



On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface

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ABSTRACT

Let M_c be the moduli space of semistable torsion-free sheaves of rank 2 with Chern classes $c_1 = 0$ and $c_2 = c$ over a K3 surface with generic polarization. When $c = 2n \geq 4$ is even, M_c is a singular projective variety which admits a symplectic form, called the Mukai form, on the smooth part. A natural question raised by O’Grady asks if there exists a desingularization on which the Mukai form extends everywhere nondegenerately. In this paper we show that such a desingularization does not exist for many even integers c by computing the stringy Euler numbers.

1. Introduction

Let X be a projective K3 surface with generic polarization $\mathcal{O}_X(1)$ and let $M_c = M(2, 0, c)$ be the moduli space of semistable torsion-free sheaves on X of rank 2, with Chern classes $c_1 = 0$ and $c_2 = c$. When $c = 2n \geq 4$ is even, M_c is a singular projective variety. Recently O’Grady raised the following question [Ogr99, (0.1)].

Question 1.1. Does there exist a symplectic desingularization of M_{2n} ?

In [Ogr99], he analyzed Kirwan’s desingularization \widehat{M}_c of M_c and proved that \widehat{M}_c can be blown down twice and that as a result he obtained a symplectic desingularization \widetilde{M}_c of M_c in the case when $c = 4$. This turns out to be a new irreducible symplectic variety.

When $c \geq 6$, O’Grady conjectures that there is no smooth symplectic model of M_c (see [Ogr99, p. 50]). The purpose of this paper is to provide a partial answer to Question 1.1.

THEOREM 1.2. *There is no symplectic desingularization of M_{2n} if $na_n/(2n - 3)$ is not an integer where a_n is the Euler number of the Hilbert scheme $X^{[n]}$ of n points in X .*

It is well known that a_n is given by the equation

$$\sum_{n=0}^{\infty} a_n q^n = \prod_{n=1}^{\infty} 1/(1 - q^n)^{24}.$$

By direct computation, one can check that $na_n/(2n - 3)$ is not an integer for $n = 5, 6, 8, 11, 12, 13, 15, 16, 17, 18, 19, 20, \dots$

The idea of the proof is to use properties of the stringy Euler numbers. If there is an irreducible symplectic desingularization \widetilde{M}_c of M_c , then the stringy Euler number of M_c is equal to the ordinary Euler number of \widetilde{M}_c because the canonical divisors of both \widetilde{M}_c and M_c are trivial (Theorem 2.2).

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In particular, we deduce that the stringy Euler number $e_{\text{st}}(M_c)$ must be an integer. Therefore, Theorem 1.2 is a consequence of the following.

PROPOSITION 1.3. *The stringy Euler number $e_{\text{st}}(M_{2n})$ is of the form*

$$\frac{na_n}{2n-3} + \text{integer}.$$

We prove this proposition in § 3 after a brief review of preliminaries.

One motivation for Question 1.1 is to find a mathematical interpretation of Vafa–Witten’s formula [VW94, (4.17)] which says that the ‘Euler characteristic’ of M_{2n} is

$$e^{\text{VW}}(M_{2n}) = a_{4n-3} + \frac{1}{4}a_n.$$

Because $k/4 \neq l/(2n-3)$ for $1 \leq k \leq 3, 1 \leq l < 2n-3$, we deduce the following from Proposition 1.3.

COROLLARY 1.4. *The stringy Euler number $e_{\text{st}}(M_{2n})$ is not Vafa–Witten’s Euler characteristic $e^{\text{VW}}(M_{2n})$ in general.*

Independently, Kaledin and Lehn [KL04a] proved that there is no symplectic desingularization of M_{2n} for any $n \geq 3$ by a very different method.

2. Preliminaries

In this section, we recall the definition and basic facts about stringy Euler numbers. The references are [Bat98, DL99].

Let W be a variety with at worst *log-terminal* singularities, i.e.:

- W is \mathbb{Q} -Gorenstein;
- for a resolution of singularities $\rho : V \rightarrow W$ such that the exceptional locus of ρ is a divisor D whose irreducible components D_1, \dots, D_r are smooth divisors with only normal crossings, we have

$$K_V = \rho^*K_W + \sum_{i=1}^r a_i D_i$$

with $a_i > -1$ for all i , where D_i runs over all irreducible components of D . The divisor $\sum_{i=1}^r a_i D_i$ is called the *discrepancy divisor*.

For each subset $J \subset I = \{1, 2, \dots, r\}$, define $D_J = \bigcap_{j \in J} D_j$, $D_\emptyset = Y$ and $D_J^0 = D_J - \bigcup_{j \in I-J} D_j$. Then the stringy E -function of W is defined by

$$E_{\text{st}}(W; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \tag{2.1}$$

where

$$E(Z; u, v) = \sum_{p,q} \sum_{k \geq 0} (-1)^k h^{p,q}(H_c^k(Z; \mathbb{C})) u^p v^q$$

is the Hodge–Deligne polynomial for a variety Z . Note that the Hodge–Deligne polynomials have:

- the additive property: $E(Z; u, v) = E(U; u, v) + E(Z-U; u, v)$ if U is a smooth open subvariety of Z ;
- the multiplicative property: $E(Z; u, v) = E(B; u, v)E(F; u, v)$ if Z is a locally trivial F -bundle over B .

DEFINITION 2.1. The stringy Euler number is defined as

$$e_{\text{st}}(W) = \lim_{u,v \rightarrow 1} E_{\text{st}}(W; u, v) = \sum_{J \subset I} e(D_J^0) \prod_{j \in J} \frac{1}{a_j + 1} \tag{2.2}$$

where $e(D_J^0) = E(D_J^0; 1, 1)$.

The ‘change of variable formula’ (Theorem 6.27 in [Bat98], Lemma 3.3 in [DL99]) implies that the function E_{st} is independent of the choice of a resolution and the following holds.

THEOREM 2.2 [Bat98, Theorem 3.12]. *Suppose W is a \mathbb{Q} -Gorenstein algebraic variety with at worst log-terminal singularities. If $\rho : V \rightarrow W$ is a crepant desingularization (i.e. $\rho^*K_W = K_V$) then $E_{\text{st}}(W; u, v) = E(V; u, v)$. In particular, $e_{\text{st}}(W) = e(V)$ is an integer.*

3. Proof of Proposition 1.3

We fix a generic polarization of X as in [Ogr99, p. 50]. The moduli space M_{2n} has a stratification

$$M_{2n} = M_{2n}^s \sqcup (\Sigma - \Omega) \sqcup \Omega$$

where M_{2n}^s is the locus of stable sheaves and

$$\Sigma \cong (X^{[n]} \times X^{[n]})/\text{involution}$$

is the locus of sheaves of the form $I_Z \oplus I_{Z'}$ ($[Z], [Z'] \in X^{[n]}$) while

$$\Omega \cong X^{[n]}$$

is the locus of sheaves $I_Z \oplus I_Z$. Kirwan’s desingularization $\rho : \widehat{M}_{2n} \rightarrow M_{2n}$ is obtained by blowing up M_c first along the deepest stratum Ω , next along the proper transform of the middle stratum Σ and finally along the proper transform of a subvariety Δ in the exceptional divisor of the first blow-up which is the locus of \mathbb{Z}_2 quotient singularities [Kir85]. This is indeed a desingularization by [Ogr99, Proposition 1.8.3].

Let $D_1 = \widehat{\Omega}$, $D_2 = \widehat{\Sigma}$, $D_3 = \widehat{\Delta}$ be the (proper transforms of the) exceptional divisors of the three blow-ups. Then they are smooth divisors with only normal crossings and the discrepancy divisor of $\rho : \widehat{M}_{2n} \rightarrow M_{2n}$ is [Ogr99, (6.1)]

$$(6n - 7)D_1 + (2n - 4)D_2 + (4n - 6)D_3.$$

Therefore the singularities are terminal for $n \geq 2$ and from (2.2) the stringy Euler number of M_{2n} is given by

$$\begin{aligned} e(M_{2n}^s) + e(D_1^0) \frac{1}{6n - 6} + e(D_2^0) \frac{1}{2n - 3} + e(D_3^0) \frac{1}{4n - 5} + e(D_{12}^0) \frac{1}{6n - 6} \frac{1}{2n - 3} \\ + e(D_{23}^0) \frac{1}{2n - 3} \frac{1}{4n - 5} + e(D_{13}^0) \frac{1}{6n - 6} \frac{1}{4n - 5} + e(D_{123}^0) \frac{1}{6n - 6} \frac{1}{2n - 3} \frac{1}{4n - 5}. \end{aligned} \tag{3.1}$$

We need to compute the (virtual) Euler numbers of D_J^0 for $J \subset \{1, 2, 3\}$. Let (E, ω) be a symplectic vector space of dimension $c = 2n$. Let $\text{Gr}^\omega(k, c)$ be the Grassmannian of k -dimensional subspaces of E isotropic with respect to the symplectic form ω (i.e. the restriction of ω to the subspace is zero).

LEMMA 3.1. *For $k \leq n$, the Euler number of $\text{Gr}^\omega(k, 2n)$ is $2^k \binom{n}{k}$.*

Proof. Consider the incidence variety

$$\{(a, b) \in \text{Gr}^\omega(k - 1, 2n) \times \text{Gr}^\omega(k, 2n) \mid a \subset b\}.$$

This is a $\mathbb{P}^{2n-2k+1}$ -bundle over $\text{Gr}^\omega(k-1, 2n)$ and a \mathbb{P}^{k-1} -bundle over $\text{Gr}^\omega(k, 2n)$. The formula follows from an induction on k . \square

Let $\hat{\mathbb{P}}^5$ be the blow-up of \mathbb{P}^5 (projectivization of the space of 3×3 symmetric matrices) along \mathbb{P}^2 (the locus of rank 1 matrices). We have the following from [Ogr99, § 6] and [Ogr97, § 3].

PROPOSITION 3.2.

- (1) D_1 is a $\hat{\mathbb{P}}^5$ -bundle over a $\text{Gr}^\omega(3, 2n)$ -bundle over $X^{[n]}$.
- (2) D_2^0 is a \mathbb{P}^{2n-4} -bundle over a \mathbb{P}^{2n-3} -bundle over $(X^{[n]} \times X^{[n]} - X^{[n]})/\text{involution}$.
- (3) D_3 is a $\mathbb{P}^{2n-4} \times \mathbb{P}^2$ -bundle over a $\text{Gr}^\omega(2, 2n)$ -bundle over $X^{[n]}$.
- (4) $D_1 \cap D_2$ is a $\mathbb{P}^2 \times \mathbb{P}^2$ -bundle over $\text{Gr}^\omega(3, 2n)$ -bundle over $X^{[n]}$.
- (5) $D_2 \cap D_3$ is a $\mathbb{P}^{2n-4} \times \mathbb{P}^1$ -bundle over a $\text{Gr}^\omega(2, 2n)$ -bundle over $X^{[n]}$.
- (6) $D_1 \cap D_3$ is a $\mathbb{P}^2 \times \mathbb{P}^{2n-5}$ -bundle over a $\text{Gr}^\omega(2, 2n)$ -bundle over $X^{[n]}$.
- (7) $D_1 \cap D_2 \cap D_3$ is a $\mathbb{P}^1 \times \mathbb{P}^{2n-5}$ -bundle over a $\text{Gr}^\omega(2, 2n)$ -bundle over $X^{[n]}$.

For instance, (1) is just Proposition 6.2 of [Ogr99] and (2) is Proposition 3.3.2 of [Ogr97], while (3) is Lemma 3.5.4 in [Ogr97].

From Proposition 3.2 and Lemma 3.1, we have the following by the additive and multiplicative properties of the (virtual) Euler numbers:

$$\begin{aligned}
 e(D_1^0) &= 0, & e(D_2^0) &= (2n-3)(2n-2)\frac{1}{2}(a_n^2 - a_n), \\
 e(D_3^0) &= 2^2 \binom{n}{2} a_n, & e(D_{12}^0) &= 3 \cdot 2^3 \binom{n}{3} a_n \\
 e(D_{23}^0) &= 2 \cdot 2^2 \binom{n}{2} a_n, & e(D_{13}^0) &= (2n-4)2^2 \binom{n}{2} a_n \\
 e(D_{123}^0) &= 2(2n-4)2^2 \binom{n}{2} a_n.
 \end{aligned}$$

Hence from the formula (3.1), the stringy Euler number of M_{2n} is given by

$$e_{\text{st}}(M_{2n}) = e(M_{2n}^s) + (n-1)(a_n^2 - a_n) + n \frac{2n-2}{2n-3} a_n = \frac{n a_n}{2n-3} + \text{integer}$$

since $e(M_{2n}^s)$ is an integer. So we have proved Proposition 1.3.

Remark 3.3. For the moduli space of rank 2 bundles over a smooth projective curve, the stringy E-function and the stringy Euler number are computed in [Kie03] and [KL04b].

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