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PREFRATTINI GROUPS

PETER FÖRSTER

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Abstract

We define and investigate $\mathfrak{H}$-prefrattini subgroups for Schunck classes $\mathfrak{H}$ of finite soluble groups, and solve a problem of Gaschütz concerning the structure of $\mathfrak{H}$-prefrattini groups for $\mathfrak{H} = \{1\}$.

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Introduction

In the theory of saturated formations $\mathfrak{F}$ of finite soluble groups, one associates with each finite soluble group G three conjugacy classes of subgroups of G, namely $\mathfrak{F}$-projectors, $\mathfrak{F}$-normalizers and $\mathfrak{F}$-prefrattini subgroups. (The latter were introduced by Hawkes in [12].) Schunck (in [15]) characterized those classes $\mathfrak{H}$ of finite soluble groups such that every finite soluble group possesses $\mathfrak{H}$-projectors, and found that for those classes (which are now known as Schunck classes) the $\mathfrak{H}$-projectors always form a class of conjugate subgroups. Meanwhile several investigations of Schunck classes and their projectors have been carried out, while $\mathfrak{H}$-normalizers and $\mathfrak{H}$-prefrattini subgroups have not even been defined (except that unpublished work of Schaller [14] contains a definition of $\mathfrak{H}$-prefrattini subgroups). Therefore, in the present paper, we define $\mathfrak{H}$-prefrattini subgroups of a finite soluble group (see Section 2), and then go on to prove some of their elementary properties (Sections 2, 3). Our approach here differs from the one of Hawkes as well as from Schaller's approach: in fact, our definition is a simple generalization of the one Gaschütz gave in [10], and is based on Gaschütz's notion of a crown (see Section 1), which turned out to be of fundamental importance in the theory of Schunck classes (see Baer, Förster [1], Chapter 3).

The next part of this paper consists of an answer to Gaschütz's question (see [10]) whether the structure of those groups that occur as prefrattini subgroups in finite soluble groups is restricted: we can show that every finite soluble group is a quotient of a prefrattini group; nevertheless, there are many examples of groups which are not themselves prefrattini groups.

The final section of this note reveals a property of $\mathcal{H}$-prefrattini subgroups (as proved by Chambers [16] for saturated formations $\mathcal{H}$) as a more general property of so-called weakly system permutable subgroups H of finite soluble groups G: we show that H permutes with suitable conjugates of arbitrary normally embedded subgroups of G. ($\mathcal{H}$-prefrattini subgroups H for Schunck classes $\mathcal{H}$ are shown to enjoy this property; see Section 3.)

All groups considered here are assumed to be finite and soluble.

1. Gaschütz's concept of a crown

A Schunck class $\mathcal{H}$ is, by definition, a class of finite soluble groups satisfying

$$\mathcal{H} = Q\mathcal{H}\,(= \{G/N \mid G \in \mathcal{H}, N \trianglelefteq G\}), \quad \text{and}$$
$$G \in \mathfrak{S}, \quad \mathrm{Pr}(G) \subseteq \mathcal{H} \Rightarrow G \in \mathcal{H};$$

here $\mathfrak{S}$ denotes the class of all (finite soluble) groups and $\mathrm{Pr}(G)$ is the class of all primitive factor groups of G. For basic facts concerning Schunck classes the reader is referred to Schunck's paper [15] as well as to 1.2 of [5] and 1.1, 1.2 of [4]. Note that in these papers Schunck classes are termed "gesättigte Homomorphe" (saturated homomorphs).

Related to the notion of a primitive group (that is, due to the hypothesis of solubility of all groups considered here, a group with unique minimal normal subgroup which, in addition, is complemented by a maximal subgroup) there is Gaschütz's concept of a crown as introduced in [10], which describes the totality of primitive factor groups of a given type in a group. For a non-equivalent definition of a crown see Baer, Förster [1] (Chapter 3). The definition given there was made to work for non-soluble (finite) groups too. Our definition of $\mathcal{H}$-prefrattini subgroups (see 2.1 below) could have been made with this more general concept of a crown as well, the proofs of their properties, however, would have become less elegant.

1.1 Definition. Let H/K be a complemented chief factor of a group G. Put $C = \mathbf{C}_G(H/K)$ and $R = \bigcap\{T \trianglelefteq G \mid T \leqslant C, C/T \cong_G H/K\}$. Then C/R is called *the crown of G associated with H/K*.

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Clearly, C/R is non-trivial, as is seen by considering $T = \mathbf{core}_G(U)$ for some complement U in G of H/K.

Combining the results of Gaschütz's paper [10] with a characterization of pronormality (see [13]), one obtains:

1.2 LEMMA. *Let notation be as in* 1.1 *and denote by p the prime divisor of H/K. Then the following holds.*

(a) *C/R is a completely reducible $GF(p)[G]$-module with all its composition factors isomorphic to H/K.*

(b) $\mathbf{F}(G/R) = \mathbf{O}_p(G/R) = \mathbf{S}(G/R) = C/R$, $\Phi(G/R) = \mathbf{O}_{p'}(G/R) = 1$.

(c) *There exists $U \leqslant G$ such that the set of all complements in G/R of C/R is precisely the set of all G-conjugates of U/R. Moreover, to each Hall system $\Sigma = \{G_\pi \mid \pi \subseteq \pi(G)\}$, there corresponds exactly one U^x into which Σ reduces.*

(d) *If $(S/R)(C/R) = G/R$, then S/R has the form $(U^x/R)(M_1/R \times \cdots \times M_s/R)$ with $x \in G$ and M_i/R minimal normal in G/R $(i = 1, \ldots, s)$.*

(e) *If $H_0/K_0 \cong_G H/K$ is a complemented chief factor of G, then $H_0R/K_0R \cong_G H/K$ is another chief factor of G and is situated below C (and above R); furthermore, C/R is the crown associated with H_0/K_0 (and with H_0R/K_0R).*

NOTATIONS. $CR(G)$ denotes the set of all crowns of G. For a given Hall system $\Sigma = \{G_\pi \mid \pi \subseteq \pi(G)\}$, $CC(G, \Sigma)$ denotes the set $\{U \leqslant G \mid G = UC$ and $U \cap C = R$ for some $C/R \in CR(G)$, Σ reduces into $U\}$. (Note that by 1.2(c), for each $C/R \in CR(G)$, there is precisely one complement in $CC(G, \Sigma)$; in turn, each $U \in CC(G, \Sigma)$ complements precisely one crown, namely C/R where $R = \mathbf{core}_G(U)$ and $C = \mathbf{F}(G \bmod R)$.)

1.3 LEMMA. *If $U \in CC(G, \Sigma)$ complements $C/R \in CR(G)$, then $U = \bigcap\{S \leqslant G \mid R \leqslant S, G = SC, S$ maximal in G, Σ reduces into $S\}$.*

In addition, for a given chief series of G, $1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$ say, $U = \bigcap_{i=1}^n S_i$ with

$$S_i = \begin{cases} G, \text{ if } C/R \text{ is not the crown associated with } G_i/G_{i-1} \text{ (including the case} \\ \qquad \text{that } G_i/G_{i-1} \text{ is a Frattini factor),} \\ \text{an arbitrary maximal subgroup of } G \text{ subject only to the conditions (i) + (ii)} \\ \qquad \text{below, if } G_i/G_{i-1} \text{ is a complemented chief factor of } G \text{ with } C/R \\ \qquad \text{being the associated crown.} \end{cases}$$

(i) *Σ reduces into S_i;*

(ii) $\mathbf{F}(G/\mathbf{core}_G(S_i)) = G_i\mathbf{core}_G(S_i)/\mathbf{core}_G(S_i)$ *and* $G_{i-1} \leqslant \mathbf{core}_G(S_i)$.

PROOF. $U = \bigcap \{S \leqslant G \mid R \leqslant S,\ G = SC,\ S$ maximal in G, Σ reduces into $S\}$ is an obvious consequence of 1.2(a)–(d).

In order to verify the additional assertion, let the set $\{X_1/Y_1, \dots, X_m/Y_m\} = \{G_i/G_{i-1} \mid i \in \{1, \dots, n\},\ G_i/G_{i-1}$ is complemented in G with associated crown $C/R\}$ be ordered such that $X_i \leqslant Y_{i+1}$ for $i = 1, \dots, m-1$. Then by 1.2(e), we have $R \leqslant Y_1R < X_1R \leqslant Y_2R < X_2R \leqslant \cdots < X_mR \leqslant C$ with complemented chief factors X_iR/Y_iR $(i = 1, \dots, m)$; again, C/R is the crown corresponding to X_iR/Y_iR. Now, on the one hand, the generalized Jordan-Hölder-Theorem ([2], 2.6) yields that m is the number of complemented chief factors of G (in a given yet arbitrary chief series of G) with C/R as the associated crown. On the other hand, we might refine the above series $R \leqslant Y_1R < X_1R \leqslant \cdots < X_mR \leqslant C$ (namely, in case that $R < Y_1R$ or $X_iR < Y_{i+1}R$ or $X_mR < C$), eventually obtaining new chief factors (besides those m factors of the form X_iR/Y_iR) with the same associated crown (see 1.2(a)). This gives

$$R = Y_1R < X_1R = Y_2R < X_2R = \cdots < X_mR = C.$$

Finally, let S_i^* be any maximal subgroup of G complementing X_i/Y_i (and thus X_iR/Y_iR too, as follows from $R \leqslant \mathbf{core}_G(S_i^*)$, the latter being a consequence of $\mathbf{C}_G(X_i/Y_i)/\mathbf{core}_G(S_i^*) = X_i\mathbf{core}_G(S_i^*)/\mathbf{core}_G(S_i^*) \cong_G X_i/X_i \cap \mathbf{core}_G(S_k^*) = X_i/Y_i$) into which Σ reduces. By 1.2(c),(d), U is contained in S_i^*, whence $S_i^* = UC \cap S_i^* = U(C \cap S_i^*)$. Further, $C \cap S_m^* = RX_m \cap S_m^* = R(X_m \cap S_m^*) = RY_m = RX_{m-1}$, so that $S_m^* \cap S_{m-1}^* = URX_{m-1} \cap S_{m-1}^* = UX_{m-1} \cap S_{m-1}^* = U(X_{m-1} \cap S_{m-1}^*) = UY_{m-1}$. A trivial induction argument may now be applied to derive the equation $\bigcap_{i=j}^m S_j^* = UY_j$ $(= UX_{j-1}$ for $j \geqslant 2)$. We obtain $\bigcap_{i=1}^m S_i^* = UY_1 = UR = U$, thus verifying our assertion that $U = \bigcap_{i=1}^n S_i$ whenever $\{S_1, \dots, S_n\}$ is as described in the statement of our lemma.

2. $\mathcal{H}$-prefrattini subgroups

2.1 DEFINITION. Let $\mathcal{H}$ be a Schunck class and consider a Hall system Σ of a group G. Recall that a complemented chief factor of G, H/K say, is called *$\mathcal{H}$-eccentric* provided that the corresponding primitive factor group $G/\mathbf{core}_G(U) \cong (G/\mathbf{C}_G(H/K))(H/K)$ (where U complements H/K in G) is not contained in $\mathcal{H}$; H/K is called *$\mathcal{H}$-central* otherwise. We extend the notion of $\mathcal{H}$-eccentricity (and of $\mathcal{H}$-centrality) to factors $C/R \in CR(G)$ to mean that H/K is $\mathcal{H}$-eccentric ($\mathcal{H}$-central) for some (and then, by 1.2(a), (e), each) complemented chief factor H/K the associated crown of which is C/R. We define the *$\mathcal{H}$-prefrattini subgroup of G corresponding to* Σ as

$$W(G, \Sigma, \mathcal{H}) = \bigcap \{U \in CC(G, \Sigma) \mid C/R \in CR(G)$$
$$\text{is the crown complemented by } U \Rightarrow C/R \text{ is } \mathcal{H}\text{-eccentric}\}.$$

2.2 Theorem. *Let $\mathfrak{H}$ be a Schunck class and G a group with Hall system Σ_0, and consider a chief series $1 = G_0 \leqslant G_1 \leqslant \cdots \leqslant G_n = G$.*

(a) *$W(G, \Sigma_0, \mathfrak{H}) = \bigcap_{i=1}^n S_i$, where S_i is a complement in G of G_i/G_{i-1} into which Σ_0 reduces, provided that G_i/G_{i-1} is a complemented $\mathfrak{H}$-eccentric chief factor of G, and $S_i = G$ otherwise.*

(b) *$W(G/N, \Sigma_0 N/N, \mathfrak{H}) = W(G, \Sigma_0, \mathfrak{H})N/N$ whenever $N \trianglelefteq G$.*

(c) *The set $\{W(G, \Sigma, \mathfrak{H}) \mid \Sigma$ Hall system of $G\}$ is a characteristic class of conjugate cover-avoidance subgroups of G. More precisely, a chief factor of G is avoided if and only if it is a complemented $\mathfrak{H}$-eccentric factor.*

Proof. (a) is an immediate consequence of 2.1 and 1.3.

(b) follows from (a), as a chief series with $G_i = N$ for some $i \in \{1, \dots, n\}$ might be considered.

(c) Conjugacy of all Hall systems of G yields conjugacy of all $\mathfrak{H}$-prefrattini subgroups, when the assertion in 1.2(c) is taken into account. The set of all $W(G, \Sigma, \mathfrak{H})$ is invariant under automorphisms of G, since $\varphi \in \mathbf{Aut}(G)$ maps $C/R \in CR(G)$ onto $C^\varphi/R^\varphi \in CR(G)$ and $G/R \cong G/R^\varphi$. By (a), $W = W(G, \Sigma, \mathfrak{H})$ avoids each $\mathfrak{H}$-eccentric chief factor of G. In order to show that Frattini factors and $\mathfrak{H}$-central factors are covered, we may restrict ourselves to the case of a minimal normal subgroup M, as is shown by (b). Write $W = \bigcap_{i=1}^n S_i$ as in (a). For a Frattini factor M we have $M \leqslant \Phi(G) \leqslant S_i$ $(i = 1, \dots, n)$, hence $M \leqslant W$. For a complemented factor M we have $M \leqslant \mathbf{core}_G(S_i) = C_i \leqslant S_i$ whenever $S_i \neq G$, as MC_i/C_i cannot possibly be the $\mathfrak{H}$-eccentric factor $\mathbf{F}(G/C_i)$, because here M is $\mathfrak{H}$-central. Again it turns out that $M \leqslant W$.

2.3 Corollary. *If S is a maximal subgroup of G into which Σ reduces, then the following statements are equivalent*:

(i) $W(G, \Sigma, \mathfrak{H}) \leqslant S$;

(ii) *S is $\mathfrak{H}$-abnormal (that is to say, $\mathbf{F}(G/\mathbf{core}_G(S))$ is $\mathfrak{H}$-eccentric).*

(Actually, (i) and (ii) are equivalent provided only that $G_{p'} \leqslant S$, where p is the prime dividing $|G : S|$.)

2.4 Example. If $\mathfrak{H}$ happens to be a saturated formation, then $W(G, \Sigma, \mathfrak{H})$ is, of course, the $\mathfrak{H}$-prefrattini subgroup of G corresponding to Σ in the sense of Hawkes ([12], Section 3); in particular, putting $\mathfrak{H} = \{1\}$, one obtains Gaschütz's prefrattini subgroups (see [10], Section 6).

2.5 THEOREM. *$W \leqslant G$ is an $\mathfrak{H}$-prefrattini subgroup of G ($\mathfrak{H}$ an arbitrary Schunck class) if and only if W satisfies the following two conditions:*

(i) *W avoids all $\mathfrak{H}$-eccentric complemented chief factors of G and covers the remaining factors, and*

(ii) *W is weakly system permutable in G ("schwach vertauschbar" in the sense of* [9], 5.1a); *in fact, W permutes with the elements of Σ whenever $W = W(G, \Sigma, \mathfrak{H})$.*

PROOF. We give a proof partially different from Gillam's proof of his theorem on $\mathfrak{H}$-prefrattini subgroups for saturated formations $\mathfrak{F}$ (see [11], 3.2).

First, let $W = W(G, \Sigma, \mathfrak{H})$. Then (i) holds by 2.2(c). Condition (ii) is proved by induction on $|G|$. Therefore the assumption

$$N \trianglelefteq G, \quad N \leqslant W \Rightarrow N = 1$$

is justified. In particular, $\Phi(G)$ is trivial, and there is a minimal normal subgroup of G, M say, which is avoided by W. Let S be a complement in G of M such that Σ reduces into S. 2.3 together with $S \cong G/M$ and 2.2 implies $W = W(S, \Sigma \cap S, \mathfrak{H})$, and W permutes with the elements of $\Sigma \cap S$. Since each $H \in \Sigma$ either is an element of $\Sigma \cap S$ or may be written as $H = (H \cap S)M$, we are done.

For the converse implication the reader is referred to Gillam's [11].

The author's Proposition 6.5 of [8] on $\mathfrak{H}$-normalizers has an analogue for $\mathfrak{H}$-prefrattini subgroups.

2.6 THEOREM. *The following statements are equivalent in pairs.*

(i) *For every $G \in \mathfrak{S}$, $\mathfrak{H}$-prefrattini subgroups are pronormal subgroups of G.*

(ii) *For every $G \in \mathfrak{S}$, $\mathfrak{H}$-prefrattini subgroups are (strongly) system permutable ("(stark) vertauschbar" in the sense of* [8], 5.1).

(iii) $\mathfrak{H} = \mathfrak{S}$.

PROOF. (i) $\Rightarrow$ (ii) is a consequence of [8], 6.4, together with 2.5.

(ii) $\Rightarrow$ (iii): This follows from [8], 6.3, together with the trivial fact that in groups contained in $E_p^n(\mathfrak{H})$ (where $n \in \mathbf{N}$ and $E_p^n(\mathfrak{H})$ is defined as in [7], 2.15) $\mathfrak{H}$-normalizers (see [7], Section 2) and $\mathfrak{H}$-prefrattini subgroups coincide; indeed, if $G \in E_p^n(\mathfrak{H}) \setminus E_p^{n-1}(\mathfrak{H})$ ($n \in \mathbf{N}$), then $\mathbf{F}(G)$ is a crown with complement contained in $E_p^{n-1}(\mathfrak{H}) \setminus E_p^{n-2}(\mathfrak{H})$ when $n \geqslant 2$ or contained in $E_p^0(\mathfrak{H}) = \mathfrak{H}$ when $n = 1$. These considerations can be used to show that $\mathfrak{H} = \mathfrak{P}_\pi$ (π a set of primes), the class of all groups with no non-trivial π-factor group, whenever $\mathfrak{H}$ satisfies (ii). The case $\pi \neq \varnothing$ can be excluded as follows. Let $p \in \pi$ and q be distinct primes, V a faithful irreducible module over $GF(q)$ for the cyclic group C of order p^2. Consider the semidirect product S of CV and a faithful irreducible $GF(p)[CV]$-module W. Then $\Phi(C)$ is a $\mathfrak{P}_\pi$-prefrattini subgroup of S which is not system permutable: otherwise, since for any $x \in S$ the set $\{1, CW, V^x, S\}$ constitutes a Hall system of S, we should have $\Phi(C)V^x = V^x\Phi(C) \leqslant S$ for arbitrary $x \in S$; clearly, this is not the case.

(iii) $\Rightarrow$ (i) is trivial.

3. A lattice of cover-avoidance subgroups

The concept of an $\mathfrak{H}$-prefrattini subgroup, as developed here, really depends on the notion of a crown corresponding to a set of primitive groups rather than to a Schunck class. One checks easily that generalized $\mathfrak{X}$-prefrattini subgroups (as defined below for arbitrary subclasses $\mathfrak{X}$ of the class of all primitive groups $\mathfrak{P}$) enjoy exactly the same properties (summarized in 2.2 and 2.5) as $\mathfrak{H}$-prefrattini subgroups for Schunck classes $\mathfrak{H}$ do.

3.1 Definitions. Let G be a group with Hall system Σ, and let $\mathfrak{X} \subseteq \mathfrak{P}$.

(a) A complemented chief factor of G is called $\mathfrak{X}$-quasi-eccentric whenever $G/\mathbf{core}_G(U) \in \mathfrak{X}$ for some (equivalently: each) of its complements U. Otherwise the complemented chief factor is called $\mathfrak{X}$-quasi-central. These notions are defined for crowns instead of chief factors in the obvious way.

(b) The generalized $\mathfrak{X}$-prefrattini subgroup of G corresponding to Σ is defined to be

$$M(G, \Sigma, \mathfrak{X}) = \bigcap \{U \in CC(G, \Sigma) \mid U \text{ complements an } \mathfrak{X}\text{-quasi-eccentric crown of } G\}.$$

3.2 Remark. Given a Schunck class $\mathfrak{H}$, put $X(\mathfrak{H}) = \mathfrak{P} \setminus \Pr(\mathfrak{H})$, where $\Pr(\mathfrak{H}) = \mathfrak{H} \cap \mathfrak{P}$. Then we have

$X(\mathfrak{H})$-quasi-eccentric = $\mathfrak{H}$-eccentric,
$X(\mathfrak{H})$-quasi-central = $\mathfrak{H}$-central, and
$M(G, \Sigma, X(\mathfrak{H})) = W(G, \Sigma, \mathfrak{H})$.

3.3 Theorem. *Let G, Σ, $\mathfrak{X}$ be as in* 3.1 *and define $M(G, \Sigma) = \{M(G, \Sigma, \mathfrak{X}) \mid \mathfrak{X} \subseteq \mathfrak{P}\}$. Then $M(G, \Sigma) = \{M(G, \Sigma, \mathfrak{X}) \mid \mathfrak{X} \subseteq \Pr(G)\}$ is a lattice of pairwise permutable subgroups of G, with intersection as a meet operation and (permutable) product as a join. More precisely, we have*

$$M(G, \Sigma, \{P_1, \ldots, P_r\}) \cap M(G, \Sigma, \{P_{r+1}, \ldots, P_s\}) = M(G, \Sigma, \{P_1, \ldots, P_s\}), \quad \text{and}$$

$$M(G, \Sigma, \{P_1, \ldots, P_r\}) M(G, \Sigma, \{P_{r+1}, \ldots, P_s\}) = M(G, \Sigma, \{P_1, \ldots, P_r\} \cap \{P_{r+1}, \ldots, P_s\}).$$

The proof of this theorem is omitted, because it is an easy consequence of $M(G, \Sigma, \mathfrak{X}) \subseteq M(G, \Sigma, \mathfrak{Y})$ whenever $\mathfrak{Y} \subseteq \mathfrak{X} \subseteq \mathfrak{P}$ together with the following elementary lemma.

3.4 LEMMA. *Let V be a cover-avoidance subgroup of G, and let U be a subset of V such that U covers each chief factor of G covered by V. Then $U = V$.*

PROOF. By induction on $|G|$, it is easy to derive $|U| = |V|$.

3.5 COROLLARY. *Let $W(G, \Sigma) = \{W(G, \Sigma, \mathfrak{H}) \mid \mathfrak{H}$ Schunck class$\}$. Then $W(G, \Sigma)$ is a sublattice of $M(G, \Sigma)$.*

It is clear that a similar statement does not hold for saturated formations in place of Schunck classes.

PROOF. Using the notation introduced in 3.2, it is clear that

$$X(\mathfrak{H}_1) \cup X(\mathfrak{H}_2) = X(\mathfrak{H}_1 \cap \mathfrak{H}_2) \quad \text{and} \quad X(\mathfrak{H}_1) \cap X(\mathfrak{H}_2) = X(\widehat{\mathfrak{H}_1 \cup \mathfrak{H}_2});$$

here $\mathfrak{H}_1$ and $\mathfrak{H}_2$ denote Schunck classes and $\hat{\mathfrak{H}}$ stands for the unique minimal Schunck class containing the homomorph $\mathfrak{H}$, that is to say, $\hat{\mathfrak{H}} = \{G \in \mathfrak{S} \mid \Pr(G) \subseteq \mathfrak{H}\}$. Combining this with 3.3, a proof of the theorem can be established.

We conclude our investigations of generalized prefrattini subgroups by stating a criterion for prefrattini subgroups which is independent of the choice of a Hall system. A proof may be extracted from Gaschütz's [10], Section 6.

3.6 PROPOSITION. *$M \in \bigcup\{M(G, \Sigma, \{P_1, \ldots, P_n\}) \mid \Sigma$ Hall system of $G\}$ ($P_1, \ldots, P_n$ are primitive groups) if and only if the following two conditions hold.*

(i) *M covers each Frattini factor and each complemented $\{P_1, \ldots, P_n\}$-quasi-central chief factor of G;*

(ii) *for each complemented chief factor of G, H/K say, with $(G/\mathbf{C}_G(H/K))(H/K) \cong P_i$ for some $i \in \{1, \ldots, n\}$, and for each complement U of H/K, there is an $x \in G$ such that $M \leqslant U^x$.*

4. On a problem of Gaschütz

Gaschütz [10] asked whether there are any restrictions for the structure of prefrattini subgroups (that is, $\mathfrak{H}$-prefrattini subgroups for $\mathfrak{H} = \{1\}$). This question was modified by Doerk [3]: is there a formation $\mathfrak{F} \neq \mathfrak{S}$ such that the class $\mathfrak{P}_\Phi = \{P \in \mathfrak{S} \mid P$ is a prefrattini subgroup of some $G \in \mathfrak{S}\}$ is a subclass of $\mathfrak{F}$? We answer these questions by showing that even $Q\mathfrak{P}_\Phi = \mathfrak{S}$ holds.

4.1 EXAMPLE. Let p be a prime, let $n \in \mathbf{N}$, and suppose that $p^n \neq 2$. Take a maximal soluble subgroup H ($\neq 1$, as $p^n \neq 2$) of $GL(n, p)$ acting irreducibly upon the n-dimensional $GF(p)$-space V. (Such subgroups exist in each $GL(n, p)$ as is seen by considering, for example, the automorphism group which the semilinear group of order $p^n(p^n - 1)n$ induces in its minimal normal subgroup of order p^n.) Then the semidirect product $S = HV$ is not contained in $\mathcal{P}_\Phi$.

Indeed, if S is a prefrattini subgroup of G, then G possesses a Frattini p-chief factor M/N such that the chief factor V of S appears (up to isomorphism) in M/N. Clearly, we may assume without loss of generality that $N = 1$, and then we obtain $M \leqslant \mathbf{O}_p(S)$, whence M coincides with the minimal normal subgroup $V = \mathbf{O}_p(S)$ of S. We infer that $H \cong \mathbf{Aut}_S(V) \leqslant \mathbf{Aut}_G(V) \cong G/\mathbf{C}_G(V)$ and, as the latter is a soluble group, $H \cong G/\mathbf{C}_G(V)$ by choice of H. This forces that $G = H\mathbf{C}_G(V) \neq \mathbf{C}_G(V)$, which is impossible for a subgroup $H \leqslant S$ of G that covers only Frattini factors.

In contrast to this example we have the following solution of Gaschütz's problem:

4.2 THEOREM. $Q\mathcal{P}_\Phi = \mathcal{S}$.

PROOF. Put $\mathcal{H} = Q\mathcal{P}_\Phi$.

(1) $H \in \mathcal{P}_\Phi$ implies $HGF(p)[H] \in \mathcal{H}$ for each prime p:

Suppose that H is a prefrattini subgroup of G. Let G act on a p-group P such that $P/\Phi(P) \cong_G V_1 \oplus V_2$, $\Phi(P) = P' = \mathbf{Z}(P) \cong_G V_1 \otimes V_2 \cong_G V_1$ where $V_1 = GF(p)[G]$ and $V_2 = 1_G^p$ is the trivial irreducible $GF(p)[G]$-module; for existence of such a group P and its properties see 1.1 of [7]. If $S = GP$ denotes the semidirect product, then the structure of P gives the split extension

$$H_0 = H(\Phi(S) \cap P) = H(\mathrm{Rad}(GF(p)[G]) \oplus GF(p)[G])$$

as a prefrattini subgroup of S (see [7], 1.1 and 1.3). As $GF(p)[G]_H \cong_H \bigoplus_{|G:H|} GF(p)[H]$, the split extension $HGF(p)[H]$ is in $Q\{HGF(p)[G]\} \subseteq Q\mathcal{P}_\Phi$.

(2) $\mathcal{H}$ is an E_c-closed homomorph; here E_c denotes the closure operation defined by $E_c\mathcal{X} = \{G \in \mathcal{S} \mid$ there exists $N \trianglelefteq G$ such that $G/N \in \mathcal{X}$ and every chief factor of G below N is complemented$\} = \{G \in \mathcal{S} \mid$ there exists $N \trianglelefteq G$ such that $G/N \in \mathcal{X}$ and $P \cap N = 1$ for a prefrattini subgroup P of $G\}$ for each class $\mathcal{X}$ of groups:

Let $G \in E_c\mathcal{H}$. Proceeding by induction on $|G|$, we may assume that $G/N \in \mathcal{H}$ for every normal subgroup $N \neq 1$ of G. Taking N to be minimal normal in G, we see that $G = XN$ with $X \cong G/N \in \mathcal{H}$ a maximal subgroup of G complementing the elementary abelian normal subgroup N. If p denotes the prime dividing $|N|$, then N is an irreducible X-module over $GF(p)$, whence $G = XN$ is a factor group of $XGF(p)[X] \in Q\{HGF(p)[H]\} \subseteq Q\mathcal{P}_\Phi$ (see (1)), where $H \in \mathcal{P}_\Phi$ is such that X is a quotient of H.

(3) $H/N \in \mathcal{P}_\Phi$ with $N \leqslant \Phi(H)$ elementary abelian implies $H \in \mathcal{H}$:

Put $Y = H/N$ and choose a group G such that Y is a prefrattini subgroup of G. Let p be the prime divisor of $|N|$. Gaschütz [9] has shown that there exists a group $G_{\Phi,p}$ with the following two properties:

(i) $G_{\Phi,p}/A_p(G) \cong G$ for some elementary abelian normal p-subgroup $A_p(G)$ of $G_{\Phi,p}$ contained in $\Phi(G_{\Phi,p})$.

(ii) if $G^*/N^* \cong G$ with N^* an elementary abelian normal p-subgroup of G^* contained in $\Phi(G^*)$, then there is an $M \trianglelefteq G_{\Phi,p}$ such that $M \leqslant A_p(G)$ and $G_{\Phi,p}/M \cong G^*$. For a detailed exposition of this result and related lemmas (which we use below) see Baer, Förster [1], chapter 4. If $Y^*/A_p(G) \cong Y$ denotes a prefrattini subgroup of $G_{\Phi,p}/A_p(G)$, then Y^* is a prefrattini subgroup of $G_{\Phi,p}$, and is isomorphic to a split extension $Y_{\Phi,p}P$ with P being a (projective) Y-module over $GF(p)$. From this we deduce $Y_{\Phi,p} \in Q\{Y^*\} \subseteq Q\mathcal{P}_\Phi$. Consequently, $H \in \mathcal{H}$ by (i) + (ii) (applied to Y in place of G).

Finally, by means of (2) + (3), we show that $\mathcal{H} = \mathcal{S}$. Aiming for a contradiction, we consider a group $G \in \mathcal{S} \setminus \mathcal{H}$ of least order. If M is a minimal normal subgroup of G, then $G/M \in \mathcal{H}$, that is there exists $P \in \mathcal{P}_\Phi$ and $Q \trianglelefteq P$ such that $P/Q \cong G/M$. Let X be the direct product of G and P with amalgamated factor group G/M. It is well known that X possesses normal subgroups Y and Z satisfying

$$X/Y \cong G, \quad X/Z \cong P, \quad Y \cap Z = 1, \quad X/Y \times Z \cong G/M.$$

Clearly, $Z \cong_X Y \times Z/Y$ is minimal normal in X. If Z is complemented in X, then $X \in E_c\mathcal{P}_\Phi \subseteq Q\mathcal{P}_\Phi$ by (2), while $X \in Q\mathcal{P}_\Phi$ by (3) in case that $Z \leqslant \Phi(X)$. Therefore, in any case we have $G \cong X/Y \in Q\{X\} \subseteq Q\mathcal{P}_\Phi = \mathcal{H}$, the desired contradiction.

4.3 Remarks. (a) Some trivial modifications of our proof of 4.2 show that $Q\mathcal{P}_\Phi(\mathcal{H}) = \mathcal{S}$ for every Schunck class $\mathcal{H}$, where

$$\mathcal{P}_\Phi(\mathcal{H}) = \{X \in \mathcal{S} \mid X\ \mathcal{H}\text{-prefrattini subgroup of some } G \in \mathcal{S}\}.$$

(b) The method used in part (1) of the above proof also shows that $W_0\mathcal{P}_\Phi \subseteq Q\mathcal{P}_\Phi$, where the closure operation W_0 is defined by

$$W_0\mathfrak{X} = \bigcup_{n \in \mathbf{N}_0} w_0\mathfrak{X} \quad \text{with}$$

$$w_0\mathfrak{X} = \{A \wr G \mid A \text{ abelian with elementary Sylow subgroups, } G \in \mathfrak{X}\}.$$

From this it is easy to deduce the following assertions:

$$Q\mathcal{P}_\Phi = W_0 Q\mathcal{P}_\Phi = QW_0\mathcal{P}_\Phi = E_c\mathcal{P}_\Phi;$$

in fact, these assertions are true whenever $\mathcal{H}$ (in place of $Q\mathcal{P}_\Phi$) is a W_0-closed homomorph. This closure property of a homomorph $\mathcal{H}$, though being rather strong, is not sufficient to conclude that $\mathcal{H} = \mathcal{S}$, as we have shown by an example given in [6]. Thus we cannot avoid the use of Gaschütz's [9].

(For a detailed investigation of W_0-closed homomorphs see [18].)

5. Products of normally embedded subgroups with weakly system permutable subgroups

A p-subgroup $P \leqslant G$ is said to be normally embedded in G, if $P \in \mathrm{Syl}_p(P^G)$ (or, equivalently, if $P \in \mathrm{Syl}_p(N)$ for some $N \trianglelefteq G$). An arbitrary subgroup $H \leqslant G$ is said to be *p-normally embedded* in G (H p-ne G), if $H_p \in \mathrm{Syl}_p(H)$ is normally embedded in G, and H is said to be *normally embedded* in G (H ne G), if it is p-normally embedded for all primes p. If Σ is a Hall system of G, then $H \leqslant G$ is called *Σ-permutable*, if $HG_p = G_pH$ (that is, $HG_p \leqslant G$) for all Sylow groups G_p in Σ. It is well known that H ne G is Σ-permutable in G for each Hall system Σ such that $H \cap \Sigma$ is a Hall system of H. Two Σ-permutable subgroups H and K of G always satisfy $HK = KH$, provided only that both H and K are normally embedded in G, whereas this does not hold generally if H and K are assumed to be Σ-permutable only. In [16] Chambers has shown (generalizing results of Makan [17]) that each Σ-permutable normally embedded subgroup of G permutes with an $\mathcal{F}$-prefrattini subgroup of G (as well as with an $\mathcal{F}$-normalizer of G) corresponding to Σ and to a saturated formation $\mathcal{F}$. Our aim here is to generalize this result further: indeed, not only $\mathcal{H}$-prefrattini subgroups ($\mathcal{H}$ a Schunck class), but even an arbitrary Σ-permutable subgroup H of G permutes with each Σ-permutable normally embedded subgroup K of G, and HK is a cover-avoidance subgroup of G whenever H has this property. (A special case of this last statement is contained in Chambers' paper.)

The property of normally embedded subgroups that we shall need in our proof of this theorem characterizes normally embedded subgroups (of finite soluble groups—a counterexample for the case of insoluble finite groups is given by a cyclic maximal subgroup of a Sylow 2-subgroup of PSL(2, 17)):

5.1 Proposition. *Let P be a p-subgroup of G. Then P* ne *G if and only if for all pairs $Y \trianglelefteq X$ of subgroups of G with p-factor group X/Y such that P normalizes both X and Y the following two conditions hold*:

(i) $X \cap PY \trianglelefteq \mathbf{N}_G(X/Y)\ (= \mathbf{N}_G(X) \cap \mathbf{N}_G(Y))$;

(ii) $X \cap PY = Y$ *and* $\mathbf{C}_G(X/Y)/Y \leqslant X/Y$ *implies* $P \leqslant Y$.

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PROOF. First assume that P ne G and let X, Y be as above. Since $\mathbf{N}_G(X/Y)$ is a subgroup of G with PY/Y ne $\mathbf{N}_G(X/Y)/Y$, we may assume that $Y = 1$ and $X \trianglelefteq G$, that is, X is a normal p-subgroup of G. Then $P \cap X \leqslant P^G \cap X \leqslant \mathbf{O}_p(P^G) \cap X \leqslant P \cap X$ and $P \cap X = P^G \cap X \trianglelefteq G$, which proves (i). Further, if $X \cap PY = Y$ and $\mathbf{C}_G(X/Y) \leqslant X$ then (assuming again that $X \trianglelefteq G$ and $Y = 1$) we have as previously $P^G \cap X = P \cap X = 1$, from which we get that $[P, X] \leqslant [P^G, X] \leqslant P^G \cap X = 1$. Consequently, $P \leqslant \mathbf{C}_G(X) \leqslant X$ and $P = P \cap X = 1 \leqslant Y$. Thus (ii) holds as well.

Now suppose that P satisfies the above conditions. We prove P ne G by induction on $|G|$. If $N = \mathbf{O}_{p'}(G) \neq 1$, then $PN/N \in \mathrm{Syl}_p(M/N)$ for some $M \trianglelefteq G$, and we get $P \in \mathrm{Syl}_p(M)$, as desired. If $\mathbf{O}_{p'}(G) = 1$, then $L = \mathbf{F}(G) = \mathbf{O}_p(G) \neq 1$. The case $P \cap L \neq 1$ is handled by appealing to our inductive hypothesis, since $P \cap L \trianglelefteq \mathbf{N}_G(L) = G$ by (i). If $P \cap L = 1$, then (ii) applies (with $X = L$ and $Y = 1$) and gives $P = 1$, whence the assertion holds trivially.

5.2 THEOREM. *Let Σ be a Hall system of G, and let K ne G and $H \leqslant G$ be Σ-permutable subgroups.*

(a) *HK is a Σ-permutable subgroup of G.*

(b) *If H is a cover-avoidance subgroup of G, then so is HK.*

PROOF. (a) It suffices to show that HK is a subgroup of G. Since (by definition) the Sylow subgroups of a normally embedded subgroup of G are themselves normally embedded in G, we may assume that K is a p-subgroup for some prime p. We proceed by induction on $|G| + |G : H|$.

Case 1: $|G : H|$ is a power of a prime q.

In this case we have $\mathbf{O}_{q'}(G) \leqslant H$, whence the assumption that $\mathbf{O}_{q'}(G) = 1$ is justified by our inductive hypothesis. We get $\mathbf{F}(G) = \mathbf{O}_q(G)$. Moreover, without loss of generality, $q = p$, as otherwise $K \leqslant G_p \leqslant G_{q'} \leqslant H$; here G_π denotes a Hall π-subgroup of G from Σ. Now 5.1(i) yields that $K \cap \mathbf{O}_p(G) = K \cap \mathbf{F}(G) = 1$, and 5.1(ii) then gives $K = 1$, as $\mathbf{C}_G(\mathbf{F}(G)) \leqslant \mathbf{F}(G)$.

Case 2: There are (at least) two distinct primes q, r dividing $|G : H|$.

By induction, we know that G_qHK and G_rHK are subgroups of G, and so is

$$D = G_qHK \cap G_rHK = \left(G_qHK \cap G_rH\right)K = \left(G_qHK \cap G_r\right)HK.$$

If $q = p$, then $K \leqslant G_p$ and $G_qHK \cap G_r = HG_pK \cap G_r = HG_p \cap G_r = H \cap G_r$ (as the latter is a Sylow r-subgroup of H), whence $HK = (H \cap G_r)HK = D \leqslant G$; the case $r = p$ is treated similarly. Now we may suppose that $q \neq p \neq r$. From

$$|G_qHK| = |HG_qK| = \frac{|H| \cdot |G_qK|}{|H \cap G_qK|}$$

(with $G_q K$ an $\{q, p\}$- and hence r'-subgroup of G) we infer that $|G_q HK|_r = |H|_r$. Consequently, $G_q HK \cap G_r = H \cap G_r$, and we are done as previously.

(b) Again we may assume that K is a p-group. Here we proceed by induction on $|G|$. Therefore it suffices to show that each minimal normal subgroup M of G is covered or avoided by HK. If $|M|$ is a power of a prime q, then $HK \cap M$ is a normal q-subgroup of HK, and thus is contained in each Sylow q-subgroup. Thus the assertion follows in case that $q \neq p$. Now let M be a p-group, and suppose that $1 \neq HK \cap M \neq M$. If N is a minimal normal q-subgroup of G with $q \neq p$, then MN/N is covered by HK, since $HK \cap M \neq 1$ forces that MN/N is not avoided by HK. If N, too, is covered by HK, then $MN \leqslant HK$ against our choice of M. If N is avoided by HK, then it is easy to see that $|HK \cap M| = |HKN/N \cap MN/N|$, which gives the contradiction that M is contained in HK. Therefore we may assume that $\mathbf{O}_{p'}(G) = 1$. Then $\mathbf{F}(G) = \mathbf{O}_p(G)$ and $K \cap F(G) \trianglelefteq G$. The case $K \cap F(G) \neq 1$ leads to another contradiction, when the inductive hypothesis is taken into account. Finally, as K is a normally embedded p-subgroup of G, $K \cap F(G) = 1$ implies that $K = 1$. Thus $HK = H$ covers or avoids M, this being a final contradiction.

5.3 Remark. In general, it does not seem to be easy to give a description of those chief factors of G which are avoided by HK (where H, K, G are as in 5.2) in terms of H-avoided and K-avoided chief factors of G and by further conditions concerning the embedding in G of H and K. If H is an $\mathfrak{H}$-prefrattini subgroup of G for a Schunck class $\mathfrak{H}$, however, one can easily check that the result of Chambers [16] (where $\mathfrak{H}$ was supposed to be a saturated formation) is still valid; in fact, one may take generalized $\mathfrak{X}$-prefrattini subgroups here. (The proof given by Chambers is not suitable in this context, because $\mathfrak{H}$-normalizers are not available, but it is easy to give a direct proof here.)

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Fachbereich Mathematik
Johannes Gutenberg-Universität
D-6500 Mainz
F. R. Germany

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