

GROUP STRUCTURE AND PROPERTIES OF BLOCK IDEALS OF THE GROUP ALGEBRA

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1. Introduction. In this note relations between the structure of a finite group G and ringtheoretical properties of the group algebra FG over a field F with characteristic $p > 0$ are investigated. Denoting by $J(R)$ the Jacobson radical and by $Z(R)$ the centre of the ring R , our aim is to prove the following theorem generalizing results of Wallace [10] and Spiegel [9]:

THEOREM. *Let G be a finite group and let F be an arbitrary field of characteristic $p > 0$. Denoting by B_1 the principal block ideal of the group algebra FG the following statements are equivalent:*

- (i) $J(B_1) \subseteq Z(B_1)$,
- (ii) $J(B_1)$ is commutative,
- (iii) G is p -nilpotent with abelian Sylow p -subgroups.

The equivalence of (i) and (iii) was shown in [9] and of course, Theorem 2 of [10] is an immediate consequence of the theorem above. Its proof involves a heredity property of the principal block discussed in §2 as well as methods of Wallace [10] and finally a result of Morita [7] which is included with a short proof following from the other preliminaries. As a corollary Wallace's characterization of group algebras with commutative radicals is obtained and proved for arbitrary ground fields.

In the following all modules are finitely generated unitary right modules. To avoid trivial statements it is always assumed that the prime p divides the order $|G|$ of the finite group G . $O_p(G)$ is the largest normal subgroup of G with an order not divisible by p . The block ideals of the group algebra FG are the uniquely determined as two-sided ideals indecomposable direct summands of FG . As identity elements of the block ideals, the block idempotents are exactly the primitive idempotents of $Z(FG)$. A block of FG is a triple $B \leftrightarrow e \leftrightarrow \lambda$ consisting of the block ideal B , its identity element e , and the linear character λ of $Z(FG)$ with $\lambda(e) \neq 0$. An indecomposable FG -module M belongs to the block (ideal) B if $Me \neq 0$. $B_1 \leftrightarrow e_1 \leftrightarrow \lambda_1$ is the principal block of FG if e_1 does not annihilate the trivial simple FG -module. Concerning the further terminology we refer to Michler [5] and Curtis-Reiner [1].

I am indebted to Professor G. Michler for helpful discussions.

2. A heredity property of the principal block. Throughout this section F is an arbitrary field of characteristic $p > 0$ and G is a finite group. First we collect some elementary facts. We recall that for FG -modules M and N the tensor product $M \otimes_F N$ is defined by

$$(m \otimes n)g = mg \otimes ng \quad (g \in G, m \in M, n \in N).$$

LEMMA 2.1. *Let E be an FG-module of dimension one and let S be a simple FG-module. Then $E \otimes_F S$ is a simple FG-module.*

Proof. There is an $x \in E$ such that $E = xF$. If $M \neq 0$ is a submodule of $E \otimes_F S$, then every $m \in M$ can be written as $m = x \otimes y$ for a $y \in S$. Because of the action of G onto $E \otimes_F S$, the set

$$M_1 = \{y \in S \mid x \otimes y \in M\} \neq 0$$

is a submodule of S and hence $M_1 = S$. Since E has dimension one it follows that

$$M = \{x \otimes y \mid y \in M_1 = S\} = E \otimes_F S,$$

which proves the lemma.

LEMMA 2.2. *Let X, Y, M be FG-modules with $X \leq Y$. Then*

$$Y \otimes_F M / X \otimes_F M \cong_{FG} Y/X \otimes_F M.$$

The proof is easily given choosing suitable F -bases in the modules.

PROPOSITION 2.3. *Let U_1 be the indecomposable projective FG-module associated with the trivial representation (i.e. $U_1/U_1 \cdot J(FG) \cong F$) and let U be any indecomposable projective FG-module. If all composition factors of U_1 have dimension one, then the dimensions of all composition factors of U are equal.*

Proof. Set $U/U \cdot J(FG) = V$. It is well known that U is a direct summand of the projective FG-module $U_1 \otimes_F V$. Let

$$U_1 = U_{11} > U_{12} > U_{13} > \dots > U_{1n} = 0$$

be a composition series of U_1 . By Lemmas 2.1 and 2.2

$$U_1 \otimes_F V = U_{11} \otimes_F V > U_{12} \otimes_F V > \dots > U_{1n} \otimes_F V = 0$$

is a composition series of $U_1 \otimes_F V$ and all composition factors have the same dimension, namely $\dim_F V$. This proves the assertion.

An immediate consequence is

COROLLARY 2.4. *Let F be an arbitrary field of characteristic $p > 0$ and let G be a finite group. The following statements are equivalent:*

- (i) *All irreducible representations of G over F belonging to the same block have the same F -dimension.*
- (ii) *All irreducible representations of G over F belonging to the principal block have F -dimension one.*

(iii) All composition factors of the indecomposable projective FG -module associated with the trivial representation have F -dimension one.

Proof. (iii) implies (i) by 2.3 bearing in mind the fact that the indecomposable projective modules in a block are linked.

DEFINITION. The group algebras of finite groups characterized by Corollary 2.4 are in the following called *group rings with dimension property*.

REMARK. If the underlying field F is a splitting field for the group G then FG has the dimension property if and only if in the decomposition of any block ideal into indecomposable projective modules all isomorphism types appear with the same multiplicity. Morita [7] calls such rings *quasi-primary decomposable*.

In the following sections the subsequent observation of Spiegel [9] will be used.

LEMMA 2.5 ([9]). Let G be a p -solvable group and let $H = O_p(G)$. Then

$$e_1 = \frac{1}{|H|} \sum_{h \in H} h$$

is the principal block idempotent of FG .

Proof. ([9]). $e_1FG \cong FG/H$ implies that e_1FG is the principal block ideal because FG/H is by Fong's Theorem [2, 3A] indecomposable as a two-sided ideal.

3. Group rings with dimension property. In this section K is always a splitting field of characteristic $p > 0$ for all subgroups and factor groups of the finite group G (e.g. K can be chosen to be algebraically closed). By 2.4, KG has the dimension property if and only if the same is true for the principal block ideal of KG .

The next result, for which we give a short proof for the sake of completeness, is due to Morita [7]; for another proof see [4].

PROPOSITION 3.1. Let $H = O_p(G)$ and let P be a Sylow p -subgroup of G . The following statements are equivalent:

- (i) HP is normal in G and G/HP is abelian.†
- (ii) The principal block ideal B_1 of KG has the dimension property.
- (iii) KG has the dimension property.

Proof. (i) \Rightarrow (ii). Set $\bar{G} = G/H$ and $\bar{P} = HP/H \cong P$. By hypothesis, \bar{G} is a group with normal Sylow p -subgroup \bar{P} and abelian factor group. Hence $K\bar{G}$ has only irreducible representations of dimension one. In particular $K\bar{G}$ has the dimension property. By 2.5, $B_1 \cong K\bar{G}$ and (ii) is proved.

† For example, these properties are satisfied if G is supersolvable.

(ii) \Leftrightarrow (iii). This follows from 2.4.

(iii) \Rightarrow (i). (cf. Morita [7], p. 188) With the notation of the first part of this proof $B_1 \cong K\tilde{G}$. Since all irreducible representations of B_1 have dimension one, the right regular representation \tilde{G} of \tilde{G} on $K\tilde{G}$ consists of lower triangular matrices. It is easily verified that exactly those of these triangular matrices with only elements equal to one in the main diagonal form a Sylow p -subgroup \tilde{P} of $\tilde{G} \cong \tilde{G}$. Furthermore \tilde{P} is normal in \tilde{G} and \tilde{G}/\tilde{P} is abelian. This proves (i).

The following lemma is an immediate consequence of the fact that every group ring over a field is a symmetric algebra.

LEMMA 3.2. *Let $B \leftrightarrow e \leftrightarrow \lambda$ be a block of the group algebra FG over an arbitrary field F with positive defect. Then $J(B)^2 = 0$ if and only if there exists an (up to isomorphisms) unique projective indecomposable B -module of composition length 2.*

The next result is due to Wallace[10]. Its proof is included for the sake of completeness.

LEMMA 3.3 ([10]). *If the radical $J = J(B)$ of the block ideal B of the group algebra KG is commutative, then either $J(B)^2 = 0$ or every composition factor of an arbitrary indecomposable projective B -module has K -dimension one.*

Proof. If $J(B)^2 \neq 0$, let V be an indecomposable projective B -module. Since KG is a quasi-Frobenius ring, it is sufficient to show that the socle $\text{soc } V$ of V has dimension one. By 3.2 $V \cdot J^2 \neq 0$. Thus every element of $\text{soc } V$ is a linear combination of elements $x \cdot y$ with $x, y \in J$. Let g, h be arbitrary elements of G . Then

$$(xy)(gh) = x[(yg)h] = [(xh)y]g = (yx)(hg) = (xy)(hg).$$

This implies, by Burnside's theorem ([1], 27.4), that

$$\text{End}_K(\text{soc } V) = \text{End}_{KG}(\text{soc } V),$$

and hence, by Schur's lemma, that

$$\text{End}_K(\text{soc } V) \cong K.$$

Therefore $\dim_K(\text{soc } V) = 1$, and the lemma is proved.

PROPOSITION 3.4. *If the radical $J(B_1)$ of the principal block ideal B_1 of the group algebra KG is commutative, then KG has the dimension property.*

Proof. Since the principal block has highest defect, then $J(B_1) \neq 0$. By 2.4, it suffices to show that all composition factors of an indecomposable projective module V belonging to the principal block have dimension one. Hence the assertion follows from 3.3 and 3.2.

4. Proof of the theorem.

(A part of the step (ii) \Rightarrow (iii) is a modification of an argument in [10, p. 2]).

(i) \Rightarrow (ii) clear.

(ii) \Rightarrow (iii). Let K be the algebraic closure of F . Then

$$K \otimes_F B_1 = B_1^* \otimes \dots \otimes B_s^*,$$

with block ideals B_1^*, \dots, B_s^* of KG ; we may assume that B_1^* is the principal block ideal of KG . Since $FG/J(FG)$ is a separable algebra (cf. [5], 12.11), we conclude that

$$K \otimes_F J(B_1) = J(K \otimes_F B_1)$$

by [5, 12.12]. Hence $J(B_1^*)$ is commutative also. By 3.2, KG has the dimension property. Let $H = O_p(G)$ and let P be a Sylow p -subgroup of G . 3.1 implies that $A = HP$ is normal in G and that G/HP is abelian. It remains to show that $HP = G$. Because G is p -solvable, 2.5 yields that

$$e = \frac{1}{|H|} \sum_{h \in H} h$$

is the principal block idempotent of KG and of FG (in particular, $s = 1$). The nilpotent ideal $eJ(FP)$ of FA is contained in $J(FA)$. Hence by Villamayor's Theorem (cf. [5], 11.8),

$$e J(FP) = e J(FG) = J(B_1).$$

Therefore $eJ(FP)$ is commutative.

To apply Burnside's criterion for p -nilpotence, choose arbitrary $s \in P$ and $u \in N_G(P)$. Putting $usu^{-1} = s_1 \in P$, we have

$$\begin{aligned} e(1-s)(1-s_1) &= e(1-s)e \cdot u(1-s)u^{-1} = [e(1-s)u \cdot e(1-s)]u^{-1} \\ &= e(1-s)^2. \end{aligned}$$

As P is a right transversal of H in A , this implies that

$$(1-s)(1-s_1) = (1-s)^2;$$

hence

$$s_1 - s s_1 = s - s^2.$$

If $p \neq 2$, this yields $s_1 = s$. This is true for $p = 2$ also, because otherwise $ss_1 = s$ and hence $s_1 = 1 = s$.

Therefore always $s_1 = usu^{-1} = s$ is valid, i.e. P is contained in the centre of its normalizer. Burnside's criterion now finishes the proof of (iii).

(iii) \Rightarrow (i). With the preceding notation, 2.5 again gives $B_1 \cong FG/H \cong FP$ and (i) follows.

5. Group algebras with commutative radicals. As an application of the theorem above we prove in this section the result of Wallace [10] characterizing group algebras with commutative radicals.

PROPOSITION 5.1. *Let F be an arbitrary field of characteristic $p \neq 2$ and let the radical $J(FG)$ of the group algebra FG be commutative. Denoting by G' the commutator subgroup and by P a Sylow p -subgroup of G the following hold: p does not divide $|G'|$ and, setting*

$$f = \frac{1}{|G'|} \sum_{x \in G'} x,$$

we have $J(FG) = J(f \cdot FG)$; in particular

$$\dim_F J(FG) = \dim_F J(FG/G') = (|P|-1) \frac{|G|}{|G'| |P|}.$$

Proof. By [5, 12.12] there is no loss of generality in assuming for the proof of this assertion that F is algebraically closed. For every block $B \leftrightarrow e \leftrightarrow \lambda$ of FG with positive defect, we have $J(B)^2 \neq 0$, because otherwise, by 3.1, the Cartan matrix $C(B)$ of B has just one entry, namely 2, and this implies, by [1, 84.17], that $p = 2$, contradicting our assumption. The commutativity of $J(FG)$ implies by the theorem that G is p -nilpotent with P abelian. So p does not divide $|G'|$. Clearly $f \cdot FG \cong FG/G'$. From $FG = f \cdot FG \oplus (1-f)FG$ and the fact that the one-dimensional FG -modules and the irreducible (hence one-dimensional) modules of $FG/G' \cong f \cdot FG$ correspond in a one-to-one manner, we conclude, by 3.3, that all blocks $B \leftrightarrow e \leftrightarrow \lambda$ of FG with $e(1-f) = e$ have defect zero. Hence $(1-f)FG$ is semisimple (see e.g. [3], Cor. 3.4). This yields $J(FG) = J(f \cdot FG)$.

Denoting by $\bar{P} = G'P/G'$ the Sylow p -subgroup of the abelian group G/G' it is easily shown that $J(F\bar{P}) \cdot FG/G' = J(FG/G')$. Thus

$$\dim_F J(FG) = (|P|-1) \frac{|G|}{|G'P|}.$$

and the assertion is proved.

We recall that a finite p -nilpotent group G with a Sylow p -subgroup P and normal p -complement N is a *Frobenius group for P* , if $n \neq nq^{-1}$ for every $1 \neq q \in P$ and every $1 \neq n \in N$.

Quoting elementary facts written up for example in [5], we obtain as a corollary by means of the preceding results the following.

COROLLARY 5.2 (Wallace [10]). *Let G be a finite group and let F be a field with prime characteristic $p \neq 2$. Denoting by P a Sylow p -subgroup and by G' the commutator subgroup of G , the following statements are equivalent:*

- (i) $J(FG) \leq Z(FG)$,
- (ii) $J(FG)$ is commutative,
- (iii) p does not divide $|G'|$ and $G'P$ is a Frobenius group for P .

Proof. Clearly (i) implies (ii).

(ii) \Rightarrow (iii) follows from 5.1 and Villamayor's Theorem (cf. [5], 11.8), applying 13.6 of [5].

(iii) \Rightarrow (i) follows from 13.8 of [5].

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