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BERRY–ESSEEN BOUNDS AND THE LAW OF THE ITERATED LOGARITHM FOR ESTIMATORS OF PARAMETERS IN AN ORNSTEIN–UHLENBECK PROCESS WITH LINEAR DRIFT

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Abstract

We study the asymptotic behaviors of estimators of the parameters in an Ornstein–Uhlenbeck process with linear drift, such as the law of the iterated logarithm (LIL) and Berry–Esseen bounds. As an application of the Berry–Esseen bounds, the precise rates in the LIL for the estimators are obtained.

Keywords: Berry–Esseen bound; law of the iterated logarithm; maximum likelihood estimator; Ornstein–Uhlenbeck process; precise rate

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1. Introduction and main results

We consider the following Ornstein–Uhlenbeck (OU) process with linear drift:

$$\mathrm{d}X_t = (-\theta X_t + \gamma)\,\mathrm{d}t + \mathrm{d}W_t, \qquad X_0 = x_0. \tag{1.1}$$

Here $\theta \in (0, +\infty)$ and γ are unknown, $W = \{W_t,\ t \in [0, \infty)\}$ is a standard Brownian motion. We denote by $\mathrm{P}_{\theta,\gamma,x_0}$ the probability distribution of the solution of (1.1) on $C(\mathbb{R}_+, \mathbb{R})$.

It is known that the maximum likelihood estimators of θ and γ are given by [13, p. 64]

$$\hat{\theta}_t = \frac{-t\int_0^t X_s\,\mathrm{d}X_s + (X_t - x_0)\int_0^t X_s\,\mathrm{d}s}{t\int_0^t X_s^2\,\mathrm{d}s - (\int_0^t X_s\,\mathrm{d}s)^2} \tag{1.2}$$

and

$$\hat{\gamma}_t = \frac{-\int_0^t X_s\,\mathrm{d}s\int_0^t X_s\,\mathrm{d}X_s + (X_t - x_0)\int_0^t X_s^2\,\mathrm{d}s}{t\int_0^t X_s^2\,\mathrm{d}s - (\int_0^t X_s\,\mathrm{d}s)^2}. \tag{1.3}$$

It is also well known that $\hat{\theta}_t$ and $\hat{\gamma}_t$ are both strongly consistent estimators of θ and γ. Their asymptotic normality can be found in [13]. Moreover, Gao and Jiang [6] obtained some deviation inequalities and moderate deviations using the logarithmic Sobolev inequality [8] and the exponential martingale method. For additional references on statistical inference of diffusion processes, see [3] and [17].

For the $\gamma \equiv 0$ case, Bishwal [2] obtained the sharp Berry–Esseen bound for $\hat{\theta}_t$. Florens-Landais and Pham [5] obtained large deviations for $\hat{\theta}_t$ using the Gärtner–Ellis theorem. Bercu and Rouault [1] established the sharp large deviation properties of $\hat{\theta}_t$, while Guillin and Liptser [9] and Gao *et al.* [7] obtained the moderate deviations.

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In this paper we first study the joint law of the iterated logarithm (LIL) for $\hat{\theta}_t$ and $\hat{\gamma}_t$.

Theorem 1.1. *Let $\hat{\theta}_t$ and $\hat{\gamma}_t$ be as defined in (1.2) and (1.3). We have, for any $\nu = \binom{\nu_1}{\nu_2} \in \mathbb{R}^2$, under $\mathrm{P}_{\theta,\gamma,x_0}$,*

$$\limsup_{t\to+\infty} \sqrt{\frac{t}{2\log\log t}}\,\nu^\tau \begin{pmatrix} \hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma \end{pmatrix} = -\liminf_{t\to+\infty} \sqrt{\frac{t}{2\log\log t}}\,\nu^\tau \begin{pmatrix} \hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma \end{pmatrix} = (\nu^\tau L\nu)^{1/2} \quad \textit{almost surely (a.s.)},$$

where

$$L = \begin{pmatrix} 2\theta & 2\gamma \\ 2\gamma & \dfrac{2\gamma^2+\theta}{\theta} \end{pmatrix}.$$

Respectively taking $\nu = \binom{1}{0}$ and $\nu = \binom{0}{1}$, we can immediately obtain the following result.

Corollary 1.1. *Under $\mathrm{P}_{\theta,\gamma,x_0}$,*

$$\limsup_{t\to+\infty} \frac{\hat{\theta}_t - \theta}{\sqrt{(4\theta/t)\log\log t}} = -\liminf_{t\to+\infty} \frac{\hat{\theta}_t - \theta}{\sqrt{(4\theta/t)\log\log t}} = 1 \quad \textit{a.s.}$$

and

$$\limsup_{t\to+\infty} \frac{\hat{\gamma}_t - \gamma}{\sqrt{((4\gamma^2+2\theta)/\theta t)\log\log t}} = -\liminf_{t\to+\infty} \frac{\hat{\gamma}_t - \gamma}{\sqrt{((4\gamma^2+2\theta)/\theta t)\log\log t}} = 1 \quad \textit{a.s.}$$

Then, a natural question is: what are the precise rates of the LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$? For a sequence of independent and identically distributed nondegenerate random variables $\{X, X_n, n \ge 1\}$ with $\mathrm{E}\,X = 0$ and $\mathrm{E}\,X^2 = \sigma^2$, this question has been studied explicitly by many authors. Gut and Spătaru [11] established that, for $S_n := \sum_{i=1}^n X_i$,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \sum_{n=1}^{+\infty} \frac{1}{n\log n} \mathrm{P}(|S_n| \ge \varepsilon\sigma\sqrt{n\log\log n}) = 1,$$

which is the precise rate in the LIL for S_n. Pang *et al.* [16] developed similar results for the self-normalized sums S_n/V_n^2 with $V_n^2 = \sum_{i=1}^n X_i^2$, i.e.

$$\lim_{\varepsilon\to 0} \varepsilon^{2b+1} \sum_{n=1}^{+\infty} \frac{(\log\log n)^b}{n\log n} \mathrm{P}(|S_n| \ge (\varepsilon+\alpha_n)\sqrt{2V_n^2\log\log n}) = \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)},$$

where $\alpha_n = O(1/\log\log n)$, $b > -1$, and N stands for the standard normal distribution. More details can be found in [10] and [15].

Motivated by the above remarks, in this paper we also consider the precise rates of the LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$. To this end, we obtain the following Berry–Esseen bounds for $\hat{\theta}_t$ and $\hat{\gamma}_t$.

Theorem 1.2. *For any $\nu \in \mathbb{R}^2$, $\nu \ne 0$, and $\delta > 0$, we have*

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}_{\theta,\gamma,x_0}\left(\sqrt{\frac{t}{\nu^\tau L\nu}}\,\nu^\tau \begin{pmatrix} \hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma \end{pmatrix} \le x \right) - \mathrm{P}(N \le x) \right| = O(t^{-(1+\delta)/2(3+2\delta)}),$$

where the matrix L is as defined in Theorem 1.1.

Respectively taking $\nu = \binom{1}{0}$ and $\nu = \binom{0}{1}$, we obtain the following result.

Corollary 1.2. *For any* $\delta > 0$,

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}_{\theta,\gamma,x_0}\left(\sqrt{\frac{t}{2\theta}}(\hat{\theta}_t - \theta) \le x \right) - \mathrm{P}(N \le x) \right| = O(t^{-(1+\delta)/2(3+2\delta)})$$

and

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}_{\theta,\gamma,x_0}\left(\sqrt{\frac{\theta t}{\theta + 2\gamma^2}}(\hat{\gamma}_t - \gamma) \le x \right) - \mathrm{P}(N \le x) \right| = O(t^{-(1+\delta)/2(3+2\delta)}).$$

By the above Berry–Esseen bounds, we can obtain the precise rates of the LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$.

Theorem 1.3. *Assume that* $\alpha_t = O(1/\log\log t)$. *Then, for* $b > -1$, $0 \ne \nu \in \mathbb{R}^2$, *and the matrix* L *defined in Theorem 1.1, we have*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{2b+2} \int_{\mathrm{e}^{\mathrm{e}}}^{+\infty} \frac{(\log\log t)^b}{t\log t} \mathrm{P}_{\theta,\gamma,x_0}\left(\left| \nu^\tau \begin{pmatrix} \hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma \end{pmatrix} \right| \ge (\varepsilon + \alpha_t)\sqrt{\frac{2\nu^\tau L\nu \log\log t}{t}} \right) \mathrm{d}t$$
$$= \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.$$

Now, we easily obtain the following result.

Corollary 1.3. *Assume that* $\alpha_t = O(1/\log\log t)$. *Then, for* $b > -1$, *we have*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{2b+2} \int_{\mathrm{e}^{\mathrm{e}}}^{+\infty} \frac{(\log\log t)^b}{t\log t} \mathrm{P}_{\theta,\gamma,x_0}\left(|\hat{\theta}_t - \theta| \ge (\varepsilon + \alpha_t)\sqrt{\frac{4\theta}{t}\log\log t} \right) \mathrm{d}t = \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}$$

and

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{2b+2} \int_{\mathrm{e}^{\mathrm{e}}}^{+\infty} \frac{(\log\log t)^b}{t\log t} \mathrm{P}_{\theta,\gamma,x_0}\left(|\hat{\gamma}_t - \gamma| \ge (\varepsilon + \alpha_t)\sqrt{\frac{4\gamma^2 + 2\theta}{\theta t}\log\log t} \right) \mathrm{d}t$$
$$= \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.$$

The paper is organized as follows. In Section 2 we first recall some properties of the OU process (1.1), and then give the proof of Theorem 1.1 by the method of separation. In Section 3, the Berry–Esseen bounds are established by the deviation inequalities for the quadratic functional (see [6]). We give the proof of Theorem 1.3 in Section 4. Throughout this paper, C_0, C_1, C_2, and C_3, depending only on ν, θ, γ, and the initial point x_0, denote positive constants whose values can differ from place to place.

2. LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$

In this section we prove Theorem 1.1 by the method of separation. We first recall some properties of the OU process (1.1). We also refer the reader to [6].

2.1. Some properties of the OU process

It is well known that (1.1) has the following solution:

$$X_t = \left(x_0 - \frac{\gamma}{\theta}\right)\mathrm{e}^{-\theta t} + \frac{\gamma}{\theta} + \mathrm{e}^{-\theta t}\int_0^t \mathrm{e}^{\theta s}\,\mathrm{d}W_s.$$

Consequently, it is easily seen that, for any $t \geq 0$, under $\mathrm{P}_{\theta,\gamma,x_0}$,

$$X_t \sim N(\mu_t, \sigma_t),$$

where

$$\mu_t = \Big(x_0 - \frac{\gamma}{\theta}\Big)\mathrm{e}^{-\theta t} + \frac{\gamma}{\theta}, \qquad \sigma_t = \frac{1}{2\theta}(1 - \mathrm{e}^{-2\theta t}).$$

Set

$$\hat{\mu}_t = \frac{1}{t}\int_0^t X_s \,\mathrm{d}s \quad \text{and} \quad \hat{\sigma}_t^2 = \frac{1}{t}\int_0^t X_s^2 \,\mathrm{d}s - \hat{\mu}_t^2. \tag{2.1}$$

Then, under $\mathrm{P}_{\theta,\gamma,x_0}$,

$$\hat{\mu}_t \sim N\bigg(\frac{1}{\theta t}\Big(x_0 - \frac{\gamma}{\theta}\Big)(1 - \mathrm{e}^{-\theta t}) + \frac{\gamma}{\theta}, \frac{1}{\theta^2 t^2}\Big(t - \frac{1}{2\theta}(\mathrm{e}^{-2\theta t} - 1) + \frac{2}{\theta}(\mathrm{e}^{-\theta t} - 1)\Big)\bigg) \tag{2.2}$$

and

$$\bigg|\mathrm{E}_{\theta,\gamma,x_0}(\hat{\sigma}_t^2) - \frac{1}{2\theta}\bigg| \leq \frac{1}{\theta^2 t}\bigg(\frac{2}{\theta} + \theta\Big(x_0 - \frac{\gamma}{\theta}\Big)^2\bigg). \tag{2.3}$$

Since

$$V(x) := \int_0^x \exp\bigg\{-2\int_0^y (-\theta u + \gamma)\,\mathrm{d}u\bigg\}\,\mathrm{d}y = \int_0^x \mathrm{e}^{-2\gamma y + \theta y^2}\,\mathrm{d}y \to \pm\infty \quad \text{as } x \to \pm\infty$$

and

$$G := \int_{-\infty}^{+\infty} \exp\bigg\{2\int_0^y (-\theta u + \gamma)\,\mathrm{d}u\bigg\}\,\mathrm{d}y = \int_{-\infty}^{+\infty} \mathrm{e}^{2\gamma y - \theta y^2}\,\mathrm{d}y < +\infty,$$

it follows from Theorem 1.16 of [13, p. 40] that the OU process $\{X_t,\ t \geq 0\}$ defined by (1.1) has ergodic properties with the invariant distribution $N(\gamma/\theta, 1/2\theta)$. Together with (2.2), (2.3), and Theorem 1.16 of [13, p. 40], we have the following result.

Lemma 2.1. *As $t \to +\infty$, under $\mathrm{P}_{\theta,\gamma,x_0}$, for any $\beta \in \mathbb{R}$,*

$$\hat{\mu}_t \to \frac{\gamma}{\theta}, \qquad \hat{\sigma}_t^2 \to \frac{1}{2\theta}, \qquad \frac{1}{t}\int_0^t (\beta - X_s)^2\,\mathrm{d}s \to \frac{1}{2\theta} + \frac{1}{\theta^2}(\gamma - \theta\beta)^2 \quad \textit{a.s.}$$

2.2. LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$

Letting $\hat{\mu}_t$ and $\hat{\sigma}_t^2$ be as defined in (2.1), simple calculations lead to

$$\nu^\tau \begin{pmatrix} \hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma \end{pmatrix} = \frac{\nu^\tau M_t}{t} + \nu^\tau R_t, \tag{2.4}$$

where

$$M_t = \begin{pmatrix} \displaystyle\int_0^t (2\gamma - 2\theta X_s)\,\mathrm{d}W_s \\ \displaystyle\int_0^t \Big(\frac{2\gamma^2 + \theta}{\theta} - 2\gamma X_s\Big)\,\mathrm{d}W_s \end{pmatrix}$$

and

$$R_t = \frac{1}{t\hat{\sigma}_t^2}\begin{pmatrix} W_t\Big(\hat{\mu}_t - \dfrac{\gamma}{\theta}\Big) + (1 - 2\theta\hat{\sigma}_t^2)\displaystyle\int_0^t \Big(\frac{\gamma}{\theta} - X_s\Big)\,\mathrm{d}W_s \\ \hat{\mu}_t W_t\Big(\hat{\mu}_t - \dfrac{\gamma}{\theta}\Big) + (\hat{\mu}_t - 2\gamma\hat{\sigma}_t^2)\displaystyle\int_0^t \Big(\frac{\gamma}{\theta} - X_s\Big)\,\mathrm{d}W_s \end{pmatrix}.$$

Proof of Theorem 1.1. By Lemma 2.1, we have, under $\mathrm{P}_{\theta,\gamma,x_0}$,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t(2\gamma-2\theta X_s)^2\,\mathrm{d}s=2\theta\quad\text{a.s.},\tag{2.5}$$

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\bigg(\frac{\theta+2\gamma^2}{\theta}-2\gamma X_s\bigg)^2\,\mathrm{d}s=\frac{2\gamma^2+\theta}{\theta}\quad\text{a.s.},\tag{2.6}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t(2\gamma-2\theta X_s)\bigg(\frac{\theta+2\gamma^2}{\theta}-2\gamma X_s\bigg)\,\mathrm{d}s=2\gamma\quad\text{a.s.}\tag{2.7}$$

It follows from Theorem 4 of [14] that

$$\begin{aligned}\limsup_{t\to+\infty}\frac{\nu^\tau M_t}{\sqrt{2\langle\nu^\tau M\rangle_t\log\log\langle\nu^\tau M\rangle_t}}&=-\liminf_{t\to+\infty}\frac{\nu^\tau M_t}{\sqrt{2\langle\nu^\tau M\rangle_t\log\log\langle\nu^\tau M\rangle_t}}\\&=1\quad\text{a.s.},\end{aligned}$$

where

$$\langle\nu^\tau M\rangle_t=\nu^\tau\langle M\rangle_t\nu$$

with

$$\langle M\rangle_t=\begin{pmatrix}\int_0^t(2\gamma-2\theta X_s)^2\,\mathrm{d}s & \int_0^t(2\gamma-2\theta X_s)\Big(\frac{\theta+2\gamma^2}{\theta}-2\gamma X_s\Big)\,\mathrm{d}s\\ \int_0^t(2\gamma-2\theta X_s)\Big(\frac{\theta+2\gamma^2}{\theta}-2\gamma X_s\Big)\,\mathrm{d}s & \int_0^t\Big(\frac{\theta+2\gamma^2}{\theta}-2\gamma X_s\Big)^2\,\mathrm{d}s\end{pmatrix}.$$

Consequently, by (2.5), (2.6), and (2.7), we have, under $\mathrm{P}_{\theta,\gamma,x_0}$,

$$\lim_{t\to+\infty}\frac{\langle\nu^\tau M\rangle_t}{t}=\nu^\tau L\nu,$$

where

$$L=\begin{pmatrix}2\theta & 2\gamma\\ 2\gamma & \dfrac{2\gamma^2+\theta}{\theta}\end{pmatrix}.$$

Therefore,

$$\limsup_{t\to+\infty}\frac{\nu^\tau M_t}{\sqrt{2t\log\log t}}=-\liminf_{t\to+\infty}\frac{\nu^\tau M_t}{\sqrt{2t\log\log t}}=(\nu^\tau L\nu)^{1/2}\quad\text{a.s.}\tag{2.8}$$

Now, we show that the remainder $\nu^\tau R_t$ on the right-hand side of (2.4) can be neglected in the sense of the LIL. In fact, since

$$\limsup_{t\to+\infty}\frac{W_t}{\sqrt{2t\log\log t}}=-\liminf_{t\to+\infty}\frac{W_t}{\sqrt{2t\log\log t}}=1\quad\text{a.s.}$$

by Lemma 2.1 and (2.8), we have, under $\mathrm{P}_{\theta,\gamma,x_0}$, as $t\to+\infty$,

$$\frac{W_t(\hat\mu_t-\gamma/\theta)}{t\hat\sigma_t^2\sqrt{(4\theta/t)\log\log t}}=\bigg(\hat\mu_t-\frac{\gamma}{\theta}\bigg)\frac{1}{\hat\sigma_t^2}\frac{W_t}{\sqrt{4\theta t\log\log t}}\to0\quad\text{a.s.},\tag{2.9}$$

$$\frac{(1-2\theta\hat\sigma_t^2)\int_0^t(\gamma/\theta-X_s)\,\mathrm{d}W_s}{t\hat\sigma_t^2\sqrt{(4\theta/t)\log\log t}}=(1-2\theta\hat\sigma_t^2)\frac{1}{\hat\sigma_t^2}\frac{\int_0^t(\gamma/\theta-X_s)\,\mathrm{d}W_s}{\sqrt{4\theta t\log\log t}}\to0\quad\text{a.s.},\tag{2.10}$$

$$\frac{\hat{\mu}_t W_t(\hat{\mu}_t - \gamma/\theta)}{t\hat{\sigma}_t^2\sqrt{((4\gamma^2+2\theta)/\theta t)\log\log t}} = \left(\hat{\mu}_t - \frac{\gamma}{\theta}\right)\hat{\mu}_t \frac{1}{\hat{\sigma}_t^2}\frac{W_t}{\sqrt{((4\gamma^2+2\theta)/\theta)t\log\log t}} \to 0 \quad \text{a.s.}, \tag{2.11}$$

and

$$\frac{(\hat{\mu}_t - 2\gamma\hat{\sigma}_t^2)\int_0^t(\gamma/\theta - X_s)\,\mathrm{d}W_s}{t\hat{\sigma}_t^2\sqrt{((4\gamma^2+2\theta)/\theta t)\log\log t}} = (\hat{\mu}_t - 2\gamma\hat{\sigma}_t^2)\frac{1}{\hat{\sigma}_t^2}\frac{\int_0^t(\gamma/\theta - X_s)\,\mathrm{d}W_s}{\sqrt{((4\gamma^2+2\theta)/\theta)t\log\log t}} \to 0 \quad \text{a.s.} \tag{2.12}$$

Together with (2.8)–(2.12), we can complete the proof of Theorem 1.1.

From the proof of Theorem 1.1, we can immediately obtain the following result.

Corollary 2.1. *For $\hat{\theta}_t$ and $\hat{\gamma}_t$ defined as in (1.2) and (1.3), we have, under $\mathrm{P}_{\theta,\gamma,x_0}$,*

$$\limsup_{t\to+\infty}\sqrt{\frac{\langle\nu^\tau M\rangle_t}{2\log\log\langle\nu^\tau M\rangle_t}}\,\nu^\tau\begin{pmatrix}\hat{\theta}_t - \theta\\ \hat{\gamma}_t - \gamma\end{pmatrix} = -\liminf_{t\to+\infty}\sqrt{\frac{\langle\nu^\tau M\rangle_t}{2\log\log\langle\nu^\tau M\rangle_t}}\,\nu^\tau\begin{pmatrix}\hat{\theta}_t - \theta\\ \hat{\gamma}_t - \gamma\end{pmatrix} = \nu^\tau L\nu \quad \textit{a.s.}$$

3. Berry–Esseen bounds of $\hat{\theta}_t$ and $\hat{\gamma}_t$

In this section we give the proof of Theorem 1.2. Before giving our results, we need to mention the following useful results from Gao and Jiang [6].

Lemma 3.1. (Lemma 2.3 and Lemma 2.5 of [6].) *There exist finite positive constants C_0, C_1, and C_2 such that, for all $r > 0$ and all $T \geq 1$,*

$$\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\int_0^t X_s^2\,\mathrm{d}s - \mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t X_s^2\,\mathrm{d}s\right)\right| \geq rt\right) \leq C_0\exp\{-C_1 rt\min\{1, C_2 r\}\},$$
$$\mathrm{P}_{\theta,\gamma,x_0}(|\hat{\sigma}_t^2 - \mathrm{E}_{\theta,\gamma,x_0}(\hat{\sigma}_t^2)| \geq r) \leq C_0\exp\{-C_1 rt\min\{1, C_2 r\}\},$$

and, for each fixed $\beta \in \mathbb{R}$,

$$\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\int_0^t (X_s - \beta)\,\mathrm{d}W_s\right| \geq rt\right) \leq C_0\exp\{-C_1 rt\min\{1, C_2 r\}\}.$$

The following result is Lemma 2 of [4].

Lemma 3.2. *Let X and Y be any two random variables on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then, for any $\eta > 0$, we have*

$$\sup_{x\in\mathbb{R}}|\mathrm{P}(X+Y\leq x) - \Phi(x)| \leq \sup_{x\in\mathbb{R}}|\mathrm{P}(X\leq x) - \Phi(x)| + \mathrm{P}(|Y| > \eta) + \frac{\eta}{\sqrt{2\pi}},$$

where $\Phi(x)$ stands for the standard normal distribution function.

3.1. Proof of Theorem 1.2

From (2.4) and Lemma 3.2, we have, for any $\eta > 0$,

$$\sup_{x\in\mathbb{R}}\left|\mathrm{P}_{\theta,\gamma,x_0}\left(\sqrt{\frac{t}{\nu^\tau L\nu}}\nu^\tau\begin{pmatrix}\hat{\theta}_t-\theta\\ \hat{\gamma}_t-\gamma\end{pmatrix}\le x\right)-\mathrm{P}(N\le x)\right|$$
$$\le \eta+\sup_{x\in\mathbb{R}}\left|\mathrm{P}_{\theta,\gamma,x_0}\left(\frac{\nu^\tau M_t}{\sqrt{t\nu^\tau L\nu}}\le x\right)-\mathrm{P}(N\le x)\right|+\mathrm{P}_{\theta,\gamma,x_0}\left(\sqrt{\frac{t}{\nu^\tau L\nu}}|\nu^\tau R_t|>\eta\right)$$
$$:=\eta+I_1(t)+I_2(t,\eta).$$

For $I_2(t,\eta)$, it can be easily seen that

$$\begin{aligned}I_2(t,\eta)\le{}&\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\frac{\nu_1W_t(\hat{\mu}_t-\gamma/\theta)}{t\hat{\sigma}_t^2}\right|\ge\frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\right)\\&+\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\frac{\nu_1(1-2\theta\hat{\sigma}_t^2)\int_0^t(\gamma/\theta-X_s)\,\mathrm{d}W_s}{t\hat{\sigma}_t^2}\right|\ge\frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\right)\\&+\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\frac{\nu_2\hat{\mu}_tW_t(\hat{\mu}_t-\gamma/\theta)}{t\hat{\sigma}_t^2}\right|\ge\frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\right)\\&+\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\frac{\nu_2(\hat{\mu}_t-2\gamma\hat{\sigma}_t^2)\int_0^t(\gamma/\theta-X_s)\,\mathrm{d}W_s}{t\hat{\sigma}_t^2}\right|\ge\frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\right)\\:={}&I_{21}(t,\eta)+I_{22}(t,\eta)+I_{23}(t,\eta)+I_{24}(t,\eta).\end{aligned}$$

Now, we have to estimate $I_1(t)$ and $I_{2i}(t,\eta)$, $i=1,2,3,4$.

To estimate $I_1(t)$, we need the following lemma from Theorem 2 of [12].

Lemma 3.3. *Consider a fixed locally square-integrable martingale $\tilde{M}_t$, $t\ge0$. Then, for any $\delta>0$, there exists a finite constant C_δ depending only on δ such that, for $L_{t,2\delta}+N_{t,2\delta}\le1$,*

$$\sup_{x\in\mathbb{R}}|\mathrm{P}(\tilde{M}_t\le x)-\mathrm{P}(N\le x)|\le C_\delta(L_{t,2\delta}+N_{t,2\delta})^{1/(3+2\delta)},$$

where

$$L_{t,2\delta}=\mathrm{E}\left(\sum_{0\le s\le t}|\Delta\tilde{M}_s|^{2+2\delta}\right),\qquad N_{t,2\delta}=\mathrm{E}(|\langle\tilde{M}\rangle_t-1|^{1+\delta}),$$

and $\langle\tilde{M}\rangle$ is the predictable quadratic process of $\tilde{M}$, $\Delta\tilde{M}_t=\tilde{M}_t-\tilde{M}_{t-}$ with $\tilde{M}_{t-}=\lim_{s\uparrow t}\tilde{M}_s$.

Then, we can obtain the estimation of $I_1(t)$.

Lemma 3.4. *For any $\delta>0$ and $\beta\in\mathbb{R}$,*

$$\sup_{x\in\mathbb{R}}\left|\mathrm{P}_{\theta,\gamma,x_0}\left(\frac{\nu^\tau M_t}{\sqrt{t\nu^\tau L\nu}}\le x\right)-\mathrm{P}(N\le x)\right|=O(t^{-(1+\delta)/2(3+2\delta)}).$$

Proof. Let $\tilde{M}_s=\nu^\tau M_s/\sqrt{t\nu^\tau L\nu}$, $0\le s\le t$. Then, it is a continuous martingale, which implies that

$$L_{t,2\delta}=\mathrm{E}_{\theta,\gamma,x_0}\left(\sum_{0\le s\le t}|\Delta\tilde{M}_s|^{2+2\delta}\right)=0,\qquad\langle\tilde{M}\rangle_s=\frac{\langle\nu^\tau M\rangle_s}{t\nu^\tau L\nu}.$$

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By Lemma 3.3 and Fubini's theorem, we obtain

$$\begin{aligned}
\sup_{x\in\mathbb{R}}&\left|\mathrm{P}_{\theta,\gamma,x_0}\left(\frac{\nu^\tau M_t}{\sqrt{t\nu^\tau L\nu}}\le x\right)-\mathrm{P}(N\le x)\right|^{3+2\delta}\\
&\le C_\delta(t\nu^\tau L\nu)^{-1-\delta}\,\mathrm{E}_{\theta,\gamma,x_0}\,|\langle\nu^\tau M\rangle_t-t\nu^\tau L\nu|^{1+\delta}\\
&\le C_\delta 2^\delta(t\nu^\tau L\nu)^{-1-\delta}|\nu^\tau\,\mathrm{E}_{\theta,\gamma,x_0}(\langle M\rangle_t-tL)\nu|^{1+\delta}\\
&\quad+C_\delta 2^\delta\int_0^{+\infty}\mathrm{P}_{\theta,\gamma,x_0}(|\nu^\tau(\langle M\rangle_t-\mathrm{E}_{\theta,\gamma,x_0}\langle M\rangle_t)\nu|\ge t\nu^\tau L\nu x^{1/(1+\delta)})\,\mathrm{d}x.
\end{aligned}$$

On the one hand, by (2.2) and (2.3),

$$\left|\mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t X_s\,\mathrm{d}s\right)-\frac{\gamma}{\theta}t\right|=\frac{|x_0-\gamma/\theta|}{\theta}(1-\mathrm{e}^{-\theta t})\le\frac{|x_0-\gamma/\theta|}{\theta}$$

and

$$\begin{aligned}
\left|\mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t X_s^2\,\mathrm{d}s\right)-\frac{t}{2\theta}-\frac{\gamma^2}{\theta^2}t\right|&\le\frac{1}{\theta^2}\left(\frac{2}{\theta}+\theta\left(x_0-\frac{\gamma}{\theta}\right)^2\right)+\frac{2\gamma}{\theta^2}\left|x_0-\frac{\gamma}{\theta}\right|\\
&\quad+\frac{\gamma}{\theta t}\left(x_0-\frac{\gamma}{\theta}\right)^2,
\end{aligned}$$

which implies that

$$(t\nu^\tau L\nu)^{-1-\delta}|\nu^\tau\,\mathrm{E}_{\theta,\gamma,x_0}(\langle M\rangle_t-tL)\nu|^{1+\delta}=O(t^{-1-\delta}). \tag{3.1}$$

On the other hand, by (2.2) and Lemma 3.1, we have, for any fixed constants $\alpha_1,\alpha_2,\beta\in\mathbb{R}$ and $0<C<+\infty$,

$$\begin{aligned}
\mathrm{P}_{\theta,\gamma,x_0}&(|Q_t(\alpha_1,\alpha_2,\beta,x)|\ge Ctx^{1/(1+\delta)})\\
&\le\mathrm{P}\left(|\beta(\alpha_1+\alpha_2)|\left|\int_0^t X_s\,\mathrm{d}s-\mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t X_s\,\mathrm{d}s\right)\right|\ge\frac{Ctx^{1/(1+\delta)}}{2}\right)\\
&\quad+\mathrm{P}\left(|\alpha_1\alpha_2|\left|\int_0^t X_s^2\,\mathrm{d}s-\mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t X_s^2\,\mathrm{d}s\right)\right|\ge\frac{Ctx^{1/(1+\delta)}}{2}\right)\\
&\le 2\exp\left\{-\frac{C^2t^2x^{2/(1+\delta)}}{4(1+\beta^2(\alpha_1+\alpha_2)^2)}\right\}+C_0\exp\{-C_1tx^{1/(1+\delta)}\min\{1,C_2x^{1/(1+\delta)}\}\},
\end{aligned}$$

where

$$Q_t(\alpha_1,\alpha_2,\beta,x)=\int_0^t(\beta-\alpha_1X_s)(\beta-\alpha_2X_s)\,\mathrm{d}s-\mathrm{E}_{\theta,\gamma,x_0}\left(\int_0^t(\beta-\alpha_1X_s)(\beta-\alpha_2X_s)\,\mathrm{d}s\right).$$

Consequently,

$$\int_0^{+\infty}\mathrm{P}_{\theta,\gamma,x_0}(|Q_t(\alpha_1,\alpha_2,\beta,x)|\ge Ctx^{1/(1+\delta)})\,\mathrm{d}x=O(t^{-(1+\delta)/2}),$$

which implies that

$$\int_0^{+\infty}\mathrm{P}_{\theta,\gamma,x_0}(|\nu^\tau(\langle M\rangle_t-\mathrm{E}_{\theta,\gamma,x_0}\langle M\rangle_t)\nu|\ge t\nu^\tau L\nu x^{1/(1+\delta)})\,\mathrm{d}x=O(t^{-(1+\delta)/2}). \tag{3.2}$$

Using (3.1) and (3.2), we can complete the proof of the lemma.

We now estimate $I_{21}(\eta, t)$ and $I_{23}(\eta, t)$.

Lemma 3.5. *There exists some constant $T > \mathrm{e}^{\mathrm{e}}$ such that, for any $t \geq T$,*

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\frac{W_t(\hat{\mu}_t - \gamma/\theta)}{t\hat{\sigma}_t^2}\bigg| \geq \frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\bigg) \leq C_0\mathrm{e}^{-C_1 t^{1/2}} + 2\mathrm{e}^{-C_1\eta^2 t^{1/2}} \tag{3.3}$$

and

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\frac{\hat{\mu}_t W_t(\hat{\mu}_t - \gamma/\theta)}{t\hat{\sigma}_t^2}\bigg| \geq \frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\bigg) \leq C_0\mathrm{e}^{-C_1 t^{1/2}} + 2\mathrm{e}^{-C_1\eta^2 t^{1/2}}. \tag{3.4}$$

Proof. We only give the proof of (3.4), as (3.3) can be proved similarly. We can see that, for $t \geq 1$,

$$\begin{aligned}
&\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\frac{\hat{\mu}_t W_t(\hat{\mu}_t - \gamma/\theta)}{t\hat{\sigma}_t^2}\bigg| \geq \frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\bigg)\\
&\quad\leq \mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\sigma}_t^2 - \frac{1}{2\theta}\bigg| \geq \frac{1}{4\theta}\bigg) + \mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\mu}_t W_t\bigg(\hat{\mu}_t - \frac{\gamma}{\theta}\bigg)\bigg| \geq \frac{\sqrt{\nu^\tau L\nu t}}{16\theta}\eta\bigg)\\
&\quad\leq \mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\sigma}_t^2 - \frac{1}{2\theta}\bigg| \geq \frac{1}{4\theta}\bigg) + \mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\mu}_t - \frac{\gamma}{\theta}\bigg| \geq \frac{|\gamma|+1}{2\theta t^{1/4}}\bigg)\\
&\qquad+ \mathrm{P}_{\theta,\gamma,x_0}\bigg(\frac{|W_t|}{\sqrt{t}} \geq \frac{\theta\sqrt{\nu^\tau L\nu}}{12(1+|\gamma|)^2}\eta t^{1/4}\bigg).
\end{aligned}$$

Choosing $T > \mathrm{e}^{\mathrm{e}}$ such that, for any $t \geq T$,

$$\frac{1}{\theta^2 t}\bigg(\frac{2}{\theta} + \theta\bigg(x_0 - \frac{\gamma}{\theta}\bigg)^2\bigg) \leq \frac{1}{8\theta}, \qquad \frac{1}{\theta t}\bigg|x_0 - \frac{\gamma}{\theta}\bigg|(1 - \mathrm{e}^{-\theta t}) \leq \frac{|\gamma|+1}{4\theta t^{1/4}},$$

then, by (2.2), (2.3), and Lemma 3.1, we have

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\sigma}_t^2 - \frac{1}{2\theta}\bigg| \geq \frac{1}{4\theta}\bigg) \leq \mathrm{P}_{\theta,\gamma,x_0}\bigg(|\hat{\sigma}_t^2 - \mathrm{E}_{\theta,\gamma,x_0}(\hat{\sigma}_t^2)| \geq \frac{1}{8\theta}\bigg) \leq \frac{C_0}{2}\mathrm{e}^{-C_1 t^{1/2}} \tag{3.5}$$

and

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\hat{\mu}_t - \frac{\gamma}{\theta}\bigg| \geq \frac{|\gamma|+1}{2\theta t^{1/4}}\bigg) \leq \mathrm{P}_{\theta,\gamma,x_0}\bigg(|\hat{\mu}_t - \mathrm{E}_{\theta,\gamma,x_0}(\hat{\mu}_t)| \geq \frac{|\gamma|+1}{4\theta t^{1/4}}\bigg) \leq \frac{C_0}{2}\mathrm{e}^{-C_1 t^{1/2}}. \tag{3.6}$$

Since $W_t/t^{1/2} \sim N(0, 1)$, then

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\frac{|W_t|}{\sqrt{t}} \geq \frac{\theta\sqrt{\nu^\tau L\nu}}{12(1+|\gamma|)^2}\eta t^{1/4}\bigg) \leq 2\mathrm{e}^{-C_1\eta^2 t^{1/2}}. \tag{3.7}$$

Finally, (3.4) immediately follows from (3.5), (3.6), and (3.7).

By Lemma 3.1 and a similar method as used in the proof of Lemma 3.5, we can obtain the following estimations of $I_{22}(\eta, t)$ and $I_{24}(\eta, t)$.

Lemma 3.6. *There exists some constant $T > \mathrm{e}^{\mathrm{e}}$ such that, for any $t \geq T$,*

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\frac{(1 - 2\theta\hat{\sigma}_t^2)\int_0^t(\gamma/\theta - X_s)\,\mathrm{d}W_s}{t\hat{\sigma}_t^2}\bigg| \geq \frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\bigg) \leq C_0\mathrm{e}^{-C_1 t^{1/2}} + 2\mathrm{e}^{-C_1\eta^2 t^{1/2}} \tag{3.8}$$

and

$$\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\frac{(\hat{\mu}_t - 2\gamma\hat{\sigma}_t^2)\int_0^t (\gamma/\theta - X_s)\,\mathrm{d}W_s}{t\hat{\sigma}_t^2}\bigg| \geq \frac{\eta}{4}\sqrt{\frac{\nu^\tau L\nu}{t}}\bigg) \leq C_0\mathrm{e}^{-C_1 t^{1/2}} + 2\mathrm{e}^{-C_1\eta^2 t^{1/2}}. \tag{3.9}$$

We now continue with the proof of Theorem 1.2. Let

$$\eta = \frac{1}{\sqrt{C_1}}\bigg(\frac{\log^2 t}{t}\bigg)^{1/4}$$

in (3.3), (3.4), (3.8), and (3.9). We deduce that

$$I_1(t) = O(t^{-(1+\delta)/2(3+2\delta)}), \qquad I_{2i}(t,\eta) = O(t^{-1}) \quad \text{for } i = 1, 2, 3, 4,$$

which completes the proof of Theorem 1.2.

4. Precise rates of the LIL for $\hat{\theta}_t$ and $\hat{\gamma}_t$

We first state the result for a normal variable, which can be easily deduced from Proposition 3.1 of [16].

Lemma 4.1. *Assume that* $\alpha_t = O(1/\log\log t)$. *Then, for* $b > -1$, *we have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon^{2b+2}\int_{\mathrm{e}^\mathrm{e}}^{+\infty}\frac{(\log\log t)^b}{t\log t}\mathrm{P}(|N| \geq (\varepsilon+\alpha_t)\sqrt{2\log\log t})\,\mathrm{d}t = \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.$$

We can now give the proof of Theorem 1.2 by the Berry–Esseen bound.

Proof of Theorem 1.2. By Theorem 1.2,

$$\int_{\mathrm{e}^\mathrm{e}}^{+\infty}\frac{(\log\log t)^b}{t\log t}\Delta_t\,\mathrm{d}t < +\infty,$$

where

$$\Delta_t = \sup_{x\in\mathbb{R}}\bigg|\mathrm{P}_{\theta,\gamma,x_0}\bigg(\sqrt{\frac{t}{\nu^\tau L\nu}}\nu^\tau\begin{pmatrix}\hat{\theta}_t - \theta\\ \hat{\gamma}_t - \gamma\end{pmatrix} \leq x\bigg) - \mathrm{P}(N \leq x)\bigg|.$$

Then we have, by Lemma 4.1,

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\varepsilon^{2b+2}\int_{\mathrm{e}^\mathrm{e}}^{+\infty}\frac{(\log\log t)^b}{t\log t}\mathrm{P}_{\theta,\gamma,x_0}\bigg(\bigg|\nu^\tau\begin{pmatrix}\hat{\theta}_t - \theta\\ \hat{\gamma}_t - \gamma\end{pmatrix}\bigg| \geq (\varepsilon+\alpha_t)\sqrt{\frac{2\nu^\tau L\nu\log\log t}{t}}\bigg)\,\mathrm{d}t\\
&\quad = \lim_{\varepsilon\downarrow 0}\varepsilon^{2b+2}\int_{\mathrm{e}^\mathrm{e}}^{+\infty}\frac{(\log\log t)^b}{t\log t}\mathrm{P}(|N| \geq (\varepsilon+\alpha_t)\sqrt{2\log\log t})\,\mathrm{d}t\\
&\quad = \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.
\end{aligned}$$

By the deviation inequality for the quadratic functional, we can also obtain the precise rate in the LIL in Corollary 2.1.

Corollary 4.1. *Assume that* $\alpha_t = O(1/\log\log t)$. *Then, for* $b > -1$, *we have*

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{2b+2} \int_{e^e}^{+\infty} \frac{(\log\log t)^b}{t\log t} \mathrm{P}_{\theta,\gamma,x_0}\left(\left|\nu^\tau \begin{pmatrix}\hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma\end{pmatrix}\right| \geq \nu^\tau L\nu(\varepsilon + \alpha_t)\sqrt{\frac{2\log\log\langle\nu^\tau M\rangle_t}{\langle\nu^\tau M\rangle_t}}\right) \mathrm{d}t$$
$$= \frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.$$

Proof. For large enough t and any fixed $1 > \eta > 0$,

$$\mathrm{P}_{\theta,\gamma,x_0}\left(\left|\nu^\tau \begin{pmatrix}\hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma\end{pmatrix}\right| \geq (\varepsilon + \alpha_t)\sqrt{\frac{2\nu^\tau L\nu\log\log t}{t(1-\eta)}}\right) - \mathrm{P}(|\langle\nu^\tau M\rangle_t - t\nu^\tau L\nu| > \eta t\nu^\tau L\nu)$$
$$\leq \mathrm{P}_{\theta,\gamma,x_0}\left(\left|\nu^\tau \begin{pmatrix}\hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma\end{pmatrix}\right| \geq \nu^\tau L\nu(\varepsilon + \alpha_t)\sqrt{\frac{2\log\log\langle\nu^\tau M\rangle_t}{\langle\nu^\tau M\rangle_t}}\right)$$
$$\leq \mathrm{P}_{\theta,\gamma,x_0}\left(\left|\nu^\tau \begin{pmatrix}\hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma\end{pmatrix}\right| \geq (\varepsilon + \alpha_t)\sqrt{\frac{2\nu^\tau L\nu\log\log t}{t(1+\eta)}}\right)$$
$$+ \mathrm{P}(|\langle\nu^\tau M\rangle_t - t\nu^\tau L\nu| > \eta t\nu^\tau L\nu).$$

Thus, it follows from Theorem 1.3 and the proof of Lemma 3.4 that

$$(1-\eta)^{b+1}\frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}$$
$$\leq \lim_{\varepsilon\downarrow 0} \varepsilon^{2b+2} \int_{e^e}^{+\infty} \frac{(\log\log t)^b}{t\log t}$$
$$\times \mathrm{P}_{\theta,\gamma,x_0}\left(\left|\nu^\tau \begin{pmatrix}\hat{\theta}_t - \theta \\ \hat{\gamma}_t - \gamma\end{pmatrix}\right| \geq \nu^\tau L\nu(\varepsilon + \alpha_t)\sqrt{\frac{2\log\log\langle\nu^\tau M\rangle_t}{\langle\nu^\tau M\rangle_t}}\right) \mathrm{d}t$$
$$\leq (1+\eta)^{b+1}\frac{\mathrm{E}\,|N|^{2b+2}}{2^{b+1}(b+1)}.$$

The proof is completed by letting $\eta \to 0$.

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