ell = 1$ with $\boldsymbol{k} = -\boldsymbol{k}_3$. This leads to

$$
\begin{aligned}
&-\left\{\tanh(k_3h)\left(\Gamma^{(1)}_{h,12} + \Gamma^{(2)}_{12}\right) + (\omega_1 + \omega_2)\left(k_3\tanh(k_3h)\omega_2 + \Gamma^{(3)}_{h,12}\right)\right\} \\
&\quad -\left\{\tanh(k_3h)\left(\Gamma^{(1)}_{h,21} + \Gamma^{(2)}_{21}\right) + (\omega_2 + \omega_1)\left(k_3\tanh(k_3h)\omega_1 + \Gamma^{(3)}_{h,21}\right)\right\} \\
&= \tanh(k_3h)\left(\frac{\omega_1\omega_2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)}{k_3k_1k_2\tanh(k_1h)\tanh(k_2h)} + \frac{\omega_1\omega_2}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)\right. \\
&\quad \left. - k_3\omega_2\omega_3 + \frac{\omega_3\omega_1\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{k_1\tanh(k_1h)\tanh(k_3h)}\right) \\
&\quad + \tanh(k_3h)\left(\frac{\omega_2\omega_1(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)}{k_3k_2k_1\tanh(k_2h)\tanh(k_1h)} + \frac{\omega_2\omega_1}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)\right. \\
&\quad \left. - k_3\omega_1\omega_3 + \frac{\omega_3\omega_2\boldsymbol{k}_3\cdot\boldsymbol{k}_2}{k_2\tanh(k_2h)\tanh(k_3h)}\right).
\end{aligned} \tag{B16}
$$

To this end, the nonlinear coefficient γ_3^F is given by

$$\gamma_3^F = k_3 \tanh(k_3 h)\left(\frac{\omega_1\omega_2 \boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)} + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right). \tag{B17}$$

Coefficient of $\mathrm{e}^{\mathrm{i}\theta_1}$.
Here, we have contribution from resonant terms with $j=3$, $\ell=2$ coming from (B10) and $j=2$, $\ell=3$ arising from (B11). In both cases, $\boldsymbol{k}=\boldsymbol{k}_1$. Thus

$$\begin{aligned} &-\left\{\tanh(k_1h)\left(\Gamma^{(1)}_{h,32}-\Gamma^{(2)}_{32}\right)-(\omega_3-\omega_2)\left(k_1\tanh(k_1h)\omega_2+\Gamma^{(3)}_{h,32}\right)\right\}\\ &\quad+\tanh(k_1h)\left(\Gamma^{(1)}_{h,23}-\Gamma^{(2)}_{23}\right)+(\omega_2-\omega_3)\left(k_1\tanh(k_1h)\omega_3-\Gamma^{(3)}_{h,23}\right)\\ &=-\tanh(k_1h)\left(\frac{\omega_3\omega_2(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)}{k_1k_3k_2\tanh(k_3h)\tanh(k_2h)}-\frac{\omega_3\omega_2}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)\right.\\ &\quad\left.-k_1\omega_1\omega_2-\frac{\omega_1\omega_3\boldsymbol{k}_1\cdot\boldsymbol{k}_3}{k_3\tanh(k_3h)\tanh(k_1h)}\right)\\ &\quad+\tanh(k_1h)\left(\frac{\omega_2\omega_3(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)}{k_1k_2k_3\tanh(k_2h)\tanh(k_3h)}-\frac{\omega_2\omega_3}{k_1}(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)\right.\\ &\quad\left.-k_1\omega_1\omega_3+\frac{\omega_1\omega_2\boldsymbol{k}_1\cdot\boldsymbol{k}_2}{k_2\tanh(k_2h)\tanh(k_1h)}\right). \end{aligned} \tag{B18}$$

After some algebra, we get

$$\gamma_1^F = k_1 \tanh(k_1 h)\left(\frac{\omega_1\omega_2 \boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)} + \frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)} + \frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)} + \omega_1\omega_2 - \omega_1\omega_3 - \omega_2\omega_3\right). \tag{B19}$$

Coefficient of $\mathrm{e}^{\mathrm{i}\theta_2}$.
In this case, the only terms that contribute to resonance are those with $j=3$, $\ell=1$ (B10) and $j=1$, $\ell=3$ (B11). In addition, we find $\boldsymbol{k}=\boldsymbol{k}_2$ and

$$\begin{aligned} &-\left\{\tanh(k_2h)\left(\Gamma^{(1)}_{h,31}-\Gamma^{(2)}_{31}\right)-(\omega_3-\omega_1)\left(k_2\tanh(k_2h)\omega_1+\Gamma^{(3)}_{h,31}\right)\right\}\\ &\quad+\tanh(k_2h)\left(\Gamma^{(1)}_{h,13}-\Gamma^{(2)}_{13}\right)+(\omega_1-\omega_3)\left(k_2\tanh(k_2h)\omega_3-\Gamma^{(3)}_{h,13}\right)\\ &=\tanh(k_2h)\left(-\frac{\omega_3\omega_1(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)}{k_2k_3k_1\tanh(k_3h)\tanh(k_1h)}+\frac{\omega_3\omega_1}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)\right.\\ &\quad\left.+k_2\omega_1\omega_2+\frac{\omega_2\omega_3\boldsymbol{k}_2\cdot\boldsymbol{k}_3}{k_3\tanh(k_3h)\tanh(k_2h)}\right) \end{aligned}$$

$$+\tanh(k_2h)\left(\frac{\omega_1\omega_3(\boldsymbol{k}_1\cdot\boldsymbol{k}_3)(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)}{k_2k_1k_3\tanh(k_1h)\tanh(k_3h)}-\frac{\omega_1\omega_3}{k_2}(\boldsymbol{k}_2\cdot\boldsymbol{k}_3)\right.$$
$$\left.-k_2\omega_2\omega_3+\frac{\omega_2\omega_1\boldsymbol{k}_2\cdot\boldsymbol{k}_1}{k_1\tanh(k_1h)\tanh(k_2h)}\right). \tag{B20}$$

Thus, the nonlinear coefficient γ_2^F reads

$$\gamma_2^F=k_2\tanh(k_2h)\left(\frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)}+\frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)}\right.$$
$$\left.+\frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)}+\omega_1\omega_2-\omega_1\omega_3-\omega_2\omega_3\right). \tag{B21}$$

Coefficient of $\mathrm{e}^{\mathrm{i}\theta_3}$.
All contributing terms to the resonant condition $\theta_1+\theta_2=\theta_3$ arise from (B9) with $j=1$, $\ell=2$ as well as $j=2$, $\ell=1$ with $\boldsymbol{k}=\boldsymbol{k}_3$. This leads to

$$\tanh(k_3h)\left(\varGamma_{h,12}^{(1)}+\varGamma_{12}^{(2)}\right)-(\omega_1+\omega_2)\left(k_3\tanh(k_3h)\omega_2-\varGamma_{h,12}^{(3)}\right)$$
$$+\tanh(k_3h)\left(\varGamma_{h,21}^{(1)}+\varGamma_{21}^{(2)}\right)-(\omega_2+\omega_1)\left(k_3\tanh(k_3h)\omega_1-\varGamma_{h,21}^{(3)}\right)$$
$$=\tanh(k_3h)\left(\frac{\omega_1\omega_2(\boldsymbol{k}_1\cdot\boldsymbol{k}_2)(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)}{k_3k_1k_2\tanh(k_1h)\tanh(k_2h)}+\frac{\omega_1\omega_2}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_2)\right.$$
$$\left.-k_3\omega_2\omega_3+\frac{\omega_3\omega_1\boldsymbol{k}_3\cdot\boldsymbol{k}_1}{k_1\tanh(k_1h)\tanh(k_3h)}\right)$$
$$+\tanh(k_3h)\left(\frac{\omega_2\omega_1(\boldsymbol{k}_2\cdot\boldsymbol{k}_1)(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)}{k_3k_2k_1\tanh(k_2h)\tanh(k_1h)}+\frac{\omega_2\omega_1}{k_3}(\boldsymbol{k}_3\cdot\boldsymbol{k}_1)\right.$$
$$\left.-k_3\omega_1\omega_3+\frac{\omega_3\omega_2\boldsymbol{k}_3\cdot\boldsymbol{k}_2}{k_2\tanh(k_2h)\tanh(k_3h)}\right). \tag{B22}$$

To this end, the nonlinear coefficient γ_3^F is

$$\gamma_3^F=k_3\tanh(k_3h)\left(\frac{\omega_1\omega_2\boldsymbol{e}_1\cdot\boldsymbol{e}_2}{\tanh(k_1h)\tanh(k_2h)}+\frac{\omega_1\omega_3\boldsymbol{e}_1\cdot\boldsymbol{e}_3}{\tanh(k_1h)\tanh(k_3h)}\right.$$
$$\left.+\frac{\omega_2\omega_3\boldsymbol{e}_2\cdot\boldsymbol{e}_3}{\tanh(k_2h)\tanh(k_3h)}+\omega_1\omega_2-\omega_1\omega_3-\omega_2\omega_3\right). \tag{B23}$$

References

Ablowitz, M.J. 2011 *Nonlinear Dispersive Waves: Asymptotic Analysis And Solitons*. Cambridge University Press.

Ablowitz, M.J. & Clarkson, P.A. 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, vol. 149. Cambridge University Press.

Ablowitz, M.J., Fokas, A.S. & Musslimani, Z.H. 2006 On a new non-local formulation of water waves. *J. Fluid Mech.* **562**, 313–343.

Ablowitz, M.J. & Haberman, R. 1975*a* Nonlinear evolution equations–two and three dimensions. *Phys. Rev. Lett.* **35** (18), 1185.

Ablowitz, M.J. & Haberman, R. 1975*b* Resonantly coupled nonlinear evolution equations. *J. Math. Phys.* **16** (11), 2301–2305.

Ablowitz, M.J. & Haut Terry, S. 2009 Coupled nonlinear Schrödinger equations from interfacial fluids with a free surface. *Theor. Math. Phys.* **159**, 689–697.
Ablowitz, M.J., Luo, X.-D. & Musslimani, Z.H. 2023 Three-wave interaction equations: classical and nonlocal. *SIAM J. Math. Anal.* (in press).
Ablowitz, M.J. & Musslimani, Z.H. 2013 Integrable nonlocal nonlinear Schrödinger equation. *Phys. Rev. Lett.* **110** (6), 064105.
Ablowitz, M.J. & Musslimani, Z.H. 2016 Inverse scattering transform for the integrable nonlocal nonlinear Schrödinger equation. *Nonlinearity* **29** (3), 915.
Ablowitz, M.J. & Musslimani, Z.H. 2017 Integrable nonlocal nonlinear equations. *Stud. Appl. Maths* **139**, 7–59.
Ablowitz, M.J. & Musslimani, Z.H. 2019 Integrable nonlocal asymptotic reductions of physically significant nonlinear equations. *J. Phys.* A **52** (15), 15LT02.
Ablowitz, M.J. & Musslimani, Z.H. 2021 Integrable space–time shifted nonlocal nonlinear equations. *Phys. Lett.* A **409**, 127516.
Ablowitz, M.J. & Segur, H. 1979 On the evolution of packets of water waves. *J. Fluid Mech.* **92** (4), 691–715.
Ablowitz, M.J. & Segur, H. 1981 *Solitons and the Inverse Scattering Transform*. SIAM.
Benjamin, T.B. & Feir, J.E. 1967 The disintegration of wave trains on deep water. Part 1. Theory. *J. Fluid Mech.* **27** (3), 417–430.
Benney, D.J. 1962 Non-linear gravity wave interactions. *J. Fluid Mech.* **14** (4), 577–584.
Benney, D.J. & Newell, A.C. 1967 The propagation of nonlinear envelopes. *J. Math. Phys.* **46**, 133–139.
Bretherton, F.P. 1964 Resonant interactions between waves, the case of discrete oscillations. *J. Fluid Mech.* **20** (3), 457–479.
Buryak, A.V., Di Trapani, P., Skryabin, D.V. & Trillo, S. 2002 Optical solitons due to quadratic nonlinearities: from basic physics to futuristic applications. *Phys. Rep.* **370** (2), 63–235.
Case, K.M. & Chiu, S.C. 1977 Three-wave resonant interactions of gravity-capillary waves. *Phys. Fluids* **20** (5), 742–745.
Craik, A.D.D. 1988 *Wave Interactions and Fluid Flows*. Cambridge University Press.
Dyachenko, A.I., Dyachenko, S.A., Lushnikov, P.M. & Zakharov, V.E. 2021 Short branch cut approximation in two-dimensional hydrodynamics with free surface. *Proc. R. Soc.* A **477** (2249), 20200811.
Dyachenko, A.I., Korotkevich, A.O. & Zakharov, V.E. 2003 Decay of the monochromatic capillary wave. *J. Expl Theor. Phys. Lett.* **77** (9), 477–481.
Dyachenko, S.A., Lushnikov, P.M. & Korotkevich, A.O. 2016 Branch cuts of stokes wave on deep water. Part I: numerical solution and padé approximation. *Stud. Appl. Maths* **137** (4), 419–472.
Dyachenko, A.I., Zakharov, V.E. & Kuznetsov, E.A. 1996 Nonlinear dynamics of the free surface of an ideal fluid. *Plasma Phys. Rep.* **22** (10), 829–840.
Falcon, E., Laroche, C. & Fauve, S. 2002 Observation of depression solitary surface waves on a thin fluid layer. *Phys. Rev. Lett.* **89** (20), 204501.
Fokas, A.S. 2006 Integrable nonlinear evolution partial differential equations in 4+ 2 and 3+ 1 dimensions. *Phys. Rev. Lett.* **96** (19), 190201.
Gardner, C.S., Greene, J.M., Kruskal, M.D. & Miura, R.M. 1967 Method for solving the Korteweg–Devries equation. *Phys. Rev. Lett.* **19** (19), 1095.
Gibbons, G.W., Page, D.N. & Pope, C.N. 1990 Einstein metrics on S3, R3 and R4 bundles. *Commun. Math. Phys.* **127** (3), 529–553.
Hammack, J.L. & Henderson, D.M. 1993 Resonant interactions among surface water waves. *Annu. Rev. Fluid Mech.* **25** (1), 55–97.
Hasselmann, K. 1967 Nonlinear interactions treated by the methods of theoretical physics (with application to the generation of waves by wind). *Proc. R. Soc. Lond.* A **299** (1456), 77–103.
Henderson, D.M. & Hammack, J.L. 1987 Experiments on ripple instabilities. Part 1. Resonant triads. *J. Fluid Mech.* **184**, 15–41.
Kadomtsev, B.B. & Petviashvili, V.I. 1970 On the stability of solitary waves in weakly dispersing media. In *Doklady Akademii Nauk*, vol. 192, pp. 753–756. Russian Academy of Sciences.
Kaup, D.J. 1976 The three-wave interaction? A nondispersive phenomenon. *Stud. Appl. Maths* **55** (1), 9–44.
Kaup, D.J. 1981 The solution of the general initial value problem for the full three dimensional three-wave resonant interaction. *Physica* D **3** (1–2), 374–395.
Korotkevich, A.O., Dyachenko, A.I. & Zakharov, V.E. 2016 Numerical simulation of surface waves instability on a homogeneous grid. *Physica* D **321**, 51–66.
Lushnikov, P.M. 2016 Structure and location of branch point singularities for stokes waves on deep water. *J. Fluid Mech.* **800**, 557–594.

MASON, L.J. & WOODHOUSE, N.M.J. 1996 *Integrability, Self-Duality, and Twistor Theory*. Oxford University Press.

MCGOLDRICK, L.F. 1965 Resonant interactions among capillary-gravity waves. *J. Fluid Mech.* **21** (2), 305–331.

MCGOLDRICK, L.F. 1970 An experiment on second-order capillary gravity resonant wave interactions. *J. Fluid Mech.* **40** (2), 251–271.

NOVIKOV, S., MANAKOV, S.V., PITAEVSKII, L.P. & ZAKHAROV, V.E. 1984 *Theory of Solitons: The Inverse Scattering Method*. Springer Science & Business Media.

PERLIN, M., HENDERSON, D. & HAMMACK, J. 1990 Experiments on ripple instabilities. Part 2. Selective amplification of resonant triads. *J. Fluid Mech.* **219**, 51–80.

PHILLIPS, O.M. 1960 On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions. *J. Fluid Mech.* **9** (2), 193–217.

PHILLIPS, O.M. 1966 *The Dynamics of the Upper Ocean*. Cambridge University Press.

SIMMONS, W.F. 1969 A variational method for weak resonant wave interactions. *Proc. R. Soc. Lond.* A **309** (1499), 551–577.

STOKES, G.G. 1847 On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441–473.

WARD, R.S. 1977 On self-dual gauge fields. *Phys. Lett.* A **61** (2), 81–82.

WHITHAM, G.B. 1974 *Linear and Nonlinear Waves*. John Wiley & Sons.

YANG, B. & YANG, J. 2021 General rogue waves in the three-wave resonant interaction systems. *IMA J. Appl. Maths* **86** (2), 378–425.

YANG, B. & YANG, J. 2022 Rogue waves in (2+ 1)-dimensional three-wave resonant interactions. *Physica* D **432**, 133160.

ZAKHAROV, V.E. 1968 Stability of periodic waves of finite amplitude on the surface of a deep fluid. *J. Appl. Mech. Tech. Phys.* **9** (2), 190–194.

ZAKHAROV, V.E. 1998 Weakly nonlinear waves on the surface of an ideal finite depth fluid. *Am. Math. Soc. Transl.*, 167–197.

ZAKHAROV, V.E. & MANAKOV, S.V. 1975 The theory of resonance interaction of wave packets in nonlinear media. *Zh. Eksp. Teor. Fiz.* **69**, 1654–1673.

ZAKHAROV, V.E. & SHABAT, A. 1972 Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**, 62–69.