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ANALYTIC EQUIVALENCE RELATIONS SATISFYING HYPERARITHMETIC-IS-RECURSIVE

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Abstract

We prove, in ZF + $\boldsymbol{\Sigma}^1_2$-determinacy, that, for any analytic equivalence relation E, the following three statements are equivalent: (1) E does not have perfectly many classes, (2) E satisfies hyperarithmetic-is-recursive on a cone, and (3) relative to some oracle, for every equivalence class $[Y]_E$ we have that a real X computes a member of the equivalence class if and only if $\omega_1^X \geqslant \omega_1^{[Y]}$. We also show that the implication from (1) to (2) is equivalent to the existence of sharps over ZF.

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1. Introduction

In 1955, Spector [Spe55] proved that every well ordering of ω with a hyperarithmetic presentation has a computable presentation. This theorem has been of great importance in recursion theory and in lightface descriptive set theory. In this paper, we prove that Spector's theorem can be extended to very general circumstances which apply to a variety of known cases, unearthing a more general phenomenon that is behind all of them.

Some years ago, the author [Mon05, Mon07] showed that Spector's theorem can be extended to the class of all linear orderings if we replace isomorphism by biembeddability: every hyperarithmetic linear ordering is biembeddable with a computable one. Notice that, among well orderings, the notions of isomorphism and biembeddability coincide, so Spector's theorem is a special case of this more general result. Not much later, Greenberg and the author showed the same result

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for biembeddability of p-groups [GM08]. Let us remark that, for both countable linear orderings and countable p-groups, the number of equivalence classes under biembeddability is $\aleph_1$, as proved by Laver [Lav71], and Barwise and Eklof [BE71], respectively. Some time later, the author showed that any counterexample to Vaught's conjecture, that is, a theory which has $\aleph_1$ but not continuum many models, if it exists, would also satisfy the same property [Mon13], giving a computability theoretic statement equivalent to Vaught's conjecture. After all these examples we started to think that something more general was going on.

DEFINITION 1.1. We say that an equivalence relation E on the reals, 2^ω, satisfies *hyperarithmetic-is-recursive* if every hyperarithmetic real is E-equivalent to a computable one.

Our main result says that any analytic equivalence relation with less than continuum many equivalence classes *essentially* satisfies hyperarithmetic-is-recursive. We say 'essentially' because one can always build a nonnatural equivalence relation for which this is not true. To overcome this problem, we ask for the equivalence relation to satisfy hyperarithmetic-is-recursive relative to almost every oracle, where 'almost every' is in the sense of Martin's measure. If we have a natural equivalence relation at hand, one would expect to be able to prove either that it satisfies hyperarithmetic-is-recursive or that it does not, and in either case, one would expect that this proof to relativize to every oracle. Therefore, restricting oneself to almost every oracle should not make a difference on natural equivalence relations.

By *Martin's measure* we mean the $\{0, 1\}$-measure, where a set of reals has *Martin's measure* 1 if it contains a cone, where a *cone* is a set of reals of the form $\{X \in 2^\omega : X \geqslant_T Y\}$ for some Y called the *base of the cone*. Martin showed that this is a measure on the degree-invariant sets of reals of complexity Γ, assuming Γ-determinacy, where Γ is a complexity class like for instance Borel, analytic, etc.

DEFINITION 1.2. We say that an equivalence relation on 2^ω satisfies *hyperarithmetic-is-recursive on a cone* if there is a $C \in 2^\omega$ (the base of the cone) such that, for every X which computes C, every X-hyperarithmetic real is equivalent to an X-computable one.

Here is our main theorem.

THEOREM 1.3 (ZF + Σ^1_2-Det). *Let E be an analytic equivalence relation on 2^ω. The following are equivalent.*

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(H1) *There is no perfect set all of whose elements are E-inequivalent.*

(H2) *E satisfies hyperarithmetic-is-recursive on a cone.*

(H3) *There is an oracle relative to which, for every $Y \in 2^\omega$, the* degree spectrum *of its equivalence class, $Sp([Y]_E)$, is of the form $\{X \in 2^\omega : \omega_1^X \geqslant \alpha\}$ for some ordinal $\alpha \in \omega_1$.*

Burgess [**Bur78**] showed that, given an analytic equivalence relation, either it has at most $\aleph_1$ many equivalence classes, or it has perfectly many classes (i.e., there is a perfect set of E-inequivalent reals). Thus, if the continuum hypothesis is false, saying that E does not have perfectly many classes is equivalent to saying that it has $\leqslant \aleph_1$ many classes. The existence of such a perfect set is absolute (it is Σ_2^1) and does not depend on the continuum hypothesis.

The *degree spectrum* of an equivalence class is the analog of the degree spectrum of a structure, a notion widely studied in Computable Structure Theory. It gives us a way of measuring the complexity of the equivalence class in terms of how difficult it is to compute a member. More precisely, define

$$Sp([Y]_E) = \{X \in 2^\omega : \exists W \leqslant_T X \ (W \ E \ Y)\}.$$

The set $\{X \in 2^\omega : \omega_1^X \geqslant \alpha\}$ is the set of all reals that can compute copies of all ordinals below α. It is a very particular set, and the fact that the spectrum of any equivalence class would have this form seems to be a very strong statement. Let us remark that the relativized version of the spectrum is defined as $Sp^Z([Y]_E) = \{X \in 2^\omega : \exists W \leqslant_T X \oplus Z \ (W \ E \ Y)\}$, and that the set $\{X : \omega_1^X \geqslant \alpha\}$ relativized to Z becomes $\{X : \omega_1^{X\oplus Z} \geqslant \alpha\}$.

Let us observe that this result applies to all the examples mentioned before. For instance, let $X \ E_{\omega_1} \ Y$ if either neither of X and Y codes a well ordering of ω, or the orderings they code are isomorphic. This is a Σ_1^1 equivalence relation with one equivalence class for each countable ordinal, and one equivalence class for all the reals not coding a well ordering. It has $\aleph_1$ equivalence classes, and by Spector's theorem it satisfies hyperarithmetic-is-recursive. We can do the same with biembeddability of linear orderings or p-groups, which we know have $\aleph_1$ equivalence classes. So, Theorem 1.3 tells us that they satisfy hyperarithmetic-is-recursive on a cone. The proofs in [**Mon05**, **GM08**] proved these results relative to every oracle, and not just on a cone. Our general proof does not say anything about what happens relative to every oracle. However, if the relation is natural enough, we expect the behavior to be the same relative to every oracle and relative to almost every oracle. The proofs in [**Mon05**, **GM08**] still require a deep analysis of the embeddability relation among linear orderings and p-groups used for those results. In [**Mon13**], the author showed that any counterexample to Vaught's

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conjecture must satisfy hyperarithmetic-is-recursive on a cone, and that result follows directly from Theorem 1.3. However, the proof in [**Mon13**] is much more constructive, and analyzes the structure among the models of a counterexample to Vaught's conjecture, something we do not get from the proof in this paper.

Theorem 1.3 uses ZF + Σ^1_2-determinacy. That (H3) implies (H2), and that (H2) implies (H1), can be proved in just ZF. The use of ZF + Σ^1_2-determinacy is only necessary to show that (H1) implies (H3). That (H1) implies (H2) only requires Σ^1_1-determinacy, which is equivalent to the existence of sharps (fo rall X ($X^\sharp$ exists)), as proved by Harrington [**Har78**]. We show that the use of Σ^1_1-determinacy is actually necessary.

THEOREM 1.4 (ZF). *The following statements are equivalent.*

(O1) *Every lightface Σ^1_1 equivalence relation without perfectly many classes satisfies hyperarithmetic-is-recursive on a cone.*

(O2) $0^\sharp$ *exists.*

This theorem will be proved in Section 3.

An interesting remark about our main theorem 1.3 is that it shows how cardinality issues get reflected at the hyperarithmetic/computable level.

2. The proof of the main theorem

We start by proving the following effective version of Burgess' theorem [**Bur79**, Corollary 1].

LEMMA 2.1. *For every Σ^1_1 equivalence relation E there is a decreasing nested sequence of equivalence relations $\{E_\alpha : \alpha \in \omega_1\}$ such that E_α is $\Sigma^0_{\alpha+1}$ uniformly in α, and $E = \bigcap_{\alpha\in\omega_1} E_\alpha$.*

Proof. Using Kleene's normal form, let T be a computable subtree of $2^{<\omega} \times \omega^{<\omega} \times 2^{<\omega}$ such that, for all X, Y, if we let

$$T_{X,Y} = \{\sigma \in \omega^{<\omega} : (X \upharpoonright |\sigma|, \sigma, Y \upharpoonright |\sigma|) \in T\},$$

then $X \; E \; Y$ if and only if $T_{X,Y}$ is ill founded. The first wrong idea would be to let $E_\alpha = \{(X, Y) : rk(T_{X,Y}) \geqslant \alpha\}$, which is known to be $\Sigma^0_{\alpha+1}$ uniformly in α and satisfies $E = \bigcap_{\alpha\in\omega_1} E_\alpha$. Unfortunately E_α might not be transitive or symmetric. In Burgess' proof [**Bur79**], he shows that, for a club of ordinals α, E_α is an equivalence relation, which is all he needs to get his result. This is not enough for our more effective version.

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To get the symmetry property, let us replace T by the tree $T \cup \{(\tau, \sigma, \rho) : (\rho, \sigma, \tau) \in T\}$. This way we get that $T_{X,Y} = T_{Y,X}$, and we still have that $X \mathrel{E} Y \iff \neg WF(T_{X,Y})$.

We will modify the tree even further to get transitivity. For each $k \geqslant 1$ and $X, Y \in 2^\omega$, let

$$T^k_{X,Y} = \{(\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \tau_{k-1}, \sigma_k) \in \omega^n \times 2^n \times \cdots \times 2^n \times \omega^n : \\ n \in \omega, (X \restriction n, \sigma_1, \tau_1) \in T, (\tau_1, \sigma_2, \tau_2) \in T, \ldots, (\tau_{k-1}, \sigma_k, Y \restriction n) \in T\}.$$

Note that $T^1_{X,Y} = T_{X,Y}$. Let $\hat{T}_{X,Y} = \sum_{k\in\omega} T^k_{X,Y}$, that is, the disjoint union of all the $T^k_{X,Y}$ identifying the roots of all these trees. We note that $X \mathrel{E} Y \iff \neg WF(\hat{T}_{X,Y})$: this is because, if there is a path through one of the $T^k_{X,Y}$, then we would have $(Z_1, X_1, \ldots, X_{k-1}, Z_k)$ such that, for all i, $Z_{i+1} \in T_{X_i,X_{i+1}}$, where $X_0 = X$ and $X_k = Y$, and hence $X = X_0 \mathrel{E} X_1 \mathrel{E} X_2 \mathrel{E} \cdots \mathrel{E} X_k = Y$. On the other hand, if $X \mathrel{E} Y$, then $T^1_{X,Y}$ is ill founded, and hence so is $\hat{T}_{X,Y}$.

We are now ready to define E_α as follows. Let

$$X \mathrel{E_\alpha} Y \iff rk(\hat{T}_{X,Y}) \geqslant \alpha.$$

We still have that $X \mathrel{E} Y \iff (\forall \alpha < \omega_1)\, X \mathrel{E_\alpha} Y$, that these relations are nested, and that they are uniformly $\Sigma^0_{\alpha+1}$. We now claim that each E_α is an equivalence relation. They are reflexive just because E is. It is not hard to see that $rk(\hat{T}_{X,Y}) = rk(\hat{T}_{Y,X})$, and hence that E_α is symmetric.

To prove transitivity, suppose that $X \mathrel{E_\alpha} Y \mathrel{E_\alpha} Z$. Then, since $rk(\hat{T}_{X,Y}) = \sup\{rk(T^k_{X,Y}) : k \in \omega\}$, for every $\beta < \alpha$ there exist $k, l \in \omega$, $rk(T^k_{X,Y}) \geqslant \beta$, and $rk(T^l_{Y,Z}) \geqslant \beta$. We claim that $rk(T^{k+l}_{X,Z}) \geqslant \beta$, which would imply that $rk(\hat{T}_{X,Y}) \geqslant \alpha$, and hence that $X \mathrel{E_\alpha} Z$, as needed. For each $(\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_k) \in T^k_{X,Y}$ and $(\hat{\sigma}_1, \hat{\tau}_1, \hat{\sigma}_2, \hat{\tau}_2, \ldots, \hat{\sigma}_l) \in T^l_{Y,Z}$ of the same length n, we note that $(\sigma_1, \tau_1, \sigma_2, \tau_2, \ldots, \sigma_k, Y \restriction n, \hat{\sigma}_1, \hat{\tau}_1, \hat{\sigma}_2, \hat{\tau}_2, \ldots, \hat{\sigma}_l) \in T^{k+l}_{X,Z}$. This is an order-preserving embedding from $\{(\rho, \pi) \in T^k_{X,Y} \times T^l_{Y,Z} : |\rho| = |\pi|\}$ into $T^{k+l}_{X,Z}$. It follows that $rk(T^{k+l}_{X,Z}) \geqslant \min\{rk(T^k_{X,Y}), rk(T^l_{Y,Z})\}$, and hence that $rk(T^{k+l}_{X,Z}) \geqslant \beta$, as wanted. □

REMARK 2.2. Notice that, if $X \mathrel{E_{\omega_1^{X\oplus Y}}} Y$, then $X \mathrel{E} Y$. This is because $\hat{T}_{X,Y}$ is computable in $X \oplus Y$, and hence, if it is well founded, it has rank below $\omega_1^{X\oplus Y}$.

The following is the key lemma to prove the main direction of Theorem 1.3. We will then apply Turing determinacy to the set considered in the lemma, or to a variation of it, to get what we want. Recall that, for a complexity class Γ, *Γ-Turing determinacy* says that any degree-invariant Γ-set of reals $\mathcal{S}$ which is cofinal in the Turing degrees contains a cone. (A set $\mathcal{S}$ is *degree invariant* if

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$\forall X \equiv_T Y$ $(X \in \mathcal{S} \leftrightarrow Y \in S)$, and it is *cofinal* if $\forall Z \in 2^\omega\ \exists X \geqslant_T Z$ $(X \in \mathcal{S})$.) Γ-Turing determinacy was introduced by Martin, who showed that it follows from plain Γ-determinacy [Mar68].

LEMMA 2.3 (ZF). *For every analytic equivalence relation E without perfectly many classes, the set $\mathcal{S} \subseteq 2^\omega$, defined as follows*

$$\mathcal{S} = \{X \in 2^\omega : \forall Y\ (\omega_1^{X \oplus Y} = \omega_1^X \Rightarrow X \in Sp([Y]_E)\},$$

is cofinal in the Turing degrees.

Proof. To prove that $\mathcal{S}$ is cofinal, take any Z, and let us build $X \in \mathcal{S}$ with $X \geqslant_T Z$. By relativizing the rest of the argument, let us assume that Z is computable and that E is lightface Σ^1_1, and hence that the tree $\hat{T}$ used in Lemma 2.1 is computable.

For each α, there is no perfect set of E_α-inequivalent reals, as otherwise there would be one for E. Silver [Sil80] showed that any Borel equivalence relation without perfectly many classes has countably many classes. Thus, each E_α has countably many classes. For each $\alpha \in \omega_1$, let $\langle A_{\alpha,n} : n \in \omega \rangle \subseteq 2^\omega$ be a list which contains one real of each E_α-equivalence class. (For the reader who worries about the use of choice, we will see how to avoid it later.) Let us code this whole sequence as a single subset A of $\omega_1 \times \omega \times \omega$: just let $(\alpha, n, m) \in A$ if and only if $m \in A_{\alpha,n}$. Recall Gödel's hierarchy $L_\alpha[A]$, where A is considered as a relation symbol and $L_{\alpha+1}[A]$ consist of the first-order definable subsets of $(L_\alpha[A]; \in, A \cap \alpha \times \omega \times \omega)$ (see, for instance, [Kan03, Section 1.3]). For some $\alpha \in \omega_1$ we have that $L_\alpha[A]$ is admissible, and that every $\beta < \alpha$ can be coded by a well ordering of ω within $L_\alpha[A]$. (For instance, take any α where $L_\alpha[A]$ is an elementary substructure of $L_{\omega_1^{L[A]}}[A]$.) Now, using Barwise compactness for the admissible set $L_\alpha[A]$ [Bar75, Theorem III.5.6], we get an ill-founded model $\mathcal{M} = (M; \in^{\mathcal{M}}, A^{\mathcal{M}})$ of KP whose ordinals have well-founded part equal to α, with $A^{\mathcal{M}} \restriction \alpha$ coinciding with $A \restriction \alpha$, and satisfying that every ordinal can be coded by a real. (To show this, one has to consider the infinitary theory in the language $L = \{\in, A, c\}$ saying all this, plus axioms saying that the constant symbol c is an ordinal and that any ordinal below α exists and that c is above it. Then observe that whole the set of axioms is $\Sigma_1(L_\alpha[A])$, and that, choosing c appropriately, $L_\alpha[A]$ is a model of any subset of these axioms which is a set in $L_\alpha[A]$. Thus, by Barwise compactness [Bar75, Theorem III.5.6], this theory has a model and its ordinals have well-founded part at least α. Then, using [Bar75, Theorem III.7.5], we get such a model with well-founded part exactly α.) Let $\alpha^* \in ON^{\mathcal{M}} \setminus \alpha$, and let X be a real in $\mathcal{M}$ coding α^* and $A^{\mathcal{M}} \restriction \alpha^*$. Notice that $\omega_1^X = \alpha$. (To see this, we have that $\omega_1^X \geqslant \alpha$ because X codes every initial segment of α, and $\omega_1^X \leqslant \alpha$ because every X-computable well ordering is isomorphic to an ordinal in $\mathcal{M}$.)

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We claim that $X \in \mathcal{S}$. Consider Y with $\omega_1^{X \oplus Y} \leqslant \alpha$; we must show that X computes a real E-equivalent to Y. Let us think of α^* as the well ordering of ω of type α^* which is coded by X. Let

$$P = \{\beta \in \alpha^* : (\exists W \leqslant_T X)\ W\ E_\beta\ Y\}.$$

(Let us remark that, when β is not a true ordinal, i.e., $\beta \in \alpha^* \setminus \alpha$, we can still talk about E_β using the definition from Lemma 2.1; that is, $X\ E_\beta\ Y \iff rk(\hat{T}_{X,Y}) \geqslant \beta \iff \exists f : \beta \to \hat{T}_{X,Y}\ (\forall \gamma, \delta < \beta\ (f(\gamma) \subsetneq f(\delta) \to \gamma > \delta))\}$.) The set $P \subseteq \omega$ is $\Sigma_1^1(X, Y)$, using this Σ_1^1 definition of E_β. The set P contains all the true ordinals $\beta < \alpha$ because X computes all the reals $A_{\beta,n}$, which are taken one from each E_β-equivalence class. We can now apply an overspill argument: since $\omega_1^{X \oplus Y} \leqslant \alpha$, α (viewed as the initial segment of the presentation of α^*) is not $\Sigma_1^1(X \oplus Y)$ (as, being the well-ordered part of α^* is $\Pi_1^1(X)$, and it cannot be $\Delta_1^1(X, Y)$). Thus, there must exist a nonstandard $\beta^* \in P \setminus \alpha$. Let Y^* be the witness that $\beta^* \in P$. That is, $Y^* \leqslant_T X$ and $Y^*\ E_{\beta^*}\ Y$. By the nestedness of these equivalence relations, for all true ordinals $\beta < \alpha$, $Y^*\ E_\beta\ Y$. Since $\omega_1^{Y \oplus Y^*} \leqslant \omega_1^{Y \oplus X} = \alpha$, by Remark 2.2, we have that $Y^*\ E\ Y$, as needed to get that $X \in \mathcal{S}$.

For the interested reader, let us see how to avoid the use of the axiom of choice. This proof uses the axiom of choice only to define the sequence $A_{\beta,n}$, which can be defined directly as follows. By Shoenfield's absoluteness, for each $\beta < \omega_1^L$, the sequence $\langle A_{\beta,n} : n \in \omega\rangle$ can be taken to be inside $L_{\omega_1^L}$, and hence we can define it as the $<_L$-least such that $\forall Y \exists n\ (Y\ E_\beta\ A_{\beta,n}) \forall n, m\ \neg(A_{\beta,n}\ E_\beta\ A_{\beta,m})$. This definition works inside $L_{\omega_1^L}$, and hence $(L_{\omega_1^L}; \in, A)$ is admissible, and we can let $\alpha = \omega_1^L$. (Unless the reader is worried that for this lemma we might have $\omega_1^L = \omega_1$, in which case any ordinal α with $L_\alpha[A]$ an elementary substructure of $(L_{\omega_1^L}; \in, A)$ would work.) □

We are now ready to prove the main theorem. Let us start by showing that, if E does not have perfectly many classes, then E satisfies hyperarithmetic-is-recursive on a cone.

Proof of (H1) $\Rightarrow$ (H2) *in* (ZF + Σ_1^1-*Det*). Consider the set $\mathcal{S}_1$ of the oracles relative to which E satisfies hyperarithmetic-is-recursive; that is,

$$\mathcal{S}_1 = \{X \in 2^\omega : \forall Y \leqslant_{hyp} X\ \exists W \leqslant_T X\ (W\ E\ Y)\},$$

where $Y \leqslant_{hyp} X$ means that Y is hyperarithmetic-in-X. This set is Σ_1^1, as the quantifier $\forall Y \leqslant_{hyp} X$ can be replaced by an existential quantifier over all the reals (see [**Sac90**, Exercise III.3.11]). The set $\mathcal{S}_1$ is clearly degree invariant. Also, it contains the set $\mathcal{S}$ because, by Spector's theorem, $Y \leqslant_{hyp} X \Rightarrow \omega_1^{X \oplus Y} = \omega_1^X$, and hence, by Lemma 2.3, it is cofinal in the Turing degrees. By Σ_1^1-Turing determinacy, which follows from Σ_1^1-determinacy, it contains a cone. □

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Let us now show that, if E does not have perfectly many classes, then, relative to some oracle, all the degree spectra of the E-equivalence classes are of the form $\{X : \omega_1^X \geqslant \alpha\}$.

Proof of (H1) $\Rightarrow$ (H3) *in* (ZF + Σ_2^1*-Det*). Consider the set $\mathcal{S}$ from Lemma 2.3. This set is Π_2^1 and degree invariant. (We are using that the relation $\omega_1^Y = \omega_1^X$ is Σ_1^1, as it says that every Y-computable well ordering is isomorphic to an X-computable ordering, and vice versa. It is easy to see that '$X \in Sp([Y]_E)$' is Σ_1^1.) So, by Σ_2^1-Turing-determinacy, which follows from Σ_2^1-determinacy, we have that $\mathcal{S}$ contains a cone.

Relativize the rest of the proof to the base of this cone, and hence assume that every real belongs to $\mathcal{S}$. Take $Y \in 2^\omega$. We claim that

$$Sp([Y]_E) = \{X \in 2^\omega : \omega_1^X \geqslant \omega_1^{[Y]}\},$$

where $\omega_1^{[Y]} = \min\{\omega_1^W : W \ E \ Y\}$. It is clear from the definition of $\omega_1^{[Y]}$ that, if $\omega_1^{[Y]} > \omega_1^X$, then X computes no real E-equivalent to Y. Suppose now that $\omega_1^{[Y]} \leqslant \omega_1^X$: we need to show that X computes a real E-equivalent to Y. Assume, without loss of generality, that Y is such that $\omega_1^Y = \omega_1^{[Y]}$ (otherwise, replace it by an E-equivalent real with this property).

The first step is to show that there exists a G satisfying

$$\omega_1^X = \omega_1^{X\oplus G} \quad \text{and} \quad \omega_1^G = \omega_1^{G\oplus Y} = \omega_1^Y.$$

Let us first show how we would use such a G. Since $G \in \mathcal{S}$ and $\omega_1^{G\oplus Y} = \omega_1^G$, there is a $Y_1 \leqslant_T G$ such that $Y_1 \ E \ Y$. Therefore $\omega_1^{Y_1\oplus X} \leqslant \omega_1^{G\oplus X} = \omega_1^X$. Since $X \in \mathcal{S}$, X computes Y_2 such that $Y_2 \ E \ Y_1$, and hence $Y_2 \ E \ Y$.

Now, we consider the existence of G: the proof is just a small modification of the proof of [**Mon13**, Lemma 3.6], where the same is proved under the assumption that $\omega_1^X = \omega_1^Y$ (which was also proved in [**Har78**, Lemma 2.10]). Let $\mathcal{H}_Y$ be a Y-computable copy of $\omega_1^Y \cdot (1 + \mathbb{Q})$ (the Harrison linear ordering relative to Y), and let $\mathcal{H}_X$ be an X-computable copy of the same ordering, $\omega_1^Y \cdot (1 + \mathbb{Q})$. Let f be an isomorphism between these two copies, and let g be a permutation of ω that is hyperarithmetically generic relative to X, Y, and f. Let G be the pull-back of $\mathcal{H}_Y$ through g. Exactly as in [**Mon13**, Lemma 3.6], we get that $\omega_1^{G\oplus Y} \leqslant \omega_1^Y$ by the genericity of G, and that $\omega_1^G \geqslant \omega_1^Y$ because G computes a copy of $\omega_1^Y \cdot (1 + \mathbb{Q})$. G is also the pull-back of $\mathcal{H}_X$ through $f \circ g$, which is a generic permutation of ω, and hence $\omega_1^{G\oplus X} \leqslant \omega_1^{f\circ g, X} \leqslant \omega_1^X$. □

It not hard to see that (H3) implies (H2).

Proof of (H2) $\Rightarrow$ (H1) *in ZF.* Suppose that there is a perfect tree $R \subseteq 2^{<\omega}$ all of whose paths are E-inequivalent. We need to show that, relative to every oracle

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on a cone, there is a hyperarithmetic real not E-equivalent to any computable real. By relativizing the rest of the proof, assume that this oracle and R are both computable.

First, let us observe that, for some $\alpha < \omega_1^{CK}$, all the paths through R are not only E-inequivalent, but also E_α-inequivalent: for each $X, Y \in [R] \times [R]$ with $X \neq Y$ there is an ordinal β such that $\neg(X \; E_\beta \; Y)$, namely the rank of $\hat{T}_{X,Y}$ plus 1 (where $\hat{T}_{X,Y}$ is as in Lemma 2.1). Thus, $\hat{T}$ gives us a computable map from $[R] \times [R] \setminus \{(X, X) : X \in [R]\}$ to the class of well-founded trees. By Σ_1^1-boundedness (due to Spector [**Spe55**]), the ranks of these trees are all bounded below some ordinal $\alpha \in \omega_1^{CK}$.

Let G be an $(\alpha + 1)$-Cohen-generic real (i.e., it decides every $\Sigma_{\alpha+1}^0$ formula) computable from $0^{(\alpha+2)}$, and let $R(G)$ be the path through R following G at every split. So $R(G)$ is hyperarithmetic. We claim that it is not E-equivalent to any computable real. Suppose it is, that X is computable, and that $X \; E \; R(G)$. Since all the paths are E_α-inequivalent, for any other path $Z \in [R]$, $Z \neq R(G)$, we have that $\neg(Z \; E_\alpha \; X)$. The real G can then be defined as the unique real such that $R(G) \; E_\alpha \; X$, which is a $\Sigma_{\alpha+1}^0$ formula. By $\alpha + 1$-genericity, there is a condition $p \in 2^{<\omega}$ forcing that G satisfies this formula. But then every other $\alpha + 1$-generic extending p would satisfy this formula too, contradicting the uniqueness of G. □

3. A reversal

In this section, we show that the use of Σ_1^1-determinacy in proving that (H1) implies (H2) is not only sufficient but also necessary. We do not know, however, if the use of Σ_2^1-determinacy in proving that (H1) implies (H3) is necessary.

Let us remark that, when E is a lightface-Σ_1^1 equivalence relation, our proof of (H1) $\Rightarrow$ (H2) only uses lightface Σ_1^1-determinacy, which is equivalent to the existence of $0^\sharp$. Thus, have we already proved that (O2) implies (O1) in Theorem 1.4.

Before proving the theorem, let us review a key lemma by Sami [**Sam99**]. First, define

$$\mathcal{S} = \{Y \in 2^\omega : \exists Z \in 2^\omega \; (\omega_1^Z = \omega_1^Y \forall W \leqslant_{hyp} Z \; (W \leqslant_T Y))\}.$$

Sami showed that, if $\mathcal{S}$ contains a cone, then $0^\sharp$ exists: he showed [**Sam99**, Proposition 3.8] that, if $\mathcal{S}$ contains the cone with base C, then every C-admissible ordinal is a cardinal in L, which then implies that $0^\sharp$ exists by a result of Silver [**Har78**, Section 1].

Proof of (O1) $\Rightarrow$ (O2). To prove that $0^\sharp$ exists, we will prove that the set $\mathcal{S}$ above contains a cone. For this, we will define a Σ_1^1 equivalence relation E without

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perfectly many classes, and then show that the cone relative to which E satisfies hyperarithmetic-is-recursive is contained in $\mathcal{S}$.

Let R be the set of all reals coding a structure isomorphic to $(L_\alpha(A); \in)$ for some ordinal $\alpha \in \omega_1$ and some $A \subseteq \omega$. This set is Π^1_1, since to verify that a model is a presentation of $L_\alpha(A)$ all one needs to do is check well foundedness, and then check that each level is defined from the previous ones correctly.

Consider the equivalence relation E that holds of presentations of the structures $L_{\alpha_X}(A_X)$ and $L_{\alpha_Y}(A_Y)$ if $\alpha_X = \alpha_Y$ and $\omega_1^{A_X} = \omega_1^{A_Y}$, and which lets all the reals outside R be equivalent to each other. This relation is Σ^1_1, since R is Π^1_1, deciding if $\alpha_X = \alpha_Y$ is Σ^1_1 and deciding if $\omega_1^{A_X} = \omega_1^{A_Y}$ is also Σ^1_1. This equivalence relation has $\aleph_1$ equivalence classes, one for each value of the pair $(\alpha_X, \omega_1^{A_X})$. Since this is true in any model of ZF, E cannot contain perfectly many classes (because having perfectly many classes is a Σ^1_2 statement). So, by (O1), E must satisfy hyperarithmetic-is-recursive on a cone, say with base C. Take $Y \geqslant_T C$: we need to show that $Y \in \mathcal{S}$. For each $\alpha < \omega_1^Y$ there is a presentation of $(L_\alpha(Y); \in)$ which is hyperarithmetic-in-Y. But then Y computes a real E-equivalent to this presentation, that is, a presentation of $(L_\alpha(Z), \in)$ for some Z with $\omega_1^Z = \omega_1^Y$. Let α^* be a presentation of the Harrison linear ordering [**Har68**] relative to Y, that is, a Y-computable linear ordering isomorphic $\omega_1^Y + \omega_1^Y \cdot \mathbb{Q}$. Let

$$P = \{\beta \in \alpha^* : Y \text{ computes a presentation of } (L_\beta(Z); \in) \text{ for some } Z \text{ with } \omega_1^Z = \omega_1^Y\}.$$

This set is $\Sigma^1_1(Y)$ as, given β, checking that a structure is a presentation of $(L_\beta(Z); \in)$ is hyperarithmetic, and checking if $\omega_1^Z = \omega_1^Y$ is Σ^1_1. By our comments before, the set P contains all β in the well-founded part of α^*, namely ω_1^Y. Therefore, by an overspill argument, P must contain some nonstandard $\beta^* \in \alpha^* \smallsetminus \omega_1^Y$. Let Z^* be such that Y computes a copy of $(L_{\beta^*}(Z^*); \in)$. Every real W which is hyperarithmetic in Z^* belongs to $L_\beta(Z^*)$ for some $\beta < \omega_1^Y$ and hence belongs to this presentation of $(L_{\beta^*}(Z^*); \in)$ too. Therefore, $W \leqslant_T Y$. We have shown that $\forall W \leqslant_{hyp} Z^* \, (W \leqslant_T Y))$, as needed to get that $Y \in \mathcal{S}$. □

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